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APPLICATION OF THE DRAZIN INVERSE TO THE ANALYSIS OF POINTWISE COMPLETENESS AND POINTWISE DEGENERACY OF DESCRIPTOR FRACTIONAL LINEAR CONTINUOUS–TIME SYSTEMS

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The Drazin inverse of matrices is applied to the analysis of pointwise completeness and pointwise degeneracy of fractional descriptor linear continuous-time systems. It is shown that (i) descriptor linear continuous-time systems are pointwise complete if and only if the initial and final states belong to the same subspace, and (ii) fractional descriptor linear continuous-time systems are not pointwise degenerated in any nonzero direction for all nonzero initial conditions. The discussion is illustrated with examples of descriptor linear electrical circuits.

Keywords: pointwise completeness, pointwise degeneracy, fractional systems, descriptor systems.

1. Introduction

A dynamical system described by a homogenous equation is called pointwise complete if every final state of the system can be reached by a suitable choice of its initial state. A system which is not pointwise complete is called pointwise degenerated. Pointwise completeness and pointwise degeneracy of linear continuous-time systems with delays were investigated by Choundhury (1972), Olbrot (1972), Popov (1972) and Trzasko *et al.* (2007), while pointwise completeness of fractional linear discrete-time systems was discussed by Busłowicz (2008), Kaczorek (2011), Kaczorek and Busłowicz (2009) or Kaczorek and Rogowski (2015). Pointwise completeness and pointwise degeneracy of standard and positive hybrid systems described by the general model were analyzed by Kaczorek (2010a), who also discussed the case of positive linear systems with state-feedbacks (Kaczorek, 2010b). The Drazin inverse of matrices was applied to analyze pointwise completeness and pointwise degeneracy of descriptor fractional linear systems in another work of Kaczorek (2019).

In this paper the Drazin inverse will be applied to the analysis of pointwise completeness and pointwise degeneracy of fractional descriptor linear continuous-time systems.

The paper is organized as follows. In Section 2 the basic definitions and theorems concerning fractional descriptor linear continuous-time systems and the Drazin inverse of matrices are recalled. Pointwise completeness of fractional descriptor linear continuous-time systems is investigated in Section 3 and pointwise degeneracy in Section 4. Concluding remarks are given in Section 5. The discussion is illustrated with two examples of a fractional linear electrical circuit.

The following notation will be used: $\mathbb{R}$, the set of real numbers; $\mathbb{R}^{n\times m}$, the set of $n \times m$ real matrices; $\mathbb{R}_+^{n\times m}$, the set of $n\times m$ real matrices with nonnegative entries and $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$; I_n, the $n \times n$ identity matrix. $\operatorname{Im} P$ is the image of the operator (matrix) P.

2. Autonomous fractional descriptor linear systems

Consider the autonomous fractional descriptor continuous-time linear system

$$E\frac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha} = Ax, \quad 0 < \alpha < 1, \tag{1}$$

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where $x = x(t) \in \mathbb{R}^n$ is the state vector, $E,\ A \in \mathbb{R}^{n\times n}$ and

$$\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{f}(\tau)}{(t-\tau)^\alpha}\,\mathrm{d}\tau, \qquad \dot{f}(\tau) = \frac{\mathrm{d}f(\tau)}{\mathrm{d}\tau} \tag{2}$$

is the Caputo fractional derivative while

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t, \quad \mathrm{Re}(x) > 0, \tag{3}$$

is the Gamma function.

It is assumed that $\det E = 0$, but the pencil $(E,\ A)$ is regular, i.e.,

$$\det[Es^\alpha - A] \neq 0 \ \text{ for some } s \in \mathbb{C}, \tag{4}$$

where $\mathbb{C}$ is the field of complex numbers.

Assuming that $\det[Ec - A] \neq 0$ for $c \in \mathbb{R}$ and premultiplying (1) by $[Ec - A]^{-1}$, we obtain

$$\bar{E}\frac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha} = \bar{A}x, \tag{5}$$

where

$$\bar{E} = [Ec-A]^{-1}E, \quad \bar{A} = [Ec-A]^{-1}A. \tag{6}$$

Equations (1) and (5) have the same solution x.

Definition 1. A matrix $E^D \in \mathbb{R}^{q\times n}$ is called the Drazin inverse of E if it satisfies the conditions

$$\bar{E}\bar{E}^D = \bar{E}^D\bar{E}, \tag{7}$$

$$\bar{E}^D\bar{E}\bar{E}^D = \bar{E}^D, \tag{8}$$

$$\bar{E}^D\bar{E}^{q+1} = \bar{E}^q, \tag{9}$$

where q is the index of E defined as the smallest nonnegative integer satisfying the condition

$$\operatorname{rank} E^q = \operatorname{rank} E^{q+1}. \tag{10}$$

Theorem 1. *Let*

$$P = \bar{E}\bar{E}^D, \tag{11}$$

$$Q = \bar{A}\bar{E}^D. \tag{12}$$

Then

$$P^k = P \ \textit{for} \ k = 2, 3, \dots, \tag{13}$$

$$PQ = QP = Q, \tag{14}$$

$$P\bar{E}^D = \bar{E}^D, \tag{15}$$

$$Px = x. \tag{16}$$

A proof is given by Kaczorek and Rogowski (2015).

Theorem 2. *The solution of Eqn. (5) has the form*

$$x(t) = \varphi_0(t)Pw, \tag{17}$$

where

$$\varphi_0(t) = \sum_{k=0}^{\infty} \frac{Q^k t^{k\alpha}}{\Gamma(k\alpha+1)} \tag{18}$$

and $w \in \mathbb{R}^n$ is an arbitrary vector, $x(0) = Pw = \mathrm{Im}\,P$.

The matrix $\varphi_0(t)$ defined by (18) is nonsingular for any $A \in \mathbb{R}^{n\times n}$ and $t \geq 0$ (Kaczorek and Rogowski, 2015).

3. Pointwise completeness of fractional descriptor linear systems

In this section conditions for pointwise completeness of fractional descriptor continuous-time linear systems will be established.

Definition 2. The fractional descriptor continuous-time linear system (1) is called *pointwise complete* for $t = t_f$ if for every final state $x_f = x(t_f) \in \mathbb{R}^n$ there exists an initial condition $x(0) \in \mathrm{Im}\,P$ such that

$$x_f = x(t_f) \in \mathrm{Im}\,P, \tag{19}$$

where P is defined by (11).

Theorem 3. *The fractional descriptor system (1) is pointwise complete for any $t = t_f$ and every $x_f \in \mathbb{R}^n$ if and only if the condition (19) is satisfied.*

Proof. Taking into account that $\det \varphi_0(t) \neq 0$ and the inverse matrix $\varphi_0^{-1}(t)$ exists for any t, from (17) for $t = t_f$ we have

$$x(0) = \varphi_0^{-1}(t_f)x_f. \tag{20}$$

Therefore, for every x_f there exists $x(0) \in \mathrm{Im}\,P$ such that $x(t_f) = x_f$. ■

Example 1. Consider the fractional descriptor linear electrical circuit shown in Fig. 1 with given resistance R, capacitances C_1, C_2 and source voltage e.

Using Kirchhoff's laws we may write the equations

$$e = RC_1\frac{\mathrm{d}^\alpha u_1}{\mathrm{d}t^\alpha} + u_1, \quad e = u_2, \tag{21}$$

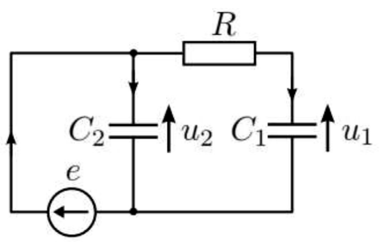

Fig. 1. Fractional electrical circuit of Example 1.

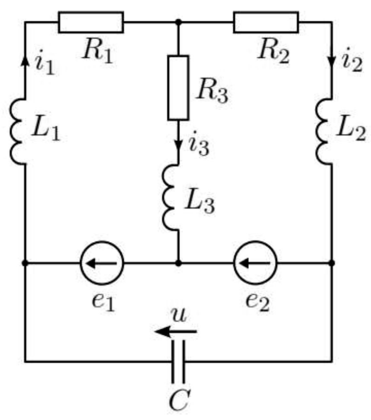

Fig. 2. Fractional electrical circuit of Example 2.

which can be presented in the form

$$E\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = A\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + Be, \tag{22}$$

where

$$E = \begin{bmatrix} RC_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{23}$$

The pencil (E, A) of the electrical circuit is regular, since

$$\det[Es^\alpha - A] = \begin{vmatrix} RC_1 s^\alpha + 1 & 0 \\ 0 & 1 \end{vmatrix} = RC_1 s^\alpha + 1 \neq 0. \tag{24}$$

Choosing $c = 0$ since $\det A = 1$ and using (23), we obtain

$$\bar{E} = -A^{-1}E = E = \begin{bmatrix} RC_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{A} = -A^{-1}A = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{25}$$

In this case we have

$$\bar{E}^D = \begin{bmatrix} \frac{1}{RC_1} & 0 \\ 0 & 0 \end{bmatrix} \tag{26}$$

and

$$P = \bar{E}\bar{E}^D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{27}$$

$$Q = \bar{A}\bar{E}^D = \begin{bmatrix} -\frac{1}{RC_1} & 0 \\ 0 & 0 \end{bmatrix}. \tag{28}$$

The solution of Eqn. (22) for $B = 0$ satisfies the condition

$$\begin{bmatrix} u_1(t_f) \\ u_2(t_f) \end{bmatrix} = \varphi_0(t_f)\begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} \in \mathrm{Im}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{29}$$

Therefore, the fractional descriptor electrical circuit is pointwise complete. ♦

Example 2. Consider the fractional descriptor linear electrical circuit shown in Fig. 2 with given resistances R_1, R_2, R_3, inductances L_1, L_2, L_3, capacitance C and source voltages e_1, e_2.

Using Kirchhoff's laws we may write the equations

$$e_1 = L_1\frac{\mathrm{d}^\alpha i_1}{\mathrm{d}t^\alpha} + R_1 i_1 + L_3\frac{\mathrm{d}^\alpha i_3}{\mathrm{d}t^\alpha} + R_3 i_3, \tag{30}$$

$$e_2 = L_2\frac{\mathrm{d}^\alpha i_2}{\mathrm{d}t^\alpha} + R_2 i_2 - L_3\frac{\mathrm{d}^\alpha i_3}{\mathrm{d}t^\alpha} - R_3 i_3, \tag{31}$$

$$i_3 = i_1 - i_2, \tag{32}$$

$$u = e_1 + e_2. \tag{33}$$

The above equations can be written in the form

$$E\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ u \end{bmatrix} = A\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ u \end{bmatrix} + B\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{34}$$

where

$$E = \begin{bmatrix} L_1 & 0 & L_3 & 0 \\ 0 & L_2 & -L_3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -R_1 & 0 & -R_3 & 0 \\ 0 & -R_2 & R_3 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}. \tag{35}$$

The matrix E is singular and the pencil (E, A) is regular, since

$$\begin{aligned} &\det[Es^\alpha - A] \\ &= \begin{vmatrix} s^\alpha L_1 + R_1 & 0 & s^\alpha L_3 + R_3 & 0 \\ 0 & s^\alpha L_2 + R_2 & -s^\alpha L_3 s - R_3 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \\ &= s^{2\alpha}[L_1(L_2 + L_3) + L_2 L_3] \\ &\quad + s^\alpha[L_1(R_2 + R_3) + L_3(R_1 + R_2) + L_2(R_1 + R_2)] \\ &\quad + R_1(R_2 + R_3) + R_2 R_3 \neq 0. \end{aligned} \tag{36}$$

We choose $c = 0$ since $\det A \neq 0$, and using (35), we obtain

$$
\begin{aligned}
\bar{E} &= -A^{-1}E \\
&= \frac{1}{R_1(R_2+R_3)+R_2R_3} \\
&\quad \times \begin{bmatrix} L_1(R_2+R_3) & L_2R_3 & L_3R_2 & 0 \\ L_1R_3 & L_2(R_1+R_3) & -L_3R_1 & 0 \\ L_1R_2 & -L_2R_1 & L_3(R_1+R_2) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{37}
$$

and

$$
\bar{A} = -A^{-1}A = -I_4.
$$

The Drazin inverse matrix of (37) has the form

$$
\bar{E}^D = \frac{1}{\Delta_L^2} \begin{bmatrix} e_{d,11} & e_{d,12} & e_{d,13} & 0 \\ e_{d,21} & e_{d,22} & e_{d,23} & 0 \\ e_{d,31} & e_{d,32} & e_{d,33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{38}
$$

where

$$
\begin{aligned}
e_{d,11} &= L_1(L_2^2R_1 + L_3^2R_1 + L_2^2R_3 \\
&\quad + L_3^2R_2 + 2L_2L_3R_1), \\
e_{d,12} &= L_2(L_3^2R_1 + L_3^2R_2 - L_1L_2R_3 + L_1L_3R_2 \\
&\quad + L_2L_3R_1), \\
e_{d,13} &= L_3(L_2^2R_1 + L_2^2R_3 + L_1L_2R_3 - L_1L_3R_2 \\
&\quad + L_2L_3R_1), \\
e_{d,21} &= L_1(L_3^2R_1 + L_3^2R_2 - L_1L_2R_3 + L_1L_3R_2 \\
&\quad + L_2L_3R_1), \\
e_{d,22} &= L_2(L_1^2R_2 + L_1^2R_3 + L_3^2R_1 + L_3^2R_2 \\
&\quad + 2L_1L_3R_2), \\
e_{d,23} &= -L_3(L_1^2R_2 + L_1^2R_3 + L_1L_2R_3 \\
&\quad + L_1L_3R_2 - L_2L_3R_1), \\
e_{d,31} &= L_1(L_2^2R_1 + L_2^2R_3 + L_1L_2R_3 \\
&\quad - L_1L_3R_2 + L_2L_3R_1), \\
e_{d,32} &= -L_2(L_1^2R_2 + L_1^2R_3 + L_1L_2R_3 \\
&\quad + L_1L_3R_2 - L_2L_3R_1), \\
e_{d,33} &= L_3(L_1^2R_2 + L_2^2R_1 + L_1^2R_3L_2^2R_3 \\
&\quad + 2L_1L_2R_3), \\
\Delta_L &= L_1(L_2+L_3) + L_2L_3
\end{aligned}
$$

and

$$
\begin{aligned}
P &= \bar{E}\bar{E}^D \\
&= \frac{1}{\Delta_L} \begin{bmatrix} L_1(L_2+L_3) & L_2L_3 & L_2L_3 & 0 \\ L_1L_3 & L_2(L_1+L_3) & -L_1L_3 & 0 \\ L_1L_2 & -L_1L_2 & L_3(L_1+L_2) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{39}
$$

$$
Q = \bar{A}\bar{E}^D = -\bar{E}^D. \tag{40}
$$

The solution of Eqn. (34) for $B = 0$ satisfies the condition

$$
\begin{bmatrix} i_1(t_f) \\ i_2(t_f) \\ i_3(t_f) \\ u(t_f) \end{bmatrix} = \varphi_0(t_f) \begin{bmatrix} i_1(0) \\ i_2(0) \\ i_3(0) \\ u(0) \end{bmatrix} \in \operatorname{Im} P. \tag{41}
$$

Therefore, the fractional descriptor electrical circuit is pointwise complete. ♦

Conclusion 1. In a fractional descriptor linear electrical circuit, by a suitable choice of the initial conditions (currents in the coils and voltages on the capacitors) belonging to $\operatorname{Im} P$, it is always possible to obtain in a given time t_f the desired values of currents in the coils and voltages on the capacitors belonging also to $\operatorname{Im} P$.

4. Pointwise degeneracy of fractional descriptor linear systems

In this section conditions for pointwise degeneracy of fractional descriptor continuous-time linear systems will be established.

Definition 3. The fractional descriptor continuous-time linear system (1) is called pointwise degenerated in the direction $v \in \mathbb{R}^n$ for $t = t_f$ if there exists a nonzero vector v such that, for all initial conditions $x(0) \in \operatorname{Im} Q$, the solution of (1) satisfies the condition

$$
v^T x_f = 0, \tag{42}
$$

where $x_f = x(t_f)$.

Theorem 4. *The fractional descriptor continuous-time linear system (1) is not pointwise degenerated in any nonzero direction $v \in \mathbb{R}^n$ and for all nonzero initial conditions $x(0) \in \operatorname{Im} Q$.*

Proof. Note that $\det \varphi_0(t) \neq 0$ for any matrix $Q = \bar{A}\bar{E}^D$ and all t_f. Substitution of $x_f = \varphi_0(t_f)x(0)$ into $v^T x_f$ yields

$$
v^T x_f = v^T \varphi_0(t_f) x(0) \neq 0 \tag{43}
$$

for all nonzero initial conditions $x(0) \in \operatorname{Im} Q$. ■

Therefore, by Theorem 4 the fractional descriptor electrical circuit shown in Fig. 1 is not pointwise degenerated in any nonzero direction $v \in \mathbb{R}^3$ for all nonzero initial conditions.

Similar results are obtained for the fractional descriptor electrical circuit shown in Fig. 2.

5. Concluding remarks

The Drazin inverse of matrices has been applied to the analysis of pointwise completeness and pointwise degeneracy of fractional descriptor linear continuous-time systems. It has been shown that

(i) the fractional descriptor linear continuous-time system is pointwise complete if and only if the initial and final states belong to the same subspace (Theorem 3);

(ii) the fractional descriptor linear continuous-time system is not degenerated in any nonzero direction for all nonzero initial conditions (Theorem 4).

The discussion has been illustrated with two examples of a fractional descriptor linear electrical circuit.

The provided analysis can be easily extended to the fractional linear discrete-time systems.

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