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ON THREE METHODS FOR BOUNDING THE RATE OF CONVERGENCE FOR SOME CONTINUOUS–TIME MARKOV CHAINS

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Consideration is given to three different analytical methods for the computation of upper bounds for the rate of convergence to the limiting regime of one specific class of (in)homogeneous continuous-time Markov chains. This class is particularly well suited to describe evolutions of the total number of customers in (in)homogeneous $M/M/S$ queueing systems with possibly state-dependent arrival and service intensities, batch arrivals and services. One of the methods is based on the logarithmic norm of a linear operator function; the other two rely on Lyapunov functions and differential inequalities, respectively. Less restrictive conditions (compared with those known from the literature) under which the methods are applicable are being formulated. Two numerical examples are given. It is also shown that, for homogeneous birth-death Markov processes defined on a finite state space with all transition rates being positive, all methods yield the same sharp upper bound.

Keywords: inhomogeneous continuous-time Markov chains, weak ergodicity, Lyapunov functions, differential inequalities, forward Kolmogorov system.

1. Introduction

In this paper we revisit the problem of finding the upper bounds for the rate of convergence of (in)homogeneous continuous-time Markov chains. Consideration is given to classic inhomogeneous birth-death processes and to special inhomogeneous chains with transitions intensities, which do not depend on the current state.

Specifically, let $\{X(t),\ t \geq 0\}$ be an inhomogeneous continuous-time Markov chain with the state space $\mathcal{X} = \{0, 1, 2, \dots, S\}$, where $1 \leq S \leq \infty$. Denote by $p_{ij}(s,t) = P\{X(t) = j \,|X(s) = i\}$, $i, j \geq 0$, $0 \leq s \leq t$, the transition probabilities of $X(t)$ and by $p_i(t) = P\{X(t) = i\}$ the probability that $X(t)$ is in state i at time t. Let $\mathbf{p}(t) = (p_0(t), p_1(t), \dots, p_S(t))^T$ be a probability distribution vector at instant t. Throughout the paper it is assumed that in a small time interval h the possible transitions and their associated probabilities are

$$p_{ij}(t, t+h) = \begin{cases} q_{ij}(t)h + \alpha_{ij}(t,h) & \text{if } j \neq i, \\ 1 - \sum\limits_{k \in \mathcal{X}, k \neq i} q_{ik}(t)h + \alpha_i(t,h) & \text{if } j = i, \end{cases}$$

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where transition intensities $q_{ij}(t) \geq 0$ are arbitrary[1] non-random functions of t, locally integrable on $[0,\infty)$, satisfying $\sup_{i\in\mathcal{X}}\left(\sum_{k\in\mathcal{X},k\neq i} q_{ik}(t)\right) \leq L < \infty$ for almost all $t \geq 0$, and $|\alpha_i(t,h)| = o(h)$ for $S < \infty$ and $\sup_{i\in\mathcal{X}}|\alpha_i(t,h)| = o(h)$ for $S = \infty$. The results of this paper are applicable to Markov chains $X(t)$ with the following transition intensities:

(i) $q_{ij}(t) = 0$ for any $t \geq 0$ if $|i-j| > 1$ and both $q_{i,i+1}(t)$ and $q_{i,i-1}(t)$ may depend on i;

(ii) $q_{i,i-k}(t) = 0$ for $k \geq 2$, $q_{i,i-1}(t)$ may depend on i and $q_{i,i+k}(t)$, $k \geq 1$, depend only on k;

(iii) $q_{i,i-k}(t) = 0$ for $k \geq 1$ depend only on k, $q_{i,i+1}(t)$ may depend on i and $q_{i,i+k}(t) = 0$, $k \geq 2$;

(iv) both $q_{i,i-k}(t)$ and $q_{i,i+k}(t)$, $k \geq 1$, depend only on k and do not depend on i.

Motivated by the application of the obtained results in the theory of queues[2], in what follows it is convenient to think of $X(t)$ as of the process describing the evolution of the total number of customers of a queueing system. Then type (i) transitions describe Markovian queues with possibly state-dependent arrival and service intensities (for example, the classical $M_t(n)/M_t(n)/1$ queue); type (ii) transitions allow consideration of Markovian queues with state-independent batch arrivals and state-dependent service intensity; type (iii) transitions lead to Markovian queues with possible state-dependent arrival intensity and state-independent batch service; type (iv) transitions describe Markovian queues with state-independent batch arrivals and batch service.

For details concerning possible applications of Markovian queues with time-dependent transitions we can refer to the work of Schwarz *et al.* (2016), which contains a broad overview and a classification of time-dependent queueing systems considered up to 2016 and also the works of Crescenzo *et al.* (2018), Giorno *et al.* (2014), Granovsky and Zeifman (2004), Schwarz *et al.* (2016), Zeifmann *et al.* (2006; 2014a), Vvedenskaya *et al.* (2018), Olwal *et al.* (2012), Wieczorek (2010), Li *et al.* (2007), Almasi *et al.* (2005), Moiseev and Nazarov (2016), Brugno *et al.* (2017), Trejo *et al.* (2019) and the references therein.

In this paper we propose three different analytical methods for the computation of the upper bounds[3] for the rate of convergence to the limiting regime (provided that it exists) of any process $X(t)$ belonging to one of the classes (i)–(iv). The first one is based on the *logarithmic norm* of a linear operator function. The second one uses simplest *Lyapunov functions* and the third one relies on *differential inequalities*. Even though the methods are not new, it is the first time it is shown how they can be applied for the analysis of Markov chains with the transition intensities specified by (i)–(iv). This constitutes the main contribution of the paper. Another is the fact that in the case of periodic intensities the bounds on the rate of convergence depend on the intensities only through their mean values over one period.

It is worth noting here that, except for the upper bounds for the rate of convergence, we may also be interested in the lower bounds, stability (perturbation) bounds or truncation bounds (with error estimation). But the exact estimates of the rate of convergence yield exact estimates of stability bounds (see, for example, the works of Kartashov (1985), Liu (2012), Mitrophanov (2003; 2004), Rudolf and Schweizer (2018), Zeifman (1985), Mitrophanov (2018) and the references therein). Moreover, as our research shows (Zeifman *et al.*, 2006; 2014a; 2018c; Zeifman and Korolev, 2014), in some cases, all these quantities can be constructed automatically, given that some good upper bounds for the rate of convergence are provided. This makes us believe that the upper bounds are of primary interest.

Estimation of the convergence rate by virtue of the methods proposed in this paper heavily relies on the notion of the reduced intensity matrix, say $B(t)$, of a Markov chain $X(t)$. The matrix $B(t)$ can be obtained by considering the probabilistic dynamics of the process $X(t)$, given by the forward Kolmogorov system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}(t) = A(t)\mathbf{p}(t), \tag{1}$$

where $A(t)$ is the transposed intensity matrix, i.e., $a_{ij}(t) = q_{ji}(t)$, $i,j \in \mathcal{X}$. Due to the normalization condition $p_0(t) = 1 - \sum_{i=1}^{S} p_i(t)$, we can rewrite[4] the system (1) as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{z}(t) = B(t)\mathbf{z}(t) + \mathbf{f}(t), \tag{2}$$

where

$$\mathbf{f}(t) = (a_{10}(t), a_{20}(t), \dots)^T,$$

$$\mathbf{z}(t) = (p_1(t), p_2(t), \dots)^T,$$

[1] It is not required (as, for example, in the work of Zeifman *et al.*, (2018c)), that $q_{i,i+k}(t)$ and $q_{i,i-k}(t)$ be monotonically decreasing in k for any $t \geq 0$.

[2] Yet the scope of the obtained results is not limited to queueing systems and includes a number of other stochastic systems occurring, for example, in medicine and biology, which satisfy the adopted assumptions.

[3] That is, bounds which guarantee that, after a certain time, say t^*, the probability characteristics of the process $X(t)$ do not depend on the initial conditions (up to a given discrepancy). Since the proposed methods are analytic, we do not compare them here from the numerical point of view (i.e., memory requirement, speed, running time, etc.).

[4] For a detailed discussion of the transformation (2), see the works of Granovsky and Zeifman (2004) or Zeifman *et al.* (2006).

$$B(t)=\begin{pmatrix} a_{11}-a_{10} & a_{12}-a_{10} & \cdots & a_{1r}-a_{10} & \cdots \\ a_{21}-a_{20} & a_{22}-a_{20} & \cdots & a_{2r}-a_{20} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{r1}-a_{r0} & a_{r2}-a_{r0} & \cdots & a_{rr}-a_{r0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3}$$

Here and henceforth each entry of $B(t)$ may depend on t but, for the sake of brevity, the argument is omitted. We note that the matrix $B(t)$ has no probabilistic meaning. All bounds of the rate of convergence to the limiting regime for $X(t)$ correspond to the same bounds of the solutions of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}(t) = B(t)\mathbf{y}(t), \tag{4}$$

because $\mathbf{y}(t) = \mathbf{z}^*(t) - \mathbf{z}^{**}(t)$ is the difference of two solutions of the system (2), and $\mathbf{y}(t) = (y_1(t), y_2(t), \dots, y_S(t))^T$ is the vector with the coordinates of arbitrary signs. As firstly noticed by Zeifman (1989), it is more convenient to study the rate of convergence using the transformed version $B^*(t)$ of $B(t)$ given by $B^*(t) = TB(t)T^{-1}$, where T is the $S \times S$ upper triangular matrix of the form

$$T = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{5}$$

Let $\mathbf{u}(t) = T\mathbf{y}(t)$. Then the system (4) can be rewritten in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}(t) = B^*(t)\mathbf{u}(t), \tag{6}$$

where $\mathbf{u}(t) = (u_1(t), u_2(t), \dots, u_S(t))^T$ is the vector with the coordinates of arbitrary signs. If one of the two matrices $B^*(t)$ or $B(t)$ is known, the other is also (uniquely) defined.

The method based on the logarithmic norm of a linear operator function and the corresponding bounds for the Cauchy operator of the reduced forward Kolmogorov system has already been applied successfully in many settings (see, e.g., Granovsky and Zeifman, 2004; Zeifman *et al.*, 2006). Moreover, Zeifman *et al.* (2018c) obtained the bounds for the rate of convergence and perturbation bounds for a process $X(t)$ belonging to classes (i)–(iv) under the assumption that $B^*(t)$ is essentially non-negative, i.e., $b^*_{ij}(t) \geq 0$, $i \neq j$, $i, j \in \mathcal{X}\backslash(0)$. The obtained bounds are tight for the non-negative difference of the initial probability distributions of $X(t)$. In this paper it is no longer assumed that $B^*(t)$ must be essentially non-negative. Thus the discussed class of eligible processes $X(t)$ is wider than the one considered by Zeifman *et al.* (2018c).

It may happen that the difference of the initial probability distributions of $X(t)$ has coordinates of different signs and/or $B^*(t)$ contains negative elements. In such situations the upper bounds provided by the method based on the logarithmic norm may not be sharp. Having alternative estimates, provided by the other two methods considered in this paper, we can choose the best one. The idea to apply Lyapunov functions for the analysis of Markov chains is not new[5] (see, e.g., Kalashnikov, 1971; Malyshev and Menshikov, 1982). Yet, to the best of our knowledge, in the setting considered they have not been applied yet (see Zeifman *et al.*, 2018a). The approach based on differential inequalities (see Zeifman *et al.*, 2019) seems to be the most general: it can be applied both in the case when $B(t)$ is essentially non-negative (and can yield the same results as the method based on the logarithmic norm) and in cases in which the other two methods are not applicable.

Usually the three methods lead to different upper bounds and the quality (sharpness) of the bounds depends on the properties of $B^*(t)$. All three methods are applicable when the state space $S < \infty$. For countable $\mathcal{X}$ the method based on Lyapunov functions no longer applies. Note also that for a $X(t)$ with a finite state space belonging to classes (i)–(iv) apparently no general method for the construction of Lyapunov functions can be suggested. Thus here consideration is given only to such $X(t)$ for which it can be guessed how Lyapunov functions can be constructed.

The paper is structured as follows. In the next section the explicit forms of the reduced intensity matrix $B^*(t)$ for each class (i)–(iv) are given. In Section 3 we review the upper bounds on the rate of convergence, obtained by the method based on the logarithmic norm. Alternative upper bounds provided by Lyapunov functions and differential inequalities for some $X(t)$ from classes (i)–(iv) are given in Sections 4 and 5. Section 6 concludes the paper.

2. Explicit forms of the reduced intensity matrix

As mentioned above, estimation of the convergence rate of $X(t)$ to the limiting regime is based on the reduced intensity matrix $B(t)$, given by (3), or its transform $B^*(t) = TB(t)T^{-1}$. In this section the explicit form of $B^*(t)$ for each class (i)–(iv) is given.

2.1. $B^*(t)$ for $X(t)$ belonging to class (i). Consider a process $X(t)$ with $a_{ij}(t) = 0$ for any $t \geq 0$ if $|i - j| > 1$, $a_{i,i+1}(t) = \mu_{i+1}(t)$ and $a_{i+1,i}(t) = \lambda_i(t)$. Then $X(t)$ is the inhomogeneous birth-death process with state-dependent transition intensities $\lambda_i(t)$ (birth)

[5] For a detailed description of the approach we can also refer to Meyn and Tweedie (1993; 2012).

and $\mu_{i+1}(t)$ (death). In the queueing theory context, $X(t)$ describes the evolution of the total number of customers in the $M_n(t)/M_n(t)/1/S$ queue. For such $X(t)$ in the case of countable state space (i.e., $S = \infty$) the matrix $B^*(t)$ has the form (7). In the case of finite state space (i.e., $S < \infty$) it has the from (8). Note that the matrix $B^*(t)$ is essentially non-negative for any $t \geq 0$, i.e., all its off-diagonal elements are non-negative for any t.

2.2. $B^*(t)$ for $X(t)$ belonging to class (ii). Consider a process $X(t)$ with $a_{ij}(t) = 0$ for $i < j - 1$, $a_{i+k,i}(t) = a_k(t)$ for $k \geq 1$ and $a_{i,i+1}(t) = \mu_{i+1}(t)$. Such $X(t)$ describes the evolution of the total number of customers in a queue with batch arrivals and single services ($a_k(t)$ are the (state-independent) intensities of group arrivals and $\mu_{i+1}(t)$ are the (state-dependent) service intensities). Such processes in the simplest forms were first considered by Nelson *et al.* (1988) and, under the assumption of decreasing $a_k(t)$, studied by Zeifman *et al.* (2018c). In the case of a countable state space (i.e. $S = \infty$) the matrix $B^*(t)$ has the form

$$B^*(t) = \begin{pmatrix} a_{11} & \mu_1 & 0 & \cdots & 0 \\ a_1 & a_{22} & \mu_2 & \cdots & 0 \\ a_2 & a_1 & a_{33} & \mu_3 & \cdots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{9}$$

In the case of a finite state space (i.e., $S < \infty$) the matrix $B^*(t)$ is given by (10). Note that the matrix $B^*(t)$ is essentially non-negative for any $t \geq 0$ if the arrival intensities $a_k(t)$ are decreasing in k.

2.3. $B^*(t)$ for $X(t)$ belonging to class (iii). Consider a process $X(t)$ with $a_{ij}(t) = 0$ for $i > j + 1$, $a_{i,i+k}(t) = b_k(t)$, $k \geq 1$ and $a_{i+1,i}(t) = \lambda_i(t)$. Such $X(t)$ describes the evolution of the total number of customers in a queue with batch services and single arrivals ($\lambda_i(t)$ are the (state-independent) arrival intensities and $b_k(t)$ are the (state-independent) intensities of service of a group of k customers). Such processes were considered to some extent by Nelson *et al.* (1988) or Li and Zhang (2017). In the case of a countable state space (i.e., $S = \infty$) the matrix $B^*(t)$is given by (11). In the case of a finite state space (i.e., $S < \infty$) the matrix $B^*(t)$ is given by (12). Note that the matrix $B^*(t)$ is essentially non-negative for any $t \geq 0$ if the service intensities $b_k(t)$ are decreasing in k.

2.4. $B^*(t)$ for $X(t)$ belonging to class (iv). Consider a process $X(t)$ with $a_{i+k,i}(t) = a_k(t)$ and $a_{i,i+k}(t) = b_k(t)$ for $k \geq 1$. Such $X(t)$ describes the evolution of the total number of customers in an inhomogeneous queue with (state-independent) batch arrivals and group services ($a_k(t)$ are the (state-independent) intensities of group arrivals and $b_k(t)$ are the (state-independent) intensities of group services). Such a process under the assumption of a decrease in k intensities $a_k(t)$ and $b_k(t)$ was studied by Zeifman *et al.* (2014a). In the case of countable state space (i.e., $S = \infty$) the matrix $B^*(t)$ has the form

$$B^* = \begin{pmatrix} a_{11} & b_1 - b_2 & b_2 - b_3 & \cdots & \cdots & \\ a_1 & a_{22} & b_1 - b_3 & \cdots & \cdots & \\ & & & & & \\ \ddots & \ddots & \ddots & \ddots & \ddots & \\ a_{r-1} & \cdots & \cdots & a_1 & a_{rr} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}. \tag{13}$$

In the case of a finite state space (i.e., $S < \infty$) the matrix $B^*(t)$ is given by (14). Note that the matrix $B^*(t)$ is essentially non-negative for any $t \geq 0$ if the intensities $a_k(t)$ and $b_k(t)$ are decreasing in k.

3. Upper bounds using the logarithmic norm

Throughout this section by $\|\cdot\|$ we denote the l_1-norm, i.e., $\|\mathbf{p}(t)\| = \sum_{i\in\mathcal{X}} |p_i(t)|$ and $\|A(t)\| = \sup_{j\in\mathcal{X}} \sum_{i\in\mathcal{X}} |a_{ij}(t)|$. Let Ω be a set of all stochastic vectors, i.e., l_1 vectors with non-negative coordinates and a unit norm. Recall that a Markov chain $X(t)$ is called *weakly ergodic* if $\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \to 0$ as $t \to \infty$ for any initial conditions $\mathbf{p}^*(0)$ and $\mathbf{p}^{**}(0)$, where $\mathbf{p}^*(t)$ and $\mathbf{p}^{**}(t)$ are the corresponding solutions of (1).

Recall that the logarithmic norm[6] of the operator function $B(t)$ is defined as

$$\gamma(B(t)) = \lim_{h\to+0} h^{-1} \left(\|I + hB(t)\| - 1\right).$$

Denote by $V(t,s) = V(t)V^{-1}(s)$ the Cauchy operator of Eqn. (4). Then $\|V(t,s)\| \leq e^{\int_s^t \gamma(B(u))\,\mathrm{d}u}$. For an operator function from l_1 to itself we have the formula

$$\gamma(B(t)) = \sup_{j\in\mathcal{X}} \Big(b_{jj}(t) + \sum_{i\in\mathcal{X}, i\neq j} |b_{ij}(t)| \Big). \tag{15}$$

Note that, if the matrix $B(t)$ is essentially non-negative, then $\gamma(B(t)) = \sup_{j\in\mathcal{X}} \left(\sum_{i\in\mathcal{X}} b_{ij}(t)\right)$.

Assume that the state space $\mathcal{X}$ is countable, i.e., $S = \infty$. Let $\{d_i,\ i \geq 1\}$ be a sequence of positive numbers and let $D = \mathrm{diag}(d_1, d_2, \dots)$ be the diagonal matrix, with the off-diagonal elements equal to zero. Setting $\mathbf{w}(t) = D\mathbf{u}(t)$ in (6), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{w}(t) = B^{**}(t)\mathbf{w}(t), \tag{16}$$

[6]A number of queueing applications of this approach were studied by Granovsky and Zeifman (2004) as well as Zeifman *et al.* (2006; 2018c).

$$B^*(t) = \begin{pmatrix} -(\lambda_0+\mu_1) & \mu_1 & 0 & \cdots & 0 & \cdots & \cdots \\ \lambda_1 & -(\lambda_1+\mu_2) & \mu_2 & \cdots & 0 & \cdots & \cdots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \\ 0 & \cdots & \cdots & \lambda_{r-1} & -(\lambda_{r-1}+\mu_r) & \mu_r & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}. \tag{7}$$

$$B^*(t) = \begin{pmatrix} -(\lambda_0+\mu_1) & \mu_1 & 0 & \cdots & 0 \\ \lambda_1 & -(\lambda_1+\mu_2) & \mu_2 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & \lambda_{S-1} & -(\lambda_{S-1}+\mu_S) \end{pmatrix}. \tag{8}$$

$$B^*(t) = \begin{pmatrix} a_{11}-a_S & \mu_1 & 0 & \cdots & 0 \\ a_1-a_S & a_{22}-a_{S-1} & \mu_2 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ a_{S-1}-a_S & \cdots & \cdots & a_1-a_2 & a_{SS}-a_1 \end{pmatrix}. \tag{10}$$

$$B^*(t) = \begin{pmatrix} -(\lambda_0+b_1) & b_1-b_2 & b_2-b_3 & \cdots & \cdots \\ \lambda_1 & -\big(\lambda_1+\sum\limits_{i\le 2} b_i\big) & b_1-b_3 & \cdots & \cdots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & \lambda_{r-1} & -\big(\lambda_{r-1}+\sum\limits_{i\le r} b_i\big)\cdots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{11}$$

$$B^*(t) = \begin{pmatrix} -(\lambda_0+b_1) & b_1-b_2 & b_2-b_3 & \cdots & b_{S-1}-b_S \\ \lambda_1 & -\big(\lambda_1+\sum\limits_{i\le 2} b_i\big) & b_1-b_3 & \cdots & b_{S-2}-b_S \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & \lambda_{S-1} & -\big(\lambda_{S-1}+\sum\limits_{i\le S} b_i\big) \end{pmatrix}. \tag{12}$$

$$B^*(t) = \begin{pmatrix} a_{11}-a_S & b_1-b_2 & b_2-b_3 & \cdots & b_{S-1}-b_S \\ a_1-a_S & a_{22}-a_{S-1} & b_1-b_3 & \cdots & b_{S-2}-b_S \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ a_{S-1}-a_S & \cdots & \cdots & a_1-a_2 & a_{SS}-a_1 \end{pmatrix}. \tag{14}$$

where $B^{**}(t) = DB(t)^* D^{-1}$. Set[7]

$$\alpha_i\,(t) = -\sum_{j=1}^{\infty} b_{ji}^{**}(t),\ i \geq 1, \tag{17}$$

and let $\alpha(t)$ and $\beta(t)$ denote the least lower and the least upper bound of the sequence of functions $\{\alpha_i(t),\ i \geq 1\}$ i.e.,

$$\alpha\,(t) = \inf_{i\geq 1} \alpha_i\,(t)\,, \quad \beta\,(t) = \sup_{i\geq 1} \alpha_i\,(t)\,. \tag{18}$$

The next theorem and corollary were proved by (Zeifman *et al.*, 2018c, Theorem 1) and are stated here for the sake of completeness.

Theorem 1. *Assume that there exists a sequence $\{d_i,\ i \geq 1\}$ of positive numbers such that $d_1 = 1$, $d = \inf_{i\geq 1} d_i > 0$ and $B^*(t)$ is essentially non-negative. Let $\alpha(t)$, defined by (18), satisfy*

$$\int_0^{\infty} \alpha(t)\,\mathrm{d}t = +\infty. \tag{19}$$

Then the Markov chain $X(t)$ is weakly ergodic and for any initial conditions $s \geq 0$, $\mathbf{w}(s)$ and any $t \geq s$ the following upper bound holds:

$$\|\mathbf{w}\,(t)\,\| \leq e^{-\int_s^t \alpha(u)\,\mathrm{d}u} \|\mathbf{w}(s)\|. \tag{20}$$

If in addition all components of the vector $\mathbf{w}\,(s)$ are non-negative, then for any $0 \leq s \leq t$ the following lower bound holds:

$$\|\mathbf{w}\,(t)\,\| \geq e^{-\int_s^t \beta(u)\,\mathrm{d}u} \|\mathbf{w}(s)\|. \tag{21}$$

Corollary 1. *Let under the assumptions of Theorem 1 the sequence $\{d_i,\ i \geq 1\}$ be such that no $\alpha_i\,(t)$ depends on i, i.e., they are the same for any i. Then $\alpha\,(t) = \beta\,(t)$ and the upper bound (20) is tight. If in addition all components of the vector $\mathbf{w}(s)$ are non-negative, for any $0 \leq s \leq t$ we have*

$$\|\mathbf{w}\,(t)\,\| = e^{-\int_s^t \alpha(u)\,\mathrm{d}u} \|\mathbf{w}\,(s)\,\|. \tag{22}$$

If the Markov chain $X(t)$ is homogeneous, then the expressions in (17) and (18) do not depend on t. In such a case the upper and lower bounds (20), (21) can be improved. The following result is due to Zeifman *et al.* (2018c, Theorem 2).

Theorem 2. *Assume that there exist a sequence $\{d_i,\ i \geq 1\}$ of positive numbers such that $d_1 = 1$, $d = \inf_{i\geq 1} d_i > 0$ and $B^*(t)$ is essentially non-negative. Let α, defined by* (18)*, be positive. Then $X(t)$ is ergodic and for any initial condition $\mathbf{w}(0)$ and any $t \geq 0$ the following upper bound holds:*

$$\|\mathbf{w}\,(t)\,\| \leq e^{-\alpha t} \|\mathbf{w}\,(0)\,\|. \tag{23}$$

If in addition all components of the vector $\mathbf{w}\,(0)$ are non-negative, then for any $t \geq 0$ the following lower bound holds:

$$\|\mathbf{w}\,(t)\,\| \geq e^{-\beta t} \|\mathbf{w}\,(0)\,\|. \tag{24}$$

If $\alpha = \beta$, then the bound (23) is tight.

Assume now that the state space is finite, i.e., $S < \infty$. Then d_i can be arbitrary positive numbers and we can find constants, say C_1 and C_2, such that

$$\|\mathbf{w}(t)\| = \|DT\mathbf{y}(t)\| \leq C_1 \|\mathbf{y}(t)\|,$$
$$\|\mathbf{y}(t)\| = \|T^{-1}D^{-1}\mathbf{w}(t)\| \leq C_2 \|\mathbf{w}(t)\|.$$

Hence Theorems 1 and 2 provide bounds on the rate of convergence in the l_1-norm. The explicit expressions for the constants can be found in the works of Granovsky and Zeifman (2004) or Zeifman *et al.* (2006). If the Markov chain $X(t)$ is homogeneous and α^* is the decay parameter, defined as

$$\lim_{t\to\infty} (p_{ij}(t) - \pi_j) = O(e^{-\alpha^* t}),$$

where $\{\pi_j,\ j \geq 0\}$ are the stationary probabilities of the chain, then $\alpha \leq \alpha^* \leq \beta$.

Notice that some additional results for finite homogeneous Markov chains $X(t)$ belonging to class (i) are provided by Doorn *et al.* (2010). In particular they proved that the exact estimate of the rate of convergence can be obtained. In the next theorem we provide an alternative proof of this fact.

Theorem 3. *Let $X(t)$ be a homogeneous birth-death process with a finite state space of size S and let all birth and death intensities be positive. Then there exists a set $\{d_i,\ 1 \leq i \leq S\}$ of positive numbers such that $\alpha = \alpha^* = \beta$, where α^* is the decay parameter of $X(t)$, and α and β are defined by* (18).

Proof. Let C be an essentially non-negative irreducible matrix such that there exists $n_0 > 0$ with $C^{n_0} > 0$. Denote by λ_0 its maximal eigenvalue. It is simple and positive. Then there exists a diagonal matrix with positive entries $D = \mathrm{diag}(d_1, \dots, d_S)$ such that all column sums for matrix $C_D = DCD^{-1}$ are equal to λ_0. Indeed, let $m = \max_{1\leq j\leq S} |c_{jj}|$.

Consider the irreducible matrix $C' = C^T + mI$. It has a simple eigenvalue $\lambda^* = \lambda_0 + m$ and the corresponding eigenvector $\mathbf{x} = (x_1, \dots, x_S)^T$ has strictly positive coordinates. Set $d_i = x_i^{-1}$, $1 \leq i \leq S$. Then $\mathbf{e} = (1, \dots, 1)^T$ is the eigenvector of the matrix $C'_D = DC'D^{-1}$. Therefore all row sums in the matrix C'_D are equal to λ^*. Thus all row sums in the matrix $C_D^T = C'_D - mI$ are equal to $\lambda^* - m = \lambda_0$, and all column sums of the matrix C_D are equal to λ_0. ■

[7] It is possible to obtain explicit expressions for $\alpha_i(t)$ for all of the classes considered (i)–(iv) (see the details in the work of Zeifman *et al.* (2018c)).

4. Upper bounds using Lyapunov functions

As mentioned in the Introduction, the method based on Lyapunov functions no longer applies in the case of a countable state space $\mathcal{X}$. In this section, under the assumption that $\mathcal{X}$ is finite, i.e., $S < \infty$, it is shown how (quadratic) Lyapunov functions can be applied to obtain the explicit upper bounds on the rate of convergence of some $X(t)$ belonging to classes (i)–(iii). Unlike the case of bounds provided by the method based on the logarithmic norm, Lyapunov functions yield bounds in the l_2-norm (Euclidean norm) and thus are somewhat weaker.

Throughout this section denote by $\|\cdot\|$ the l_2-norm, i.e., $\|\mathbf{p}(t)\| = \sqrt{\sum_{i\in\mathcal{X}} p_i(t)^2}$. Consider the system (16). Let $V(t) = \sum_{k=1}^{S} w_k^2(t)$, where $\mathbf{w}(t) = (w_1(t), w_2(t), \dots, w_S(t))^T$ is the solution of (16). By differentiating $V(t)$ we obtain

$$\begin{aligned}\frac{\mathrm{d}V(t)}{\mathrm{d}t} &= \sum_{k=1}^{S} 2w_k(t)\frac{\mathrm{d}w_k(t)}{\mathrm{d}t}\\ &= -2\sum_{i=1}^{S}\sum_{j=1}^{S}\left(-b_{ij}^{**}(t)\right) w_i(t)w_j(t).\end{aligned} \tag{25}$$

If we find a set of positive numbers $\{d_i, 1 \le i \le S\}$ and a function $\beta^*(t)$ satisfying

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \le -2\beta^*(t)V(t) \tag{26}$$

for any $\mathbf{w}(t)$, being the solution of (16), then for a $X(t)$ belonging to classes (i)–(iv) and for any initial condition $\mathbf{w}(0)$ it we have

$$\|\mathbf{w}(t)\| \le e^{-\int_s^t \beta^*(\tau)\,\mathrm{d}\tau}\|\mathbf{w}(0)\|. \tag{27}$$

For a finite homogeneous Markov chain $X(t)$ belonging to class (i) such a set $\{d_i, 1 \le i \le S\}$ is given in the next theorem.

Theorem 4. *Let $X(t)$ be a homogeneous birth-death process defined on a finite state space $\mathcal{X}$ with possibly state-dependent birth intensities λ_k and possibly state-dependent death intensities μ_k. Assume that $\lambda_k > 0$ and $\mu_k > 0$ for each $k \in \mathcal{X}$. Then there exist a set of positive numbers $\{d_i, 1 \le i \le S\}$, a positive number β^* and a set of numbers $\{\alpha_i, 1 \le i \le S\}$ such that*

$$\begin{aligned}\frac{\mathrm{d}V(t)}{\mathrm{d}t} &= -2\beta^*\sum_{k=1}^{S} w_k^2\\ &\quad - 2\sum_{k=1}^{S-1}(\alpha_k w_k - \alpha_{k+1}w_{k+1})^2.\end{aligned} \tag{28}$$

Proof. If $X(t)$ is a homogeneous birth-death process, then $B^*(t)$ does not depend on t and thus it is constant tridiagonal matrix. Let $d_1 = 1$, $d_{k+1} = d_k\sqrt{\mu_k/\lambda_k}$, $k \ge 1$. Remembering that $D = \mathrm{diag}(d_1, \dots, d_S)$ and $B^{**}(t) = DB(t)^*D^{-1}$, we immediately obtain (29). Note that B^{**} is a symmetric matrix. Setting $\Phi(t) = -0.5\,\mathrm{d}V(t)/\mathrm{d}t$ in (25), we obtain

$$\begin{aligned}\Phi(t) &= \lambda_0 w_1^2 + \mu_S w_S^2\\ &\quad + \sum_{k=1}^{S-1}(\sqrt{\mu_k}w_k - \sqrt{\lambda_k}w_{k+1})^2.\end{aligned}$$

Choose a positive number β such that $\beta < \min(\lambda_0, \lambda_1, \dots, \lambda_S)$ and put $\phi_0 = \lambda_0 - \beta$. Then the terms on the right-hand side of the previous relation can be rearranged to give

$$\begin{aligned}\Phi(t) &= \beta w_1^2 + \left(\sqrt{\mu_1+\phi_0}\,w_1 - \frac{\sqrt{\lambda_1\mu_1}}{\sqrt{\mu_1+\phi_0}}w_2\right)^2\\ &\quad + \lambda_1\left(\frac{\phi_0}{\mu_1+\phi_0}\right)w_2^2\\ &\quad + \sum_{k=2}^{S-1}(\sqrt{\mu_k}w_k - \sqrt{\lambda_k}w_{k+1})^2 + \mu_S w_S^2.\end{aligned}$$

Consider the coefficient of w_2^2. Note that it can always[8] be represented as $\beta+\phi_1$ with $\phi_1 > 0$. Thus we can rearrange the terms in the previous relation and obtain

$$\begin{aligned}\Phi(t) &= \beta(w_1^2 + w_2^2)\\ &\quad + \left(\sqrt{\mu_1+\phi_0}\,w_1 - \frac{\sqrt{\lambda_1\mu_1}}{\sqrt{\mu_1+\phi_0}}w_2\right)^2\\ &\quad + \left(\sqrt{\mu_2+\phi_1}\,w_2 - \frac{\sqrt{\lambda_2\mu_2}}{\sqrt{\mu_2+\phi_1}}w_3\right)^2\\ &\quad + \lambda_2\left(\frac{\phi_1}{\mu_2+\phi_1}\right)w_3^2\\ &\quad + \sum_{k=3}^{S-1}(\sqrt{\mu_k}w_k - \sqrt{\lambda_k}w_{k+1})^2\\ &\quad + \mu_S w_S^2.\end{aligned}$$

Proceeding in a similar manner (i.e., choosing a suitable value of β, representing each coefficient of w_k as $\beta + \phi_{k-1}$, $\phi_{k-1} > 0$, and rearranging the terms), we

[8]Indeed, if β is larger than the coefficient of w_2^2, it suffices to make one step back and choose a new value of β (satisfying $\beta < \min(\lambda_0, \lambda_1, \dots, \lambda_S)$) smaller than the current one.

$$B^{**} = \begin{pmatrix} -(\lambda_0+\mu_1) & \sqrt{\lambda_1\mu_1} & 0 & \cdots & 0 \\ \sqrt{\lambda_1\mu_1} & -(\lambda_1+\mu_2) & \sqrt{\lambda_2\mu_2} & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & \sqrt{\lambda_{S-1}\mu_{S-1}} & -(\lambda_{S-1}+\mu_S) \end{pmatrix}. \tag{29}$$

arrive at the following representation of $\Phi(t)$:

$$\begin{aligned}\Phi(t) &= \beta\sum_{k=1}^{S-1} w_k^2 \\ &+ \sum_{k=1}^{S-1}\Big(\sqrt{\mu_k+\phi_{k-1}}w_k \\ &- \frac{\sqrt{\lambda_k\mu_k}}{\sqrt{\mu_k+\phi_{k-1}}}w_{k+1}\Big)^2 \\ &+ \left(\mu_S+\lambda_{S-1}\frac{\phi_{S-2}}{\mu_{S-1}+\phi_{S-2}}\right)w_S^2.\end{aligned}$$

If the coefficient of w_S^2 is larger than β, then we can choose any β^* such that $\beta^* \in (\beta;\mu_S+\lambda_{S-1}\phi_{S-2}/(\mu_{S-1}+\phi_{S-2}))$. Therefore $\beta < \beta^* < \beta+\varepsilon$, where $\beta+\varepsilon = \mu_S+\lambda_{S-1}\phi_{S-2}/(\mu_{S-1}+\phi_{S-2})$. Set $\beta_1 = \beta+0.5\varepsilon$ and continue the process of selecting squares in the opposite direction (starting from w_S^2). If the coefficient of w_S^2 is less than β, then we can choose $\beta^* \in (\beta-\varepsilon;\beta)$. In this case we put $\beta_1 = \beta-0.5\varepsilon$ and continue the process of selecting squares, starting from w_1^2. In such a way we get a sequence of nested segments converging to β^*. ■

Note that the existence of the upper bound $\|\mathbf{w}(t)\| \le e^{-\beta^* t}\|\mathbf{w}(0)\|$ also follows from (26), (27) and Theorem 4. The inequality turns into an equality once the set of numbers $\{\alpha_i, 1\le i\le S\}$ is chosen in such a way that the second sum in (28) is equal to zero.

Let us specify the upper bound (27) for some finite homogeneous Markov chains $X(t)$ belonging to class (ii). Specifically, let in a process $X(t)$ belonging to (ii) the arrival intensities be such that $\lambda_1 = 0$ and $\lambda_k = \lambda$ for $2 \le k \le S$. From the queueing perspective this means that only arrivals in batches are possible. Then the matrix $B^*(t)$ given by (10) does not depend on t and takes the following form:

$$B^* = \begin{pmatrix} a_{11}-\lambda & \mu_1 & 0 & \cdots & 0 \\ -\lambda & a_{22}-\lambda & \mu_2 & \cdots & 0 \\ 0 & -\lambda & a_{33}-\lambda & \cdots & 0 \\ \cdots & & & & \\ 0 & 0 & 0 & \cdots & a_{SS} \end{pmatrix}. \tag{30}$$

Let $d_1 = 1$, $d_{k+1} = d_k\sqrt{\mu_k/\lambda}$, $k \ge 1$. Remembering that $D = \mathrm{diag}(d_1,\ldots,d_S)$ and $B^{**}(t) = DB(t)^*D^{-1}$, we immediately obtain (31).

For such a matrix B^{**} Eqn. (25) can be rewritten as

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} = 2\sum_{k=1}^{S}(a_{kk}-\lambda)w_k^2(t),$$

which implies the next theorem[9].

Theorem 5. *Let $X(t)$ be a homogeneous Markov chain defined on a finite state space $\mathcal{X}$ with state-independent group arrival intensities $q_{k,k+i} = \lambda$, $i \ge 2$, $q_{k,k+1} = 0$, and possibly state-dependent service intensities $q_{k,k-1} = \mu_k$, $1 \le k \le S$. Then the following bound on the rate of convergence holds:*

$$\|\mathbf{w}(t)\| \le e^{-\beta^* t}\|\mathbf{w}(0)\|, \tag{32}$$

where $\beta^* = \min(S\lambda+\mu_1,\ldots,2\lambda+\mu_{S-1},\mu_S)$, i.e., β^*, is the decay parameter (spectral gap) of the Markov chain.

Note that a similar result can be obtained for the homogeneous Markov chains $X(t)$ belonging to class (iii). The following example shows that Lyapunov functions lead to explicit uppers bounds for the rate of convergence also for finite inhomogeneous Markov chains.

Example 1. Consider the Markov process $X(t)$ that describes the evolution of the total number of customers in the $M(t)/M(t)/1/S$ queue with bulk arrivals, when all transition intensities are periodic functions of time. Let the arrival intensities be $a_1(t) = 1+\sin 2\pi t$, $a_k(t) = 2+\sin 2\pi t+\cos 2\pi t$ for $2 \le k \le S$ and all the service intensities be $\mu_k(t) = m^2(1+\cos 2\pi t)$ for $1 \le k \le S$ and some $m \ge 1$. Such $X(t)$ belongs to class (ii). By setting $d_1 = 1$, $d_{k+1} = md_k$, $k \ge 1$, we obtain the matrix $B^{**}(t)$ in the form $B^{**}(t) = (b_{ij}^{**}(t))$, where

$$b_{i,i+1}^{**}(t) = m(1+\cos 2\pi t),$$

$$b_{i,i}^{**}(t) = a_{ii}(t) - a_{S-i+1},$$

$$b_{i+1,i}^{**}(t) = -m(1+\cos 2\pi t).$$

Then

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} = 2\sum_{k=1}^{S}(a_{kk}(t)-a_{S+1-k}(t))w_k^2(t),$$

[9]Note that in the case considered we can also obtain the lower bound on the rate of convergence using the approach of Zeifman *et al.* (2018b).

$$B^{**} = \begin{pmatrix} -(S\lambda+\mu_1) & \sqrt{\lambda\mu_1} & 0 & \cdots & 0 \\ -\sqrt{\lambda\mu_1} & -((S-1)\lambda+\mu_2) & \sqrt{\lambda\mu_2} & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & -\sqrt{\lambda\mu_{S-1}} & -\mu_S \end{pmatrix}. \tag{31}$$

and from (31) it follows that for any initial condition $\mathbf{w}(0)$ the tight upper bound on the rate of convergence is

$$\|\mathbf{w}(t)\| \le e^{-\int_0^t \beta^*(\tau)\,\mathrm{d}\tau}\|\mathbf{w}(0)\|,$$

where $\beta^*(t) = 2 + \sin 2\pi t + \cos 2\pi t$. Note that, for the case considered, the method based on Lyapunov functions yields the best (among the three methods discussed in this paper) possible upper bound. It is also worth noting that we can apply the obtained upper bound for the computation of the limiting distribution of $X(t)$. For example, let $S = 199$ and $m = 90$. Then, using truncation techniques, which were developed by Zeifman *et al.* (2006; 2014b), any limiting probability characteristic of $X(t)$ can be computed with the given approximation error. In Figs. 1–8 we can see the behaviour of the conditional expected number $E(X(t)|X(0))$ of customers in the queue at instant t, and the state probabilities $p_0(t)$, $p_{99}(t)$ and $p_{199}(t)$ as functions of time t under different initial conditions $X(0)$. The approximation error is 10^{-3}. ♦

Note that one general framework for the computation of the limiting characteristics of time-dependent queueing systems is described in detail in the recent paper by Satin *et al.* (2019). Particularly, having the bounds on our rate of convergence we can compute the time instant, say t^*, starting from which probabilistic properties of $X(t)$ do not depend on the value of $X(0)$ (assuming that the process starts at time $t = 0$). Thus, for example, if the transition intensities are periodic (say, 1-time-periodic), we can truncate the process on the interval $[t^*, t^*+1]$ and solve the forward Kolmogorov system of differential equations on this interval with $X(0) = 0$. In such a way, we can build approximations for any limiting probability characteristics of $X(t)$ and estimate stability (perturbation) bounds.

5. Upper bounds using differential inequalities

As first shown by Zeifman *et al.* (2019), there are situations when the previous two methods for bounding the rate of convergence do not work well (either lead to poor upper bounds or do not yield upper bounds at all). Here we present probably the most general method, which is based on differential inequalities and which can be applied to $X(t)$ belonging to classes (i)–(iv) with a finite state space (i.e., $S < \infty$) and all transition intensity functions being analytic functions of time t.

Throughout this section we denote by $\|\cdot\|$ the l_1-norm. Consider a finite system of linear differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = A(t)\mathbf{x}(t),\ t \ge 0, \tag{33}$$

where $A(t)$ is some matrix[10] with all entries $a_{ij}(t)$ being analytic functions of t and $\mathbf{x}(t) = (x_1(t), \dots, x_S(t))^T$. Let $\mathbf{x}(t)$ be an arbitrary solution of (33). Consider an interval $[t_1, t_2]$ with fixed signs of coordinates of $\mathbf{x}(t)$ (i.e., $x_i(t) \ne 0$ for all $1 \le i \le S$ and for all $t \in [t_1, t_2]$). Choose the set of numbers $\{d_i, 1 \le i \le S\}$ such that the sign of each d_i coincides with that of $x_i(t)$. Then $d_i x_i(t) \ge 0$ for all $t \in [t_1, t_2]$ and hence $\sum_{k=1}^S d_k x_k(t) = \|\mathbf{x}(t)\|$ can be considered the l_1-norm.

Set $\mathbf{z}(t) = D\mathbf{x}(t)$ and $\tilde{A}(t) = DA(t)D^{-1}$, where $D = \mathrm{diag}(d_1, \dots, d_S)$, and consider the following system of differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{z}(t) = \tilde{A}(t)\mathbf{z}(t) \tag{34}$$

for $t \in [t_1, t_2]$. If for the chosen matrix D there exists a function[11] $\alpha_D(t)$ such that $\sum_{i=1}^S \tilde{a}_{ij}(t) \le -\alpha_D(t)$ for each $1 \le j \le S$, then the following bound holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{z}(t)\| = \sum_{j=1}^S \sum_{i=1}^S \tilde{a}_{ij}(t) z_j(t) \le -\alpha_D(t)\|\mathbf{z}(t)\|. \tag{35}$$

Choose $\alpha^*(t)$ such that $\alpha^*(t) = \min \alpha_D(t)$, where the minimum is taken over all time intervals $[t_1, t_2]$, $0 < t_1 < t_2$, with different combinations of coordinate signs of the solution $\mathbf{x}(t)$. For any such combination there exists a particular inequality $\|\mathbf{z}(t)\| \le e^{-\int_{t_1}^{t_2} \alpha^*(\tau)\,\mathrm{d}\tau}\|\mathbf{z}(t_1)\|$.

From the fact that there exist constants, say C_1 and C_2, such that $\|\mathbf{x}(t)\| \le C_1\|\mathbf{z}(t)\|$ and $\|\mathbf{z}(t)\| \le C_2\|\mathbf{x}(t)\|$ for any interval $[t_1, t_2]$, $0 < t_1 < t_2$, and any corresponding diagonal matrix D, the following result follows.

Theorem 6. *For $\alpha^*(t) = \min \alpha_D(t)$ and the corresponding constants C_1 and C_2, the following upper bound for the rate of convergence holds:*

$$\|\mathbf{x}(t)\| \le C_1 C_2 e^{-\int_0^t \alpha^*(\tau)\,\mathrm{d}\tau}\|\mathbf{x}(0)\|. \tag{36}$$

[10]This matrix $A(t)$ must not be confused with the matrix in (1).

[11]The lower index in $\alpha_D(t)$ is used to explicitly indicate that this function depends on the choice of the matrix D.

Note that, if the matrix $A(t)$ is essentially non-negative, then that based on differential inequalities yields the same results as the method based on the logarithmic norm. Thus the result of Theorem 3 can also be obtained using differential inequalities.

For some processes $X(t)$ belonging to classes (i)–(iv) the method based on differential inequalities leads to upper bounds which are better than those obtained using the both previous methods. Several such settings are illustrated below. Consider a homogeneous Markov chain $X(t)$ belonging to class (iii) with the constant arrival intensity λ and constant bulk service intensity $b_S(t) = b$ and $b_k(t) = 0$, $1 \le k \le S-1$. In this case both the method based on the logarithmic norm and that based on Lyapunov functions do not yield any upper bound, whereas with the differential inequalities we can obtain a meaningful result. Indeed, the matrix B^*, given by (12), and the matrix B^{**} take the following form:

$$B^* = \begin{pmatrix} -\lambda & 0 & 0 & \cdots & 0 & -b \\ \lambda & -\lambda & 0 & \cdots & 0 & -b \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \cdots & \lambda & -(\lambda+b) \end{pmatrix}, \tag{37}$$

$$B^{**} = \begin{pmatrix} -\lambda & 0 & 0 & 0 & \cdots & 0 & -b\frac{d_1}{d_S} \\ \lambda\frac{d_2}{d_1} & -\lambda & 0 & 0 & \cdots & 0 & -b\frac{d_2}{d_S} \\ 0 & \lambda\frac{d_3}{d_2} & -\lambda & 0 & \cdots & 0 & -b\frac{d_3}{d_S} \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & \cdots & -\lambda & -b\frac{d_{S-1}}{d_S} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\lambda-b \end{pmatrix}. \tag{38}$$

Assume that $\{d_i, 1 \le i \le S\}$ are given and put $z_k(t) = d_k x_k(t)$. Then we have

$$\sum_{i=1}^{S} \frac{\mathrm{d}z_i(t)}{\mathrm{d}t} = -\lambda \sum_{i=1}^{S-1} \left(1 - \frac{d_{i+1}}{d_i}\right) z_i(t) - \left(\lambda + b\sum_{i=1}^{S} \frac{d_i}{d_S}\right) z_S(t).$$

Since $x_i(t)$ can be of different signs, we have to consider all the possible sign changes. It is convenient to start with the case when there are no changes in signs. Let all $x_i(t)$ be positive. Set $d_i = \varepsilon^i$, $1 \le i \le S$, for some $0 < \varepsilon < 1$. Then

$$\sum_{i=1}^{S} \frac{\mathrm{d}z_i(t)}{\mathrm{d}t} = -\lambda(1-\varepsilon) \sum_{i=1}^{S-1} z_i(t) - \left(\lambda + b\sum_{i=1}^{S} \frac{1}{\varepsilon^{i-1}}\right) z_S(t),$$

and we obtain $\alpha = \lambda(1-\varepsilon)$.

Another case is when there is a single change in signs. Let all $x_1(t), \dots, x_k(t)$ be positive for some k, $1 \le k \le S-1$, and all $x_{k+1}(t), \dots, x_S(t)$ be negative. Set $d_i = \varepsilon^{S-k+i}$, $1 \le i \le k$, and $d_i = -\varepsilon^{i-k}$, $k+1 \le i \le S$. Then

$$\begin{aligned} &\sum_{i=1}^{S} z_i'(t) \\ &= -\lambda\Big(1-\frac{d_2}{d_1}\Big)z_1(t) - \lambda\Big(1-\frac{d_3}{d_2}\Big)z_2(t) \\ &\quad - \lambda\Big(1-\frac{d_4}{d_3}\Big)z_3(t) - \cdots \\ &\quad - \left(\lambda + b\left(1 + \frac{d_1}{d_S} + \frac{d_2}{d_S} + \cdots + \frac{d_{S-1}}{d_S}\right)\right) z_S(t) \\ &= -\lambda(1-\varepsilon)z_1(t) - \lambda(1-\varepsilon)z_2(t) \\ &\quad - \lambda(1-\varepsilon)z_3(t) - \cdots \\ &\quad - \lambda(1-\varepsilon)z_{k-1}(t) - \lambda(1+\frac{1}{\varepsilon^{S-1}})z_k(t) - \cdots \\ &\quad - \left(\lambda + b\left(1 - \varepsilon - \varepsilon^2 - \cdots - \varepsilon^k + \frac{1}{\varepsilon^{S-k-1}}\right.\right. \\ &\quad \left.\left. + \frac{1}{\varepsilon^{S-k-2}} + \cdots + \frac{1}{\varepsilon}\right)\right) z_S(t) \\ &\le -\lambda(1-\varepsilon)\sum_{i=1}^{S} z_i(t), \end{aligned}$$

and we have that $\alpha = \lambda(1-\varepsilon)$.

Now consider the case when there are exactly two changes in signs. Let all $x_1(t), \dots, x_k(t)$ be positive for some $1 \le k \le S-2$, all $x_{k+1}(t), \dots, x_s(t)$ be negative for some $k+1 \le s \le S-1$ and all $x_{s+1}(t), \dots, x_S(t)$ be positive. Let $d_i = \varepsilon^{S-k+i}$ for $1 \le i \le k$, $d_i = -\varepsilon^{S-s-k+i}$ for $k+1 \le i \le s$ and $d_i = \varepsilon^{i-s}$, for $s+1 \le i \le S$. We have

$$\begin{aligned} &\sum_{i=1}^{S} z_i'(t) \\ &= -\lambda\Big(1-\frac{d_2}{d_1}\Big)z_1(t) \\ &\quad - \lambda\Big(1-\frac{d_3}{d_2}\Big)z_2(t) - \lambda\Big(1-\frac{d_4}{d_3}\Big)z_3(t) - \cdots \\ &\quad - \left(\lambda + b\left(1 + \frac{d_1}{d_S} + \frac{d_2}{d_S} + \cdots + \frac{d_{S-1}}{d_S}\right)\right) z_S(t) \end{aligned}$$

$$
\begin{aligned}
&= -\lambda(1-\varepsilon)z_1(t) - \lambda(1-\varepsilon)z_2(t)\\
&\quad - \lambda(1-\varepsilon)z_3(t) - \cdots\\
&\quad - \lambda(1-\varepsilon)z_{k-1}(t) - \lambda\Big(1+\frac{1}{\varepsilon^{s-1}}\Big)z_k(t)\\
&\quad - \lambda(1-\varepsilon)z_{k+1}(t) - \cdots\\
&\quad - \lambda(1-\varepsilon)z_{s-1} - \lambda(1+\frac{1}{\varepsilon^{S-k}})z_s(t)\\
&\quad - \lambda(1-\varepsilon)z_{s+1}(t) - \cdots\\
&\quad - \big(\lambda + b\,\big(1+\varepsilon^{s-k+1}+\varepsilon^{s-k+2}+\cdots+\varepsilon^{s}\\
&\quad -\varepsilon-\varepsilon^2-\cdots-\varepsilon^{s-k}+\frac{1}{\varepsilon^{S-k-1}}\\
&\quad +\frac{1}{\varepsilon^{S-k-2}}+\cdots+\frac{1}{\varepsilon}\Big)\Big)\,z_S(t)\\
&\le -\lambda\,(1-\varepsilon)\sum_{i=1}^{S} z_i(t),
\end{aligned}
$$

and $\alpha = \lambda\,(1-\varepsilon)$. Note that the total number of sign changes does not exceed $S-1$. On each change of sign when going from $x_s(t)$ to $x_{s+1}(t)$ we set d_{s+1} equal to ε^{S-m+1}, where m is the number of the last element in the current period of consistency (i.e., when there is no change of signs). Then eventually we arrive at the following upper bound: $\|\mathbf{x}(t)\| \le C_1C_2e^{-\lambda(1-\varepsilon)t}\|\mathbf{x}(0)\|$, with $C_1C_2 = \varepsilon^{1-S}$.

The example below shows how the method based on differential inequalities can be applied for inhomogeneous Markov chains with a finite state space.

Example 2. Consider the Markov process $X(t)$ that describes the evolution of the total number of customers in the $M(t)/M^X(t)/1/S$ queue with bulk services, when all transition intensities are periodic functions of time. Let the arrival intensities be $\lambda_k(t) = \lambda(t) = 10\,(2+\sin(2\pi t))$, and the service intensities be $b_k(t) = 0$ for $1 \le k < S$, and $b_S(t) = m^{-2}\,(2+\cos 2\pi t)$ for some $m \ge 1$. Such $X(t)$ belongs to class (iii). The matrix B^{**} for such $X(t)$ has the form $B^{**}(t) = (b^{**}_{ij}(t))$, where

$$b^{**}_{i,i}(t) = -10(2+\sin(2\pi t)),$$

$$b^{**}_{i+1,i}(t) = 10(2+\sin(2\pi t))\frac{d_{i+1}}{d_i},$$

$$b^{**}_{i,S}(t) = -m^{-2}\,(2+\cos(2\pi t))\,\frac{d_i}{d_S},\quad i<S,$$

$$
\begin{aligned}
b^{**}_{SS}(t) &= -10(2+\sin(2\pi t))\\
&\quad - m^{-2}\,(2+\cos(2\pi t))\,,\quad i=S.
\end{aligned}
$$

Then for any initial condition $\mathbf{x}(0)$ we can deduce the following two upper bounds on the rate of convergence:

$$\|\mathbf{x}(t)\| \le \varepsilon^{1-S}e^{-\int_0^t(1-\varepsilon)\lambda(\tau)\,\mathrm{d}\tau}\|\mathbf{x}(0)\|,$$

$$\|\mathbf{x}(t)\| \le \varepsilon^{1-S}e^{-10(1-\varepsilon)t}\|\mathbf{x}(0)\|.$$

These bounds are not tight (the leftmost is better among the two) but the other two methods give essentially worse results. As in Example 1, these bounds can be used in the approximation of the limiting distribution of $X(t)$. For example, let $S = 40$ and $m = 1$. In Figs. 9–16 we can see the behaviour of the conditional expected number $E(X(t)|X(0))$ of customers in the queue at instant t and the state probabilities $p_0(t)$, $p_{20}(t)$ and $p_{40}(t)$ as functions of time t under different initial conditions $X(0)$. ♦

We conclude the section by emphasizing that the method of differential inequalities may lead to meaningful upper bounds for the rate of convergence even in the case of a countable state space $\mathcal{X}$. For example, consider a homogeneous countable (i.e., $S=\infty$) Markov process $X(t)$ belonging to class (iii) with constant arrival intensities λ and batch service intensities $b_2(t) = \mu > 0$ and $b_k(t) = 0$ for $k \ne 2$. Hence (12) takes the form

$$
B^* = \begin{pmatrix}
-\lambda & -\mu & \mu & \cdots & \cdots\\
\lambda & -(\lambda+\mu) & 0 & \mu & \cdots\\
\vdots & \ddots & \ddots & \ddots & \ddots\\
\vdots & \cdots & \cdots & \lambda & -(\lambda+\mu)\cdots\\
\cdots & \cdots & \cdots & \cdots & \cdots\ \cdots
\end{pmatrix}. \tag{39}
$$

In such a case, to the best of our knowledge, the method of differential inequalities is the only one, with which we can obtain the ergodicity of the chain and explicit estimates of the rate of convergence (see the details in the work by Satin *et al.* (2019)).

6. Conclusion

The three methods considered in this paper provide various alternatives for the computation of the upper bounds for the rate of convergence to the limiting regime of (in)homogeneous continuous-time Markov processes. Yet even for the four discussed classes (i)–(iv) of Markov processes a single unified framework cannot be suggested: special cases do exist when none of the methods works well.

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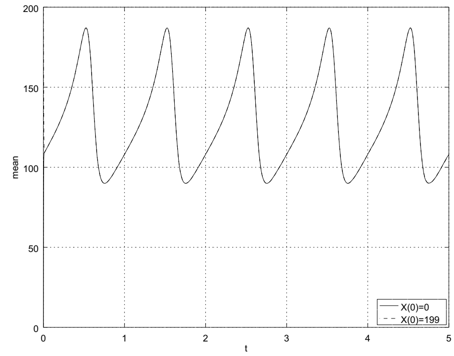

Fig. 1. Example 1: the expected number $E(X(t)|X(0))$ of customers in the queue for $t \in [0,5]$ with the initial condition $X(0) = 0$.

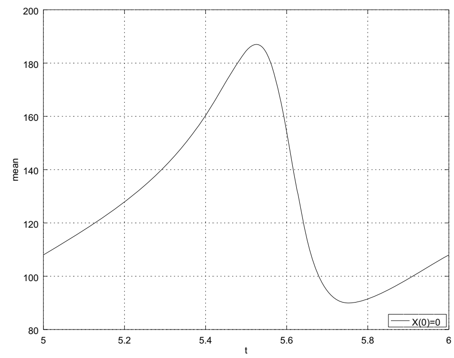

Fig. 2. Example 1: the expected number $E(X(t)|X(0))$ of customers in the queue for $t \in [5,6]$ with the initial condition $X(0) = 0$.

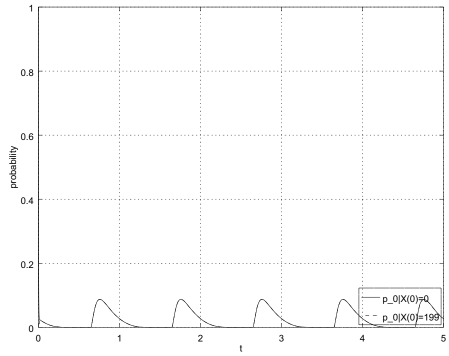

Fig. 3. Example 1: the probability $p_0(t)$ of an empty queue for $t \in [0,5]$ with the initial condition $X(0) = 0$.

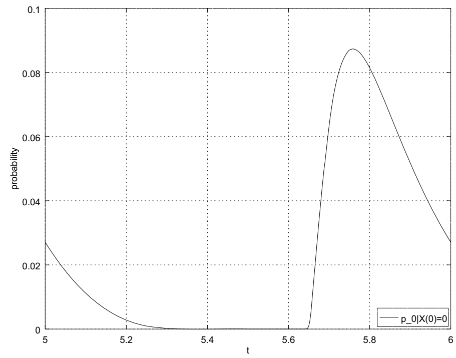

Fig. 4. Example 1: the probability $p_0(t)$ of an empty queue for $t \in [5,6]$ with the initial condition $X(0) = 0$.

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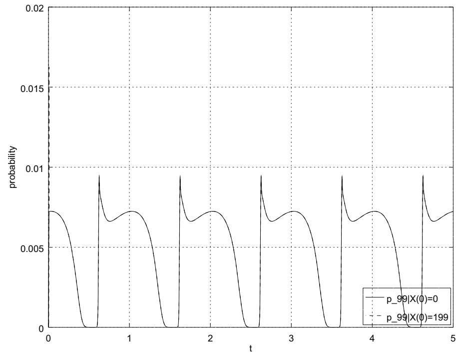

Fig. 5. Example 1: the probability $p_{99}(t)$ for $t \in [0, 5]$ with the initial condition $X(0) = 0$.

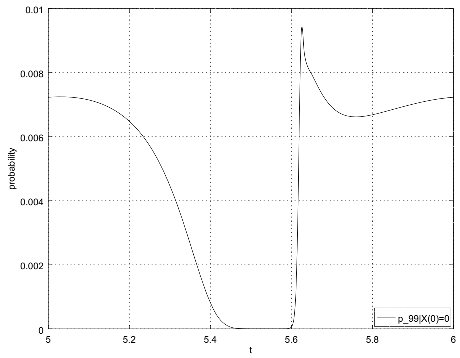

Fig. 6. Example 1: the probability $p_{99}(t)$ for $t \in [5, 6]$ with the initial condition $X(0) = 0$.

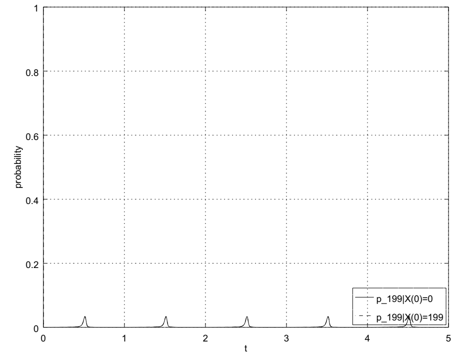

Fig. 7. Example 1: the probability $p_{199}(t)$ for $t \in [0, 5]$ with the initial condition $X(0) = 0$.

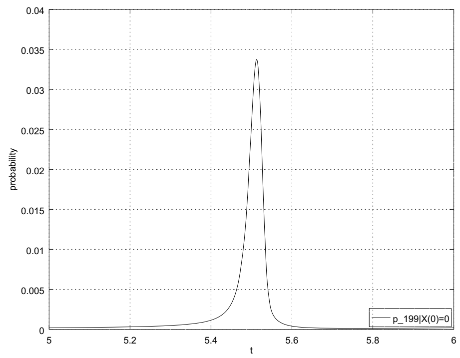

Fig. 8. Example 1: the probability $p_{199}(t)$ for $t \in [5, 6]$ with the initial condition $X(0) = 0$.

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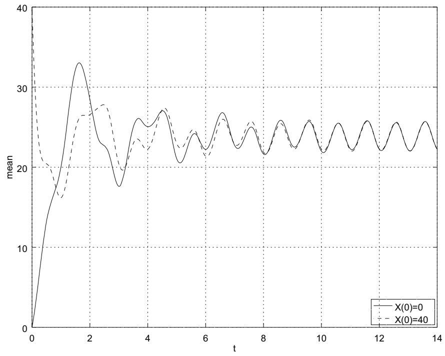

Fig. 9. Example 2: the expected number $E(X(t)|X(0))$ of customers in the queue for $t \in [0, 14]$ with the initial conditions $X(0) = 0$ and $X(0) = S$.

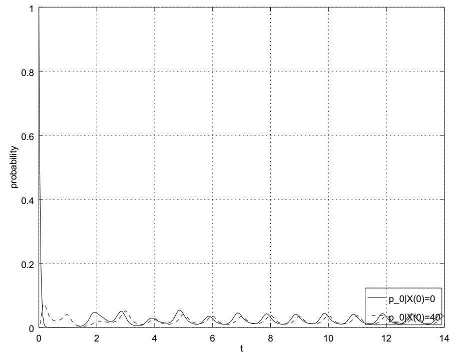

Fig. 11. Example 2: the probability $p_0(t)$ of an empty queue for $t \in [0, 14]$ with the initial conditions $X(0) = 0$ and $X(0) = S$.

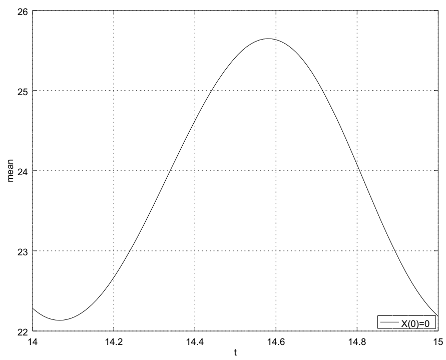

Fig. 10. Example 2: the expected number $E(X(t)|X(0))$ of customers in the queue for $t \in [14, 15]$ with the initial condition $X(0) = 0$.

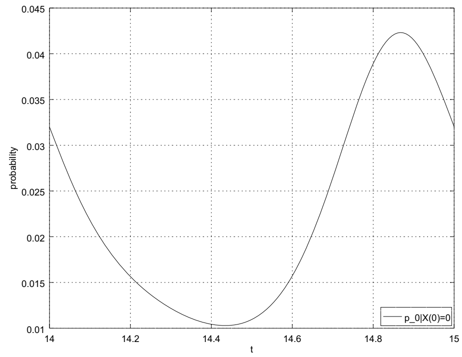

Fig. 12. Example 2: the probability $p_0(t)$ of an empty queue for $t \in [14, 15]$ with the initial condition $X(0) = 0$.

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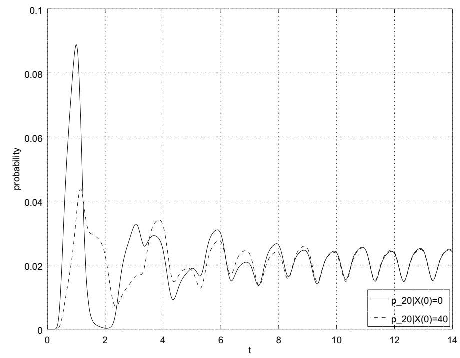

Fig. 13. Example 2: the probability $p_{20}(t)$ for $t \in [0, 14]$ with the initial conditions $X(0) = 0$ and $X(0) = S$.

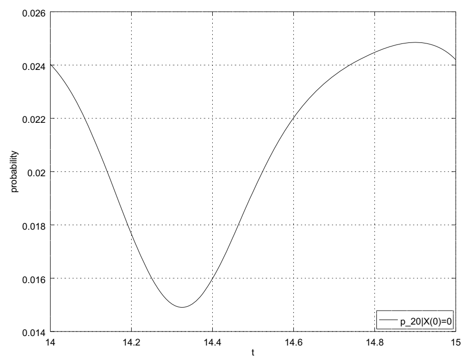

Fig. 14. Example 2: the probability $p_{20}(t)$ for $t \in [14, 15]$ with the initial condition $X(0) = 0$.

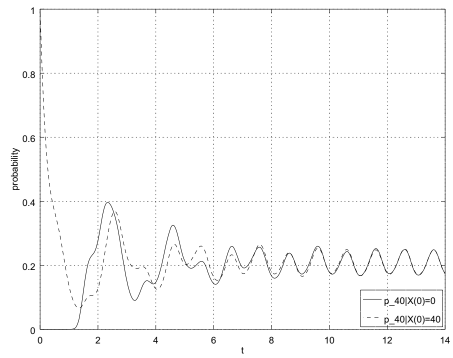

Fig. 15. Example 2: the probability $p_{40}(t)$ for $t \in [0, 14]$ with the initial conditions $X(0) = 0$ and $X(0) = S$.

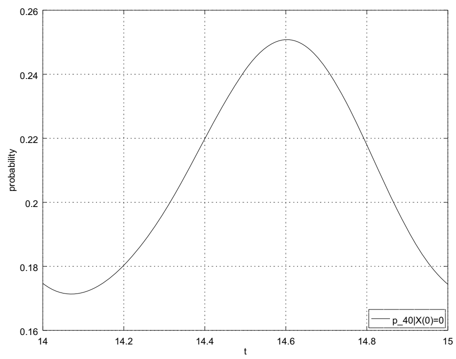

Fig. 16. Example 2: the probability $p_{40}(t)$ for $t \in [14, 15]$ with the initial condition $X(0) = 0$.

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