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POINTWISE COMPLETENESS AND POINTWISE DEGENERACY OF FRACTIONAL STANDARD AND DESCRIPTOR LINEAR CONTINUOUS–TIME SYSTEMS WITH DIFFERENT FRACTIONAL ORDERS

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Descriptor and standard linear continuous-time systems with different fractional orders are investigated. Descriptor systems are analyzed making use of the Drazin matrix inverse. Necessary and sufficient conditions for the pointwise completeness and pointwise degeneracy of descriptor continuous-time linear systems with different fractional orders are derived. It is shown that (i) the descriptor linear continuous-time system with different fractional orders is pointwise complete if and only if the initial and final states belong to the same subspace, (ii) the descriptor linear continuous-time system with different fractional orders is not pointwise degenerated in any nonzero direction for all nonzero initial conditions. Results are reported for the case of two different fractional orders and can be extended to any number of orders.

Keywords: descriptor system, fractional system, noncommensurate order, pointwise completeness, pointwise degeneracy.

1. Introduction

Descriptor (singular) linear systems have been considered in many papers and books (Borawski, 2018; Campbell *et al*., 1976; Dai, 1989; Fahmy and O'Reilly, 1989; Guang-Ren, 2010; Kaczorek, 2014; Kucera and Zagalak, 1988). In descriptor systems it is assumed that $\det E = 0$ therefore, their analysis is more complex. Standard systems are a special case of descriptor systems for which $\det E \neq 0$.

Mathematical fundamentals of fractional calculus are given in the monographs of Kaczorek (2011), Miller and Ross (1993) or Podlubny (1999). This idea was used by engineers for modeling various processes (Dzieliński *et al*., 2009; Ferreira and Machado, 2003; Kaczorek and Rogowski, 2015; Bingi *et al*., 2019; Djennoune *et al*., 2019). The positive fractional linear systems were introduced by Kaczorek (2009), while the systems consisting of n subsystems with different fractional orders were analyzed by Busłowicz (2012), Kaczorek (2010; 2011) and Sajewski (2015; 2016). Absolute stability and global stability of a class of fractional positive nonlinear systems were considered by Kaczorek (2019; 2020). The Drazin inverse matrix method for fractional descriptor continuous-time linear systems was proposed also by Kaczorek (2014).

A dynamical autonomous system is called pointwise complete if every final state of the system can be reached by a suitable choice of its initial conditions. A system which is not pointwise complete is called pointwise degenerated. These properties were studied in many works (Kaczorek, 2011; Korobov, 2017; Metel'skii and Karpuk, 2009). The pointwise completeness and pointwise degeneracy of fractional linear continuous-time systems were investigated by Kaczorek (2015) or Kaczorek and Busłowicz (2009), and for systems with different fractional orders by Trzasko (2014).

In this paper, necessary and sufficient conditions for the pointwise completeness and pointwise degeneracy of standard and descriptor continuous-time linear systems with different fractional orders will be established.

The paper is organized as follows. In Section 2 basic definitions and theorems regarding descriptor fractional continuous-time linear systems and the systems with two different fractional orders are recalled. Section 3

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gives necessary and sufficient conditions for the pointwise completeness and pointwise degeneracy of standard (nondescriptor) continuous-time linear systems with two different fractional orders. Similar conditions but for descriptor systems are given in Section 4. Concluding remarks are given in Section 5.

The following notation will be used: $\mathbb{R}$, the set of real numbers; $\mathbb{R}^{n\times m}$, the set of $n \times m$ real matrices; $\mathbb{R}_+^{n\times m}$, the set of $n \times m$ real matrices with nonnegative entries; $\mathbb{C}$, the field of complex numbers; $\mathbb{I}_n$, the $n \times n$ identity matrix.

2. Preliminaries

2.1. Fractional systems. Consider the descriptor fractional continuous-time linear system

$$E\ {}_0D_t^\alpha x(t) = Ax(t),\quad n-1 < \alpha < n,\quad n \in W = \{1, 2, \dots\},\quad (1)$$

where α is the fractional order, $x(t) \in \mathbb{R}^n$ is the state vector, $E, A \in \mathbb{R}^{n\times n}$ and

$$\begin{aligned} {}_0D_t^\alpha x(t) &= \frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^\infty \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}\,\mathrm{d}\tau, \quad (2)\\ f^n(\tau) &= \frac{\mathrm{d}^n f(\tau)}{\mathrm{d}\tau^n} \end{aligned}$$

is the Caputo definition of order $\alpha \in \mathbb{R}$ (for $0 < \alpha < 1$) of $x(t)$ and

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1}\,\mathrm{d}t \quad (3)$$

is the Euler gamma function.

If $\det E \neq 0$ (the case of standard systems), then the solution of (1) is

$$x(t) = \Phi_0(t)x(0), \quad (4a)$$

where

$$\Phi_0(t) = \sum_{k=0}^\infty \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}. \quad (4b)$$

If $\det E = 0$ and the pencil (E, A) of (1) is regular, i.e.,

$$\det[Es - A] \neq 0 \quad (5)$$

for some $s \in \mathbb{C}$, assuming that, for some chosen $c \in \mathbb{C}$, $\det[Ec - A] \neq 0$ and premultiplying (1) by $[Ec - A]^{-1}$, we obtain

$$\bar{E}\ {}_0D_t^\alpha x(t) = \bar{A}x(t), \quad (6a)$$

where

$$\bar{E} = [Ec - A]^{-1}E, \quad \bar{A} = [Ec - A]^{-1}A. \quad (6b)$$

Note that (1) and (6a) have the same solution $x(t)$.

Definition 1. (*Kaczorek, 2014*) The smallest nonnegative integer q is called the *index* of the matrix $\bar{E} \in \mathbb{R}^{n\times n}$ if

$$\operatorname{rank} \bar{E}^q = \operatorname{rank} \bar{E}^{q+1}. \quad (7)$$

Definition 2. (*Kaczorek, 2014*) A matrix $\bar{E}^D$ is called the *Drazin inverse* of $\bar{E} \in \mathbb{R}^{n\times n}$ if it satisfies the conditions

$$\bar{E}\bar{E}^D = \bar{E}^D\bar{E}, \quad (8a)$$

$$\bar{E}^D\bar{E}\bar{E}^D = \bar{E}^D, \quad (8b)$$

$$\bar{E}^D\bar{E}^{q+1} = \bar{E}^q, \quad (8c)$$

where q is the index of $\bar{E}$ defined by (7).

The Drazin inverse $\bar{E}^D$ of a square matrix $\bar{E}$ always exists and is unique. If $\det \bar{E} \neq 0$, then $\bar{E}^D = \bar{E}^{-1}$.

Lemma 1. (Kaczorek, 2014) *The matrices $\bar{E}$ and $\bar{A}$ defined by* (6b) *satisfy the following equalities:*

1. $\bar{A}\bar{E} = \bar{E}\bar{A}$ *and* $\bar{A}^D\bar{E} = \bar{E}\bar{A}^D$, $\bar{E}^D\bar{A} = \bar{A}\bar{E}^D$, $\bar{A}^D\bar{E}^D = \bar{E}^D\bar{A}^D$,
2. $\ker \bar{A} \cap \ker \bar{E} = \{0\}$,
3. $\bar{E} = T\begin{bmatrix} J & 0 \\ 0 & N \end{bmatrix}T^{-1}$,
   $\bar{E}^D = T\begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix}T^{-1}$,
   $\bar{A} = T\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}T^{-1}$,
   $\det T \neq 0$, $J \in \mathbb{R}^{n_1\times n_1}$ *is nonsingular,* $N \in \mathbb{R}^{n_2\times n_2}$ *is nilpotent,* $n_1 + n_2 = n$,
4. $(\mathbb{I}_n - \bar{E}\bar{E}^D)\bar{A}\bar{A}^D = \mathbb{I}_n - \bar{E}\bar{E}^D$ *and* $(\mathbb{I}_n - \bar{E}\bar{E}^D)(\bar{E}\bar{A}^D)^q = 0$,
5. $(\bar{E}\bar{E}^D)^k = \bar{E}\bar{E}^D$ *for* $k = 2, 3, \dots$,
6. $\bar{E}\bar{E}^D x = x$.

The solution to (1) in case $\det E = 0$ is

$$x(t) = \Phi_0(t)\bar{E}\bar{E}^D w, \quad (9a)$$

where

$$\Phi_0(t) = \sum_{k=0}^\infty \frac{(\bar{E}^D\bar{A})^k t^{k\alpha}}{\Gamma(k\alpha+1)} \quad (9b)$$

and the vector $w \in \mathbb{R}^n$ is arbitrary (Kaczorek, 2014).

2.2. Systems with different fractional orders. Consider a standard ($\det E \neq 0$) fractional linear system with two different fractional orders $\alpha \neq \beta$ described by the equation (Sajewski, 2015; 2016)

$$\begin{bmatrix} \dfrac{\mathrm{d}^\alpha x_1(t)}{\mathrm{d}t^\alpha} \\[2ex] \dfrac{\mathrm{d}^\beta x_2(t)}{\mathrm{d}t^\beta} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad (10)$$

and $p-1<\alpha<p$; $q-1<\beta<q$; $p,q\in W$, where $x_1(t)\in\mathbb{R}^{n_1}$, $x_2(t)\in\mathbb{R}^{n_2}$ and $A_{ij}\in\mathbb{R}^{n_i\times n_j}$; $i,j=1,2$.

The initial conditions for (10) have the form

$$x_1(0)=x_{10},\quad x_2(0)=x_{20},\qquad x(0)=\begin{bmatrix} x_1(0)\\ x_2(0)\end{bmatrix}. \tag{11}$$

The solution of (10) with initial conditions (11) has the form

$$\begin{bmatrix} x_1(t)\\ x_2(t)\end{bmatrix}=\Phi_0(t)\begin{bmatrix} x_1(0)\\ x_2(0)\end{bmatrix}, \tag{12a}$$

where

$$T_{k,l}=\begin{cases}\mathbb{I}_n & \text{for } k=l=0,\\ \begin{bmatrix} A_{11} & A_{12}\\ 0 & 0\end{bmatrix} & \text{for } k=1,\ l=0,\\ \begin{bmatrix} 0 & 0\\ A_{21} & A_{22}\end{bmatrix} & \text{for } k=0,\ l=1,\\ T_{10}T_{k-1,l}+T_{01}T_{k,l-1} & \text{for } k+l>0,\end{cases} \tag{12b}$$

$$\Phi_0(t)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}T_{k,l}\frac{t^{k\alpha+l\beta}}{\Gamma(k\alpha+l\beta+1)}. \tag{12c}$$

Now, consider the descriptor fractional continuous-time linear system with different fractional orders

$$E\begin{bmatrix}\dfrac{\mathrm{d}^{\alpha}x_1(t)}{\mathrm{d}t^{\alpha}}\\ \dfrac{\mathrm{d}^{\beta}x_2(t)}{\mathrm{d}t^{\beta}}\end{bmatrix}=A\begin{bmatrix} x_1(t)\\ x_2(t)\end{bmatrix} \tag{13}$$

and $p-1<\alpha<p$; $q-1<\beta<q$; $p,q\in W$, where

$$E=\begin{bmatrix} E_1 & 0\\ 0 & E_2\end{bmatrix}\in\mathbb{R}^{(n_1+n_2)\times(n_1+n_2)},$$

$$A=\begin{bmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\in\mathbb{R}^{(n_1+n_2)\times(n_1+n_2)}.$$

It is assumed that $\det E=0$ but the pencil (E,A) of (13) is regular, i.e.,

$$\det\left[\begin{bmatrix} E_1 & 0\\ 0 & E_2\end{bmatrix}\begin{bmatrix} s^{\alpha} & 0\\ 0 & s^{\beta}\end{bmatrix}-\begin{bmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\right]\neq 0 \tag{14}$$

for some $s^{\alpha},\ s^{\beta}\in\mathbb{C}$. Similarly to (1), assuming that for some chosen $c_1,c_2\in\mathbb{C}$, $\det[E\mathrm{diag}(\mathbb{I}_{n_1}c_1,\mathbb{I}_{n_2}c_2)-A]\neq 0$ and premultiplying (13) by $[E\mathrm{diag}(\mathbb{I}_{n_1}c_1,\mathbb{I}_{n_2}c_2)-A]^{-1}$, we obtain

$$\bar{E}\begin{bmatrix}\dfrac{\mathrm{d}^{\alpha}x_1(t)}{\mathrm{d}t^{\alpha}}\\ \dfrac{\mathrm{d}^{\beta}x_2(t)}{\mathrm{d}t^{\beta}}\end{bmatrix}=\bar{A}\begin{bmatrix} x_1(t)\\ x_2(t)\end{bmatrix}, \tag{15a}$$

where

$$\begin{aligned}\bar{E}&=[E\mathrm{diag}(\mathbb{I}_{n_1}c_1,\mathbb{I}_{n_2}c_2)-A]^{-1}E=\begin{bmatrix}\bar{E}_{11} & \bar{E}_{12}\\ \bar{E}_{21} & \bar{E}_{22}\end{bmatrix},\\ \bar{A}&=[E\mathrm{diag}(\mathbb{I}_{n_1}c_1,\mathbb{I}_{n_2}c_2)-A]^{-1}A=\begin{bmatrix}\bar{A}_{11} & \bar{A}_{12}\\ \bar{A}_{21} & \bar{A}_{22}\end{bmatrix}=\bar{T}_{10}+\bar{T}_{01},\\ \bar{T}_{10}&=\begin{bmatrix}\bar{A}_{11} & \bar{A}_{12}\\ 0 & 0\end{bmatrix},\\ \bar{T}_{01}&=\begin{bmatrix} 0 & 0\\ \bar{A}_{21} & \bar{A}_{22}\end{bmatrix}.\end{aligned} \tag{15b}$$

Note that (13) and (15a) have the same solution $x(t)$.

In the case of the system with two different fractional orders Definition 1 takes the following form.

Definition 3. The pair of smallest nonnegative integers q_i, $i=1,2$ is called the *index* of the matrix $\bar{E}_{ii}\in\mathbb{R}^{n_i\times n_i}$ if

$$\operatorname{rank}\bar{E}_{ii}^{q_i}=\operatorname{rank}\bar{E}_{ii}^{q_i+1} \tag{16}$$

and $q=q_1+q_2$ is the index of $\bar{E}$.

Theorem 1. *If* $\bar{T}_{k,l}\bar{E}=\bar{E}\bar{T}_{k,l}$, *then the solution of* (15a) *is*

$$x(t)=\bar{\Phi}_0(t)\bar{E}\bar{E}^{D}w, \tag{17a}$$

where

$$\bar{T}_{k,l}=\begin{cases}\mathbb{I}_n & \text{for } k=l=0,\\ \begin{bmatrix}\bar{A}_{11} & \bar{A}_{12}\\ 0 & 0\end{bmatrix} & \text{for } k=1,\ l=0,\\ \begin{bmatrix} 0 & 0\\ \bar{A}_{21} & \bar{A}_{22}\end{bmatrix} & \text{for } k=0,\ l=1,\\ \bar{T}_{10}\bar{T}_{k-1,l}+\bar{T}_{01}\bar{T}_{k,l-1} & \text{for } k+l>0,\end{cases} \tag{17b}$$

$$\bar{\Phi}_0(t)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(\bar{E}^{D})^{k+l}\bar{T}_{k,l}\frac{t^{k\alpha+l\beta}}{\Gamma(k\alpha+l\beta+1)} \tag{17c}$$

and the vector

$$w=\begin{bmatrix} w_1\\ w_2\end{bmatrix}\in\mathbb{R}^{n_1+n_2}$$

is arbitrary.

Proof. First, we shall show that if $\bar{T}_{k,l}=\bar{E}\bar{T}_{k,l}$, then

$$\bar{T}_{k,l}^{D}\bar{E}=\bar{E}\bar{T}_{k,l}^{D}, \tag{18a}$$

$$\bar{E}^{D}\bar{T}_{k,l}=\bar{T}_{k,l}\bar{E}^{D}, \tag{18b}$$

$$\bar{T}_{k,l}^{D}\bar{E}^{D}=\bar{E}^{D}\bar{T}_{k,l}^{D}. \tag{18c}$$

Postmultiplying $\bar{T}_{k,l}\bar{E} = \bar{E}\bar{T}_{k,l}$ by $(\bar{T}_{k,l}^D)^2$, we obtain for the right-hand side

$$\begin{aligned}\bar{T}_{k,l}\bar{E}(\bar{T}_{k,l}^D)^2 &= \bar{E}\bar{T}_{k,l}(\bar{T}_{k,l}^D)^2 \\ &= \bar{E}\bar{T}_{k,l}^D\bar{T}_{k,l}\bar{T}_{k,l}^D = \bar{E}\bar{T}_{k,l}^D\end{aligned} \tag{19}$$

since $\bar{T}_{k,l}^D\bar{T}_{k,l}\bar{T}_{k,l}^D = \bar{T}_{k,l}^D$ and for the left-hand side taking into account that $\bar{T}_{k,l}^D$ can be written as a polynomial $p(\bar{T}_{k,l})$

$$\begin{aligned}\bar{T}_{k,l}\bar{E}(\bar{T}_{k,l}^D)^2 &= \bar{T}_{k,l}\bar{E}[p(\bar{T}_{k,l})]^2 \\ &= \bar{T}_{k,l}[p(\bar{T}_{k,l})]^2\bar{E} = \bar{T}_{k,l}^D\bar{E}.\end{aligned} \tag{20}$$

Therefore, from (19) and (20) we have (18a). The proof of (18b) is dual. To prove (18c) we pre- and postmultiply $\bar{T}_{k,l}\bar{E} = \bar{E}\bar{T}_{k,l}$ by $(\bar{T}_{k,l}^D)^2$ and we obtain for the right-hand side

$$\begin{aligned}(\bar{T}_{k,l}^D)^2\bar{T}_{k,l}\bar{E}(\bar{T}_{k,l}^D)^2 &= (\bar{T}_{k,l}^D)^2\bar{E}\bar{T}_{k,l}(\bar{T}_{k,l}^D)^2 \\ &= (\bar{T}_{k,l}^D)^2\bar{E}\bar{T}_{k,l}^D = (\bar{T}_{k,l}^D)^3\bar{E}\end{aligned} \tag{21}$$

and for the left-hand side

$$(\bar{T}_{k,l}^D)^2\bar{T}_{k,l}(\bar{T}_{k,l}^D)^2\bar{E} = (\bar{T}_{k,l}^D)^3\bar{E}. \tag{22}$$

Applying the Laplace transform ($\mathfrak{L}$) to (15a) and taking into account that (Kaczorek, 2011)

$$\mathfrak{L}\left[\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha}\right] = s^\alpha X(s) - s^{\alpha-1}x(0), \tag{23}$$

$$\mathfrak{L}[t^\alpha] = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}, \tag{24}$$

we obtain

$$\begin{aligned}\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} &= \begin{bmatrix} \mathbb{I}_{n_1}s^\alpha - \bar{A}_{11} & -\bar{A}_{12} \\ \bar{A}_{21} & \mathbb{I}_{n_2}s^\beta - \bar{A}_{22} \end{bmatrix}^{-1} \\ &\quad\times \begin{bmatrix} \mathbb{I}_{n_1}s^{\alpha-1} & 0 \\ 0 & \mathbb{I}_{n_2}s^{\beta-1} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}.\end{aligned} \tag{25}$$

Using (15b), it can be verified that

$$\begin{aligned}&\begin{bmatrix} \mathbb{I}_{n_1}s^\alpha - \bar{A}_{11} & -\bar{A}_{12} \\ \bar{A}_{21} & \mathbb{I}_{n_2}s^\beta - \bar{A}_{22} \end{bmatrix}^{-1} \\ &\quad\times \begin{bmatrix} \mathbb{I}_{n_1}s^{\alpha-1} & 0 \\ 0 & \mathbb{I}_{n_2}s^{\beta-1} \end{bmatrix} = \mathfrak{L}[\bar{\Phi}_0(t)].\end{aligned} \tag{26}$$

This completes the proof. ■

Remark 1. If it is possible to choose $c_1 = c_2 \in \mathbb{C}$ in (15b), then the conditions of Theorem 1 are always satisfied. If $\det A \neq 0$ and we assume $c_1 = c_2 = 0$, then

$$\begin{aligned}\bar{E} &= [-A]^{-1}E, & \bar{A} &= -\mathbb{I}_n, \\ \bar{T}_{10} &= \begin{bmatrix} \mathbb{I}_{n_1} & 0 \\ 0 & 0 \end{bmatrix}, & \bar{T}_{01} &= \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{I}_{n_2} \end{bmatrix}.\end{aligned} \tag{27}$$

3. Pointwise completeness and pointwise degeneracy of fractional continuous-time linear systems with different fractional orders

In this section necessary and sufficient conditions for the pointwise completeness and pointwise degeneracy of standard (nondescriptor) continuous-time linear systems with different fractional orders will be established.

Definition 4. The standard fractional continuous-time linear system (10) is called *pointwise complete* at the point $t = t_f$ if for every final state $x_f \in \mathbb{R}^n$, there exists a boundary condition (11) such that

$$x_f = x(t_f). \tag{28}$$

Theorem 2. *The standard fractional continuous-time linear system* (10) *is pointwise complete at the point* $t = t_f$ *if and only if*

$$\operatorname{rank}\Phi_0(t_f) = n, \tag{29a}$$

where

$$\Phi_0(t_f) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} T_{k,l}\frac{t_f^{k\alpha+l\beta}}{\Gamma(k\alpha+l\beta+1)}. \tag{29b}$$

Proof. Using the solution (12) for $t = t_f$ of the fractional linear system (10) we obtain

$$x_f = x(t_f) = \Phi_0(t_f)x(0). \tag{30}$$

From (30), it follows that for given x_f, it is possible to find $x(0)$ if and only if the condition (29) is satisfied. Therefore, the fractional system (10) is pointwise complete at the point $t = t_f$ if and only if the condition (29) is satisfied. ■

Example 1. Check the pointwise completeness of the fractional system (10) for $0 < \alpha, \beta < 1$ with the nilpotent matrix

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{31}$$

The nilpotency index of the matrix (31) is equal to 2. Using (29b) and (31), we obtain

$$\begin{aligned}\Phi_0(t_f) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\frac{t_f^\alpha}{\Gamma(\alpha+1)} \\ &= \begin{bmatrix} 1 & \frac{t_f^\alpha}{\alpha} \\ 0 & 1 \end{bmatrix} \in \mathbb{R}_+^{2\times 2}.\end{aligned} \tag{32}$$

Assuming $t_f = 1$, $x_f = [\,1 \quad 1\,]^T$ and using (30), (32) we obtain

$$\begin{aligned}x(0) &= \Phi_0(t_f)^{-1}x_f \\ &= \begin{bmatrix} 1 & \frac{1}{\alpha} \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 - \frac{1}{\alpha} \\ 1 \end{bmatrix}.\end{aligned} \tag{33}$$

Therefore, the fractional system (10) with (31) is pointwise complete in the point $t = t_f = 1$. ♦

Definition 5. The standard fractional continuous-time linear system (10) is called pointwise degenerated in the direction v for $t = t_f$ if there exists a vector $v \in \mathbb{R}^n$ such that for all bounded conditions $x(0)$ the solution of (10) for $t = t_f$ satisfies the condition

$$v^T x_f = 0. \tag{34}$$

Theorem 3. *The standard fractional continuous-time linear system* (10) *is pointwise degenerated in the direction* $v \in \mathbb{R}^n$ *for* $t = t_f$ *if and only if*

$$\det \Phi_0(t_f) = 0. \tag{35}$$

Proof. From (34) and (30) we have

$$v^T \Phi_0(t_f) x(0) = 0. \tag{36}$$

Note that there exists a nonzero vector $v \in \mathbb{R}^n$ such that (36) holds, if and only if the matrix $\Phi_0(t_f)$ is singular. Therefore, the standard fractional system (10) is pointwise degenerated in the direction $v \in \mathbb{R}^n$ for $t = t_f$ if and only if the condition (35) is satisfied. ■

Example 2. (*Continuation of Example 1*) Consider the system (10) for $0 < \alpha, \beta < 1$ with the matrix (31). The matrix $\Phi_0(t_f)$ has the form (32) and for $t_f = 1$ we obtain

$$\Phi_0(1) = \begin{bmatrix} 1 & \frac{1}{\alpha} \\ 0 & 1 \end{bmatrix} \in \mathbb{R}_+^{2\times 2}. \tag{37}$$

In this case, the condition (35) is not satisfied since $\det \Phi_0(1) = 1$ and the fractional system is not pointwise degenerated in any direction $v \in \mathbb{R}^2$.

Note that the equation $v^T \Phi_0(1) = 1$ has only the zero solution $v^T = [\,0 \quad 0\,]$. ♦

4. Pointwise completeness and pointwise degeneracy of descriptor fractional continuous-time linear systems with different fractional orders

In this section, necessary and sufficient conditions for the pointwise completeness and pointwise degeneracy of descriptor continuous-time linear systems with different fractional orders will be established.

Definition 6. The descriptor fractional continuous-time linear system (13) is called pointwise complete at the point $t = t_f$ if for every final state $x_f \in \mathbb{R}^n$, there exists a boundary condition $x(0) \in \operatorname{Im} \bar{E}\bar{E}^D$ such that

$$x(t_f) = x_f \in \operatorname{Im} \bar{E}\bar{E}^D. \tag{38}$$

Theorem 4. *The descriptor fractional continuous-time linear system* (13) *is pointwise complete for* $t = t_f$ *and every* $x_f \in \operatorname{Im} \bar{E}\bar{E}^D \subset \mathbb{R}^n$ *if and only if*

$$\det \bar{\Phi}_0(t_f) \neq 0, \tag{40}$$

where

$$\bar{\Phi}_0(t) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} (\bar{E}^D)^{k+l} \bar{T}_{k,l} \frac{t^{k\alpha+l\beta}}{\Gamma(k\alpha + l\beta + 1)}. \tag{41}$$

Proof. From (17) we obtain

$$x_f = x(t_f) = \bar{\Phi}_0(t_f) x(0), \tag{42}$$

where $x(0) \in \operatorname{Im} \bar{E}\bar{E}^D$.

For given $x_f \in \operatorname{Im} \bar{E}\bar{E}^D \subset \mathbb{R}^n$ we may find $x(0)$ if and only if the condition (39) is satisfied. Therefore, the descriptor fractional system (13) is pointwise complete at the point $t = t_f$ if and only if the condition (39) is satisfied. ■

Example 3. Consider the descriptor fractional system (13) for $\alpha = 0.6$, $\beta = 0.8$ with the matrices

$$E = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$
$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \tag{43}$$
$$n_1 = 1, \ n_2 = 2.$$

We choose $c_1 = c_2 = 1$ and, using (15b) and (43), we obtain

$$\bar{E} = [E\mathrm{diag}(c_1, c_2) - A]^{-1} E = \begin{bmatrix} \bar{E}_{11} & \bar{E}_{12} \\ \bar{E}_{21} & \bar{E}_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$\bar{A} = [E\mathrm{diag}(c_1, c_2) - A]^{-1} A = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{44}$$

Next, using (17b), we obtain

$$\bar{T}_{10} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$\bar{T}_{01} = \begin{bmatrix} 0 & 0 \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{45}$$
$$\bar{T}_{11} = \bar{T}_{10}\bar{T}_{01} + \bar{T}_{01}\bar{T}_{10} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$\vdots$$

The Drazin inverse matrix of $\bar{E}$ has the form

$$\bar{E}^D = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{46}$$

and

$$\bar{E}\bar{E}^D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{47}$$

Using (17) and (45), (47) we obtain the solution $x(t)$ to (15a), where

$$x(0) \in \operatorname{Im} \bar{E}\bar{E}^D = \begin{bmatrix} x_{11}(0) \\ 0 \\ x_{21}(0) \end{bmatrix}$$

and $x_{11}(0)$, $x_{21}(0)$ are arbitrary.

Note that the matrix $\bar{\Phi}_0(t)$ is nonsingular and by Theorem 4 the descriptor fractional system with (43) is pointwise complete for $t = t_f = 1$ and every $x_f \subset \mathbb{R}^3$ of the form

$$x_f = \begin{bmatrix} x_{11}(t_f) \\ x_{21}(t_f) \end{bmatrix},$$

where $x_{11}(t_f)$, $x_{21}(t_f)$ are arbitrary. ♦

Definition 7. The descriptor fractional continuous-time linear system (13) is called *pointwise degenerated* in the direction v for $t = t_f$ if there exists a vector $v \in \mathbb{R}^n$ such that for all bounded conditions $x(0) \in \operatorname{Im} \bar{E}\bar{E}^D$ the solution of (13) for $t = t_f$ satisfies the condition

$$v^T x_f = 0. \tag{48}$$

Theorem 5. *The descriptor fractional continuous-time linear system* (13) *is pointwise degenerated in the direction* $v \in \mathbb{R}^n$ *for* $t = t_f$ *if and only if*

$$\det \bar{\Phi}_0(t_f) = 0. \tag{49}$$

Proof. From (48) and (42) for $t = t_f$ we have

$$v^T \bar{\Phi}_0(t_f) x(0) = 0. \tag{50}$$

There exists a nonzero vector $v \in \mathbb{R}^n$ such that (50) holds for all $x(0) \in \operatorname{Im}\bar{E}\bar{E}^D$ if and only if the condition (49) is satisfied. Therefore, the descriptor fractional system (13) is pointwise degenerated in the direction $v \in \mathbb{R}^n$ for $t = t_f$ if the condition (49) is satisfied. ■

Remark 2. The vector $v \in \mathbb{R}^n$ in which the descriptor fractional continuous-time linear system (13) is pointwise degenerated can be computed from the equation

$$v^T \bar{\Phi}_0(t_f) = 0. \tag{51}$$

Example 4. (*Continuation of Example 3*) Consider the system (13) for $\alpha = 0.6$, $\beta = 0.8$ with the matrices (43). In Example 3 it was shown that the matrix $\bar{\Phi}_0(t_f)$ for $t_f = 1$ is nonsingular. Therefore, the descriptor fractional system (13) with (43) is not pointwise degenerated for $t_f = 1$ in any direction $v \in \mathbb{R}^3$. ♦

5. Concluding remarks

Descriptor and standard linear continuous-time systems with different fractional orders have been analyzed. The Drazin matrix inverse has been used in the analysis of descriptor systems. Necessary and sufficient conditions for the pointwise completeness and pointwise degeneracy of descriptor continuous-time linear systems with different fractional orders have been given provided. We have proven the following:

(i) The descriptor linear continuous-time system with different fractional orders is pointwise complete if and only if the initial and final states belong to the same subspace.

(ii) The descriptor linear continuous-time system with different fractional orders is not pointwise degenerated in any nonzero direction for all nonzero initial conditions.

The discussion has been complemented with numerical examples for two different fractional orders α and β. The presented results can be extended to any number of orders.

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