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AN APPROXIMATE SOLUTION OF THE AFFINE–QUADRATIC CONTROL PROBLEM BASED ON THE CONCEPT OF OPTIMAL DAMPING

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The paper is devoted to a particular case of the nonlinear and nonautonomous control law design problem based on the application of the optimization approach. Close attention is paid to the controlled plants, which are presented by affine-control mathematical models characterized by integral quadratic functionals. The proposed approach to controller design is based on the optimal damping concept firstly developed by V.I. Zubov in the early 1960s. A modern interpretation of this concept allows us to construct effective numerical procedures of control law synthesis initially oriented to practical implementation. The main contribution is the proposition of a new methodology for selecting the functional to be damped. The central idea is to perform parameterization of a set of admissible items for this functional. As a particular case, a new method of this parameterization has been developed, which can be used for constructing an approximate solution to the classical optimization problem. Applicability and effectiveness of the proposed approach are confirmed by a practical numerical example.

Keywords: feedback, stability, damping control, functional, optimization.

1. Introduction

The wide spread of various intelligent automatic control systems now raises numerous problems related to their performance, safety and reliability. In this respect, the various approaches associated with the design of feedback control laws have already been extensively researched and presented in numerous publications (Sontag, 1998; Khalil, 2002; Slotine and Li, 1991; Lewis *et al.*, 2012; Geering, 2007; Sepulchre *et al.*, 1997). Nevertheless, the complexity of the problem is now high because of the presence of many dynamical requirements, restrictions, and conditions to be satisfied by using control systems.

Among various dynamic systems, a special place is occupied by affine-in-control plants with nonlinear and nonautonomous mathematical models. This class of plants primarily includes mobile objects such as robots, marine vehicles, aircrafts, and cars (Khalil, 2002; Slotine and Li, 1991; Fossen, 1994; Do and Pan, 2009). Moreover, it was shown by Balakrishnan (1966) that any controlled autonomous nonlinear system can be converted to the affine-control form using a specific nonlinear transformation of the state space vector. To be fair, it should be noted that it is very difficult to find such a transformation.

Let us note that one of the most effective analytical and numerical tools for feedback connection design for today is the optimization approach. This opinion is due to the flexibility and adaptability of modern optimization methods with respect to the relevant practical demands. The most significant aspects of applicability of this methodology for control systems design are reflected by Lewis *et al.* (2012), Geering (2007) and Sepulchre *et al.* (1997). A modern example of the practical application of the optimization approach is given by Wasilewski *et al.* (2019).

As for affine plants with integral quadratic performance indices, there exist several methods to find an optimal controller numerically. Nevertheless, we cannot say that the optimization approach is treated overall as a universal instrument for the practical implementation. This can be explained by some disadvantages connected with computational troubles. In particular, using Bellman's dynamic programming principle, we face difficulties related to the numerical solution of the Hamilton–Jacobi–Bellman (HJB) equation and the computer implementation of the resulting solution.

Therefore, there exists a dire necessity to develop persistently analytical and numerical methods of control law design based on optimization methodology. The essence of such a study should be especially aimed at improving the computational effectiveness of these methods.

This work is mostly devoted to the specific approach which can be used to design stabilizing controllers based on the theory of optimal damping (OD) in transient processes. This theory, firstly proposed and developed by Zubov (1962; 1966; 1978), allows constructing effective methods for control calculation with essentially reduced computational consumptions.

The main purpose of the paper is to develop a method for an approximate solution of the affine-quadratic optimization (AQO) problem, which can be put into practice using the OD concept.

The main contributions of this paper are determined by the following statements. First, we propose a new methodology for selecting the functional to be damped, taking into account the specific features of the AQO-problem. The central idea is to provide a parameterization of the set of admissible items for this functional. Second, we develop, as a particular case, a new method of such parameterization, using the control Lyapunov functions from the set of the positive definite quadratic forms. Hereby, the choice of these functions as the basis is argued by the guaranteed asymptotic stability and the desired quality of processes under small deviations from the zero equilibrium position. The practical applicability and effectiveness of the proposed method is illustrated with a controller design for a convey-crane system. The advantage of the proposed approach is determined by the main features of the OD concept, which allow calculating the control action with essentially reduced computational load that is very significant for a real-time implementation of the feedback.

The paper is organized as follows. In Section 2, the AQO-problem is posed and the numerical scheme of its solution based on Bellman's optimality principle is discussed. Section 3 is devoted to the specific features of Zubov's OD-concept and its connection with the exact solution of the AQO-problem. In Section 4, a numerical method is constructed for an approximate solution of the above-mentioned problem. This one is realized as a solution to the corresponding OD-problem. Section 5 presents a numerical example of the application of the proposed method to design a control law for a convey-crane system. Section 6 concludes the presentation by discussing the overall results of this research.

2. Affine-quadratic optimization problem

Consider an affine-control mathematical model of the plant given by the ordinary time dependent nonlinear differential equations of the form

$$\dot{\mathbf{x}} = \mathbf{f}(t,\mathbf{x}) + \mathbf{g}(t,\mathbf{x})\mathbf{u}, \tag{1}$$

where $\mathbf{x} \in E^n$ is the state vector and vector $\mathbf{u} \in E^m$ implies a control action. Here, $t \in [t_0, \infty)$, $\mathbf{g} := (\mathbf{g_1}\ \ \mathbf{g_2}\ \ \ldots\ \ \mathbf{g_m})$, $\mathbf{f}, \mathbf{g}_i : E^{n+1} \to E^n$, $i = \overline{1,m}$, the functions $\mathbf{f}$ and $\mathbf{g}_i$ are continuously differentiable for any $\mathbf{x}$ and t. Let us assume that the plant (1) is controllable by the vector $\mathbf{u}$.

The main practical problem connected with the plant (1) is to design a feedback control law (controller) of the form

$$\mathbf{u} = \mathbf{u}_0(t,\mathbf{x}), \tag{2}$$

providing a zero equilibrium position for the closed-loop connection (1), (2) and stabilizing this position, taking into account certain desirable requirements for the performance of the dynamical processes. In addition, for any t, $\mathbf{x}$, the vector $\mathbf{u}$ of control actions must be admissible, i.e., $\mathbf{u}_0(t,\mathbf{x}) \in U \subseteq E^m$, where the compact set U is initially given for plant (1).

Remark 1. As is known, usually, feedback is composed using the vector of real measurements. However, the construction of the state controller (2) serves as a mandatory composite stage of the dynamic feedback synthesis based on asymptotic observers.

In the range of stabilizing properties, uniform asymptotic stability (UAS) is usually sufficient; nevertheless, this requirement can be strengthened to global stability (UGAS) for some practical cases.

As for practical performance requests, these questions are usually reflected in the form of certain additional requirements to be satisfied by the controller (2). In most cases, these requirements can be presented as follows:

$$\mathbf{x}(t,\mathbf{x}_0,\mathbf{u}_0(\cdot)) \in X, \quad \forall t \geq t_0, \forall \mathbf{x}_0 \in B_r, \forall \mathbf{u}_0 \in U, \tag{3}$$

where the function $\mathbf{x}(t,\mathbf{x}_0,\mathbf{u}_0(\cdot))$ is the trajectory of plant (1), closed by controller (2), under the initial condition $\mathbf{x}(\mathbf{t_0}) = \mathbf{x}_0$, $B_r \subseteq E^n$ (r-neighborhood of the origin). Here, X is an admissible set for the closed-loop system motion: any violation of its bounds is forbidden.

To discuss the aforementioned problem, it is possible to use different methods of its mathematical formalization. The approach consists in formulating certain optimization problems, which are solved by the choice of the controller (2).

Most often, numerous scientific publications (Geering, 2007; Sepulchre *et al.*, 1997; Zubov, 1978) flatly connect our impressions of the process performance with values of the integral functionals of the form

$$J = J(\mathbf{u}(\cdot)) = \int_{t_0}^{\infty} F_0(t, \mathbf{x}, \mathbf{u})\, \mathrm{d}t. \tag{4}$$

Here, the integrand F_0 is positive definite, i.e.,

$$F_0(t, \mathbf{x}, \mathbf{u}) \geq 0, \quad \forall t \geq t_0, \forall \mathbf{x} \in B_r, \forall \mathbf{u} \in U. \tag{5}$$

Note that its zero values occur if and only if $\mathbf{x} = \mathbf{0}$ and $\mathbf{u} = \mathbf{0}$ simultaneously for any time t. It is assumed that, for the closed-loop systems considered, the improper integral in (4) converges.

Let us particularly note that the choice of the function F_0, as a rule, can be made informally based on an expert's opinions in a certain connection with the requirement (3).

A significant role is played by the partial case of the functional (4) in the form

$$J = J(\mathbf{u}(\cdot)) = \int_{t_0}^{\infty} \left(\mathbf{x}^T \mathbf{Q}(t)\mathbf{x} + \mathbf{u}^T \mathbf{R}(t)\mathbf{u}\right)\, \mathrm{d}t \tag{6}$$

with initially given matrices, where $\mathbf{R}(t)$ is symmetric and positive-definite, and $\mathbf{Q}(t)$ is symmetric and positive semi-definite (Geering, 2007).

Such a choice of the functional has an intuitively clear meaning: the control process is much better when the value of the functional (6) is smaller.

In this respect, the following affine-quadratic optimization problem

$$J(\mathbf{u}(\cdot)) \to \min_{\mathbf{u}\in U}, \quad \mathbf{u}_{c0}(t, \mathbf{x}) = \arg\min_{\mathbf{u}\in U} J(\mathbf{u}(\cdot)), \tag{7}$$
$$J_0 := J(\mathbf{u}_{c0}(\cdot))$$

occupies the main position, and numerous well-known approaches are widely used for its practical solution. In particular, consider certain varieties of Bellman's dynamic programming methodology (Lewis *et al.*, 2012; Geering, 2007; Zubov, 1966). As is known, to provide a feedback control design, it is necessary to carry out the following steps:

Step 1. Given the system (1), the performance index (6) and an admissible set U, form the HJB-equation

$$\frac{\partial V(t, \mathbf{x})}{\partial t} + \min_{\mathbf{u}\in U} \left\{ \frac{\partial V(t, \mathbf{x})}{\partial \mathbf{x}} [\mathbf{f}(t, \mathbf{x}) + \mathbf{g}(t, \mathbf{x})\mathbf{u}] + \mathbf{x}^T \mathbf{Q}(t)\mathbf{x} + \mathbf{u}^T \mathbf{R}(t)\mathbf{u} \right\} = 0, \tag{8}$$

where the Bellman function $V(t, \mathbf{x})$ is initially unknown.

Step 2. In accordance with (8), establish the connection between a control $\mathbf{u}$ and the Bellman function $V(t, \mathbf{x})$, providing the minimum of the expression in the brackets:

$$\begin{aligned} \mathbf{u} &= \tilde{\mathbf{u}}\, [t, \mathbf{x}, V(t, \mathbf{x})] \\ &= \arg\min_{\mathbf{u}\in U} \left\{ \frac{\partial V(t, \mathbf{x})}{\partial \mathbf{x}} [\mathbf{f}(t, \mathbf{x}) + \mathbf{g}(t, \mathbf{x})\mathbf{u}] \right. \\ &\quad \left. + \mathbf{x}^T \mathbf{Q}(t)\mathbf{x} + \mathbf{u}^T \mathbf{R}(t)\mathbf{u} \right\} \\ &= \arg\min_{\mathbf{u}\in U} \left\{ \frac{\partial V(t, \mathbf{x})}{\partial \mathbf{x}} \mathbf{g}(t, \mathbf{x})\mathbf{u} + \mathbf{u}^T \mathbf{R}(t)\mathbf{u} \right\}. \end{aligned} \tag{9}$$

Additionally supposing that $\tilde{\mathbf{u}}$ is an inner point of U, we obtain

$$\tilde{\mathbf{u}}\, [t, \mathbf{x}, V(t, \mathbf{x})] = -\frac{1}{2} \mathbf{R}^{-1}(t) \mathbf{g}^T(t, \mathbf{x}) \left(\frac{\partial V}{\partial \mathbf{x}} \right)^T. \tag{10}$$

Step 3. Substitute the obtained function $\tilde{\mathbf{u}}$ into (8), getting as a result an HJB-equation, which is not weighed down by the minimum search operation:

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \mathbf{x}} \mathbf{f}(t, \mathbf{x}) - \frac{1}{4} \frac{\partial V}{\partial \mathbf{x}} \mathbf{g}(t, \mathbf{x}) \mathbf{R}^{-1} \mathbf{g}^T(t, \mathbf{x}) \left(\frac{\partial V}{\partial \mathbf{x}} \right)^T + \mathbf{x}^T \mathbf{Q}\mathbf{x} = 0. \tag{11}$$

One can easily see that (11) is a routine PDE with respect to the initially unknown function $V(t, \mathbf{x})$.

Step 4. If the solution $V = \tilde{V}(t, \mathbf{x})$ of this equation is computed, and if function $\tilde{V}$ is continuously differentiable and satisfies the conditions $\tilde{V}(t, \mathbf{0}) = 0$, $\forall t \geq t_0$, $\tilde{V}(\infty, \mathbf{x}) = 0$ $\forall \mathbf{x} \in B_r$, then after substitution $V = \tilde{V}(t, \mathbf{x})$ into (10) obtain a desired solution to the AQO-problem as follows:

$$\mathbf{u}_0(t, \mathbf{x}) = -\frac{1}{2} \mathbf{R}^{-1}(t) \mathbf{g}^T(t, \mathbf{x}) \left(\frac{\partial \tilde{V}(t, \mathbf{x})}{\partial \mathbf{x}} \right)^T. \tag{12}$$

As for practical implementation of the mentioned scheme, we may face two difficulties: first, a solution process for the PDE (11) can be computationally too expensive; second, the obtained feedback (12) can be too complicated for a practical realization.

The aforementioned circumstances motivate us to proceed to an approximate solution of the AQO-problem, which can be implemented using the OD-concept. In general, this allows reducing computational consumption, and simplifying the control law.

3. Foundations of optimal damping control

Taking into account the presence of the difficulties mentioned above, turn to an alternative approach to a solution of the AQO-problem. This approach is based on the concept of optimal transient processes damping, which was first proposed by Zubov (1962; 1966; 1978). The essence of this concept is built upon the functional

$$L = L(t,\mathbf{x},\mathbf{u}) = V(t,\mathbf{x}) + \int_{t_0}^{t} F(\tau,\mathbf{x},\mathbf{u})\,\mathrm{d}\tau, \qquad (13)$$

which is defined on the trajectories of the plant (1). This functional is introduced to check the performance of the closed-loop connection (1), (2).

Here, the scalar function $V = V(t,\mathbf{x})$ can be used to define a distance from the current state $\mathbf{x}$ of the plant (1) to the origin. Let us assume that this function is continuously differentiable and satisfies the conditions

$$\alpha_1(\|\mathbf{x}\|) \le V(t,\mathbf{x}) \le \alpha_2(\|\mathbf{x}\|), \qquad (14)$$

$\forall t \ge t_0$, $\forall \mathbf{x} \in B_r$, and for some functions $\alpha_1, \alpha_2 \in K$ (Khalil, 2002; Slotine and Li, 1991; Hahn and Baartz, 1967).

Note that the integral item in (13) determines inherently a penalty for the closed-loop system with the help of the additionally given function F connected with the performance of the motion. Assume that this function is positive definite in the sense of (5).

The problem of optimal damping (OD) with respect to the functional (13) can be posed in the following form:

$$\begin{aligned} W = W(t,\mathbf{x},\mathbf{u}) \to \min_{\mathbf{u}\in U}, \\ \mathbf{u} = \mathbf{u}_d(t,\mathbf{x}) := \arg\min_{\mathbf{u}\in U} W(t,\mathbf{x},\mathbf{u}), \end{aligned} \qquad (15)$$

where the function W defines a rate of the functional L change in the motions of the plant (1) as follows:

$$\begin{aligned} W(t,\mathbf{x},\mathbf{u}) &:= \left.\frac{\mathrm{d}L}{\mathrm{d}t}\right|_{(1)} \\ &= \left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(1)} + F(t,\mathbf{x},\mathbf{u}) \\ &= \frac{\partial V(t,\mathbf{x})}{\partial t} + \frac{\partial V(t,\mathbf{x})}{\partial \mathbf{x}}\big[\mathbf{f}(t,\mathbf{x}) \\ &\quad + \mathbf{g}(t,\mathbf{x})\mathbf{u}\big] + \mathbf{F}(\mathbf{t},\mathbf{x},\mathbf{u}). \end{aligned} \qquad (16)$$

It is clear that the solution

$$\mathbf{u} = \mathbf{u}_d(t,\mathbf{x}) \qquad (17)$$

of the OD-problem (15) determines a feedback control (OD-controller) for the plant (1). Let us call the corresponding closed-loop system (1), (17), having a zero equilibrium, as a closed-loop OD-system.

Remark 2. The setting of the OD-problem for the functional (13) of a general form and for an arbitrary plant (1) is discussed in detail by Veremey and Sotnikova (2019).

The optimal damping concept is based on the following simple idea: the more rapidly the functional (13) decreases based on the motions of the closed-loop connection, the more significantly the process improves.

Let us particularly note that the computational scheme of the OD-problem solution is considerably simpler than the above presented dynamical programming scheme for the AQO-problem. The main advantage consists in the possibility to calculate the values of $\mathbf{u} = \mathbf{u}_d(t,\mathbf{x})$ numerically, directly using the pointwise minimization of the function $W(t,\mathbf{x},\mathbf{u})$ with the choice of $\mathbf{u} \in U$ for the current values of the variables $t, \mathbf{x}$.

The aforementioned advantage is the first reason for the OD-theory application to lessen computational load. The second reason is determined by a coincidence of the mentioned problem solutions under certain conditions, which are determined by the following result.

Theorem 1. *Let the function $\tilde{V}(t,\mathbf{x})$ be a solution to the HJB-equation (11) for the plant (1), and let there exist a unique optimal controller (12) with respect to the AQO-problem discussed above. Then this controller is simultaneously a solution to the OD-problem (15) with respect to the functional (13), where $V(t,\mathbf{x}) \equiv \tilde{V}(t,\mathbf{x})$, $F(t,\mathbf{x},\mathbf{u}) = \mathbf{x}^T\mathbf{Q}(t)\mathbf{x} + \mathbf{u}^T\mathbf{R}(t)\mathbf{u}$.*

Proof. This statement is a particular case of the corresponding Zubov (1962; 1966; 1978) theorem on a connection between the two problems. Nevertheless, let us prove the theorem, using its simple setting.

The functional (13) to be damped takes the form

$$\begin{aligned} L(t,\mathbf{x},\mathbf{u}) \equiv \tilde{V}(t,\mathbf{x}) + \int_{t_0}^{t} \big[\mathbf{x}^T(\tau)\mathbf{Q}(\tau)\mathbf{x}(\tau) \\ + \mathbf{u}^T(\tau)\mathbf{R}(\tau)\mathbf{u}(\tau)\big]\,\mathrm{d}\tau. \end{aligned} \qquad (18)$$

Then, in accordance with (16), we arrive at the equality

$$\begin{aligned} W(t,\mathbf{x},\mathbf{u}) &= \left.\frac{\mathrm{d}L}{\mathrm{d}t}\right|_{(1)} \\ &= \frac{\partial \tilde{V}(t,\mathbf{x})}{\partial t} + \frac{\partial \tilde{V}(t,\mathbf{x})}{\partial \mathbf{x}}\left[\mathbf{f}(t,\mathbf{x}) + \mathbf{g}(t,\mathbf{x})\mathbf{u}\right] \\ &\quad + \mathbf{x}^T\mathbf{Q}(t)\mathbf{x} + \mathbf{u}^T\mathbf{R}(t)\mathbf{u}. \end{aligned}$$

Assuming that the values of variables t and $\mathbf{x}$ are fixed, we obtain

$$\begin{aligned} \mathbf{u} = \mathbf{u}_d(t,\mathbf{x}) &:= \arg\min_{\mathbf{u}\in U} W(t,\mathbf{x},\mathbf{u}) \\ &= \arg\min_{\mathbf{u}\in U} \left\{ \frac{\partial \tilde{V}(t,\mathbf{x})}{\partial \mathbf{x}}\mathbf{g}(t,\mathbf{x})\mathbf{u} + \mathbf{u}^T\mathbf{R}(t)\mathbf{u} \right\}. \end{aligned}$$

From the last equality, by analogy with (9) and (10), we obtain the OD-optimal controller

$$\mathbf{u}_d(t,\mathbf{x}) = -\frac{1}{2}\mathbf{R}^{-1}(t)\mathbf{g}^T(t,\mathbf{x})\left(\frac{\partial \tilde{V}(t,\mathbf{x})}{\partial \mathbf{x}}\right)^T, \qquad (19)$$

which coincides with (12), i.e., $\mathbf{u}_d(t,\mathbf{x}) \equiv \mathbf{u}_0(t,\mathbf{x})$. ■

Let us particularly note that Theorem 1 formally reduces the AQO-problem to a solution of an essentially simpler OD-problem. However, it is natural that the direct utilization of such a transformation has no practical sense, since we need to have a solution $\tilde{V}(t,\mathbf{x})$ of the HJB-equation (11) to formulate the OD-problem; but the solution of the HJB-equation is the essence of the AQO-problem.

Nevertheless, this fundamental feature can be used both for theoretical investigations, and for construction of computational methods. For example, the aforementioned coincidence was applied by Zubov (1962; 1966; 1978) for the minimum-time problem solution carried out with the help of OD-theory.

Next, let us assume that the exact solution $\tilde{V}(t,\mathbf{x})$ of the HJB-equation (11) is not known, but we have some approximation $V(t,\mathbf{x})$ of this function. One of the most convenient variants of the choice of V is its representation by the positive definite quadratic form

$$V(t,\mathbf{x}) = \mathbf{x}^T\mathbf{P}(t)\mathbf{x} \tag{20}$$

with a symmetric matrix $\mathbf{P}(t)$.

Consider the OD-problem (15) with the functional

$$\begin{aligned} L(t,\mathbf{x},\mathbf{u}) \\ &= \mathbf{x}^T\mathbf{P}(t)\mathbf{x} + \int_{t_0}^{t} \big[\mathbf{x}^T(\tau)\mathbf{Q}(\tau)\mathbf{x}(\tau) \\ &\qquad + \mathbf{u}^T(\tau)\mathbf{R}(\tau)\mathbf{u}(\tau)\big]\,\mathrm{d}\tau \end{aligned} \tag{21}$$

to be damped. The rate function W for this case has the following form:

$$\begin{aligned} W &= W(t,\mathbf{x},\mathbf{u}) = \left.\frac{\mathrm{d}L}{\mathrm{d}t}\right|_{(1)} \qquad (22) \\ &= \left(\dot{\mathbf{x}}^T\mathbf{P}\mathbf{x} + \mathbf{x}^T\mathbf{P}\dot{\mathbf{x}} + \mathbf{x}^T\dot{\mathbf{P}}\mathbf{x}\right)_{(1)} + \mathbf{x}^T\mathbf{Q}\mathbf{x} \\ &\quad + \mathbf{u}^T\mathbf{R}\mathbf{u} = \mathbf{f}^T\mathbf{P}\mathbf{x} + \mathbf{x}^T\mathbf{P}\mathbf{f} + \mathbf{x}^T\dot{\mathbf{P}}\mathbf{x} \\ &\quad + \mathbf{2x}^T\mathbf{P}\mathbf{g}\mathbf{u} + \mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}. \end{aligned}$$

Supposing that the extreme is achieved at an interior point of U, we obtain with necessity

$$\begin{aligned} \frac{\mathrm{d}W(t,\mathbf{x},\mathbf{u})}{\mathrm{d}\mathbf{u}} &= 2\mathbf{u}^T\mathbf{R} + 2\mathbf{x}^T\mathbf{P}\mathbf{g} = 0 \\ &\Leftrightarrow \mathbf{R}\mathbf{u} + \mathbf{g}^T\mathbf{P}\mathbf{x} = \mathbf{0}, \end{aligned} \tag{23}$$

which gives us the following expression for the OD-optimal controller:

$$\mathbf{u} = \mathbf{u}_d(t,\mathbf{x}) = -\mathbf{R}^{-1}(t)\mathbf{g}^T(t,\mathbf{x})\mathbf{P}(t)\mathbf{x}. \tag{24}$$

Note that, in contrast to the exact solution (19), the controller (24) does not guarantee the stability of the origin for the closed-loop connection (1), (24). In other words, not for all matrices P and R this system is stable.

To discuss this question in detail, recall (Khalil, 2002; Slotine and Li, 1991; Hahn and Baartz, 1967) that, if the conditions (14) hold for some function $V(t,\mathbf{x})$, and if we have

$$\begin{aligned} \min_{\mathbf{u}\in U}\left[\frac{\mathrm{d}V(t,\mathbf{x},\mathbf{u})}{\mathrm{d}t}\right]_{(1)} + \alpha_3(\|\mathbf{x}\|) \le 0, \\ \forall t \ge t_0, \forall \mathbf{x} \in B_r, \end{aligned} \tag{25}$$

where $\alpha_3 \in K$, then this function is said to be a *local control Lyapunov function (local CLF) for the plant* (1).

It is known that if the CLF for the system exists, then this system is *uniformly asymptotically stabilizable (UGAS or UAS)* (Slotine and Li, 1991).

Evidently, any choice of the function V for the functional (13) to be damped should be treated as a choice of the Lyapunov function candidate. Moreover, the following statement gives a concrete expression for this choice.

Theorem 2. *Let the condition*

$$\begin{aligned} W_{d0}(t,\mathbf{x}) := W(t,\mathbf{x},\mathbf{u}_d(t,\mathbf{x})) \le -\alpha_4(\|\mathbf{x}\|), \\ \forall t \ge t_0, \forall \mathbf{x} \in B_r \end{aligned} \tag{26}$$

hold for the feedback control (24), where $\alpha_4 \in K$. Then the function $V(t,\mathbf{x}) = \mathbf{x}^T\mathbf{P}(t)\mathbf{x}$ is the CLF for the plant (1), and the zero equilibrium for the closed-loop system (1), (24) is locally uniformly asymptotically stable, i.e., the feedback (24) serves as a stabilizing controller for the plant (1).

Proof. Assume that the condition (26) holds for controller (24), i.e., we have

$$\begin{aligned} &W(t,\mathbf{x},\mathbf{u}_d(t,\mathbf{x})) \\ &\quad = \min_{\mathbf{u}\in U} W(t,\mathbf{x},\mathbf{u}) \\ &\quad = \min_{\mathbf{u}\in U}\left[\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(1)} + \mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right] \qquad (27) \\ &\quad \ge \min_{\mathbf{u}\in U}\left[\frac{\mathrm{d}V(t,\mathbf{x},\mathbf{u})}{\mathrm{d}t}\right]_{(1)} \\ &\qquad + \min_{\mathbf{u}\in U}\left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right) \\ &\quad \le -\alpha_4(\|\mathbf{x}\|). \end{aligned}$$

Since $-(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}) \le 0$, in accordance with (27), we obtain

$$\begin{aligned} &\min_{\mathbf{u}\in U}\left[\frac{\mathrm{d}V(t,\mathbf{x},\mathbf{u})}{\mathrm{d}t}\right]_{(1)} \\ &\quad \le -\alpha_4(\|\mathbf{x}\|) - \min_{\mathbf{u}\in U}\left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right) \\ &\quad \le -\alpha_4(\|\mathbf{x}\|), \end{aligned}$$

which corresponds to the definition of the local CLF V for the plant (1).

We can additionally note that, in accordance with (22) and based on (26), we have

$$\begin{aligned}\tilde{W}_d(t,\mathbf{x}) &:= \min_{\mathbf{u}\in U}\left[\frac{\mathrm{d}V(t,\mathbf{x},\mathbf{u})}{\mathrm{d}t}\right]_{(1),\,\mathbf{u}=\mathbf{u}_d(t,\mathbf{x})} \\ &= \mathbf{f}^T\mathbf{P}\mathbf{x} + \mathbf{x}^T\mathbf{P}\mathbf{f} \\ &\quad + \mathbf{x}^T\dot{\mathbf{P}}\mathbf{x} + \mathbf{2}\mathbf{x}^T\mathbf{P}\mathbf{g}\mathbf{u}_d \\ &\le -\alpha_4(\|\mathbf{x}\|) - (\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}_d^T\mathbf{P}\mathbf{u}_d) \\ &\le -\alpha_4(\|\mathbf{x}\|).\end{aligned}$$

This means (Slotine and Li, 1991) that the zero equilibrium point of the closed-loop system (1), (24) is UAS, i.e., (24) is a stabilizing controller for the plant (1). ∎

Remark 3. If all the mentioned conditions of Theorem 2 are fulfilled for the whole space, i.e., if $B_r = E^n$, $U = E^m$, and if all the aforementioned above functions α_i, $i = \overline{1,4}$, belong to the class K_∞, then the zero equilibrium point for the closed-loop system is globally uniformly asymptotically stable (UGAS) (Khalil, 2002; Slotine and Li, 1991).

4. Approximate optimal control design

As can be seen from the reasoning presented above, the choice of the matrices $\mathbf{Q}$ and $\mathbf{R}$ in (6) for the AQO-problem uniquely determines the function $V = \tilde{V}$ as a solution to the HJB-equation (11). If this function is used together with the function $F(t,\mathbf{x},\mathbf{u}) \equiv \mathbf{x}^T\mathbf{Q}(t)\mathbf{x} + \mathbf{u}^T\mathbf{R}(t)\mathbf{u}$ for the functional (13) in the OD-problem (15), then the OD-controller $\mathbf{u} = \mathbf{u}_d(t,\mathbf{x})$ provides the same optimal value $J = J_0$ as the AQO-controller $\mathbf{u} = \mathbf{u}_0(t,\mathbf{x})$.

However, if some function $V \not\equiv \tilde{V}$ is used in (13) instead of $\tilde{V}(t,\mathbf{x})$, keeping the identity $F(t,\mathbf{x},\mathbf{u}) \equiv \mathbf{x}^T\mathbf{Q}(t)\mathbf{x} + \mathbf{u}^T\mathbf{R}(t)\mathbf{u}$, then the corresponding OD-controller (24) is not a solution to the AQO-problem, i.e., this controller provides a value $J \ge J_0$ of the performance index (6).

In that case, solving the OD-problem (15) for different functions V, one could ask a question about the choice of the function V to approximate the HJB-solution $\tilde{V}(t,\mathbf{x})$ in the best way. Here, the OD-problem can be treated as an instrument for dragging the function V to the mentioned optimal solution $\tilde{V}$, yielding the convergence $J \to J_0$.

As noted above, this idea is justified only for a situation when a direct AQO-problem solution is connected with the above-mentioned difficulties: either solution of the PDE (11) is too expensive computationally, or the AQO-controller (12) is not convenient for a practical realization.

The aforementioned circumstances motivate us to proceed to an approximate solution of the AQO-problem, which can be put into practice using the OD-concept. In general, this allows reducing computational consumptions, and simplifying the control law.

To construct a specialized method of synthesis, consider the AQO-problem (7) with the functional (6), which is defined on the trajectories of the closed-loop system (1), (2).

As mentioned above, this problem is equivalent to the OD-problem of the form (15) with respect to the functional (18), where the function $\tilde{V}(t,\mathbf{x})$ is a solution to the HJB-equation (11).

The choice of the aforementioned approximation can be realized as a solution to the corresponding OD-problem. For this purpose, let us consider the space $\Re_0$ of continuously differentiable functions $V(t,\mathbf{x})$ satisfying the conditions (14).

Given the function $V^*(t,\mathbf{x})$, solve the OD-problem (15), deriving the OD-controller $\mathbf{u}_d^*(t,\mathbf{x}) := \mathbf{u}_d(t,\mathbf{x},V^*)$. Since this controller is not AQO-optimal, we obtain

$$\begin{aligned}J^* &:= J(V^*) \\ &:= J\left(\mathbf{u}_d(t,\mathbf{x},V^*)\right) \ge J\left(\mathbf{u}_0(t,\mathbf{x})\right) = J_0.\end{aligned} \tag{28}$$

If the obtained controller satisfies the conditions of Theorem 2 and if

$$\Delta J = \frac{J^* - J_0}{J_0} \le \varepsilon_J \tag{29}$$

for a given value ε_J of the admissible functional J degradation, then the controller $\mathbf{u} = \mathbf{u}_d^*(t,\mathbf{x})$ can be accepted as an approximate solution for the problem (15). Note that its approximation quality is interpreted as in (29).

Continuing this line of reasoning, it is natural to apply an optimization approach to construct the function V^*. Actually, formulate the minimization problem

$$J = J(V) := J(\mathbf{u}_d(t,\mathbf{x},V)) \to \inf_{V\in\Re_0}, \tag{30}$$

such that numerical methods of its solution generate the minimizing sequence $\{V_k(t,\mathbf{x})\} \in \Re_0$. It is obvious that the above-mentioned function V^* should be searched among the elements of the specified sequence.

Consider the set of positive definite quadratic forms (20) as $\Re_0$: it is known that all the functions $V(t,\mathbf{x}) = \mathbf{x}^T\mathbf{P}(t)\mathbf{x}$ satisfy the conditions (14) for any symmetrical matrix $\mathbf{P} \ge \mathbf{0}$.

In many practical situations, it is convenient to narrow this set by introducing its certain parameterization. To this end, one should fix a structure of the functions V, extracting the vector $\mathbf{h} \in E^p$ of parameters to be varied: $V = V(t, \mathbf{x}, \mathbf{h})$. It is supposed that parameterization is done such that

$$\mathbf{h} \in H_v \subset E^p \Rightarrow V(t, \mathbf{x}, \mathbf{h}) \in \Re_0, \tag{31}$$

where H_v is a compact set.

Note that there are no formalized suggestions for accepting the vector $\mathbf{h} \in E^p$. However, in most cases, this vector should include diagonal components of the matrix $\mathbf{P}$ (possibly, their common multiplier), which significantly affect the magnitude of the components of the control signals. This allows us to influence energy consumptions in the control processes.

By analogy with (30), it is now possible to pose the optimization problem so that its solution with respect to $\mathbf{h}$ results in an approximate optimal controller.

Given the initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0 \in B_r$ for the plant (1), consider the following computational steps

Step 1. Set the vector $\mathbf{h} \in H_v \subset E^p$ of tunable parameters.

Step 2. Specify the function $V(t, \mathbf{x}, \mathbf{h}) = \mathbf{x}^T\mathbf{P}(t, \mathbf{h})\mathbf{x}$.

Step 3. Solve the OD-problem with the following functional to be damped:

$$L(t, \mathbf{x}, \mathbf{u}, \mathbf{h}) = \mathbf{x}^T\mathbf{P}(t, \mathbf{h})\mathbf{x} + \int_{t_0}^{t} \big[\mathbf{x}^T(\tau)\mathbf{Q}(\tau)\mathbf{x}(\tau) + \mathbf{u}^T(\tau)\mathbf{R}(\tau)\mathbf{u}(\tau)\big]\,\mathrm{d}\tau; \tag{32}$$

this gives the OD-controller

$$\mathbf{u} = \mathbf{u}_d(t, \mathbf{x}, \mathbf{h}) = -\mathbf{R}^{-1}(t)\mathbf{g}^T(t, \mathbf{x})\mathbf{P}(t, \mathbf{h})\mathbf{x}.$$

Step 4. From the equations of the closed-loop system,

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}) + \mathbf{g}(t, \mathbf{x})\mathbf{u}_d(t, \mathbf{x}, \mathbf{h}). \tag{33}$$

Step 5. Solve the Cauchy problem for the system (33) with the given initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0$ that result in the motion $\mathbf{x}_d(t, \mathbf{h})$.

Step 6. Specify the function

$$\tilde{\mathbf{u}}_d(t, \mathbf{h}) = \mathbf{u}_d(t, \mathbf{x}_d(t, \mathbf{h}), \mathbf{h}).$$

Step 7. Calculate the value of the function $J_d(\mathbf{h})$ determined by the expression

$$\begin{aligned} J_d &= J_d(\mathbf{h}) \\ &= \int_{t_0}^{\infty} \big(\mathbf{x}_\mathbf{d}^T(\mathbf{t}, \mathbf{h})\mathbf{Q}(t)\mathbf{x}_\mathbf{d}(\mathbf{t}, \mathbf{h}) \\ &\quad + \tilde{\mathbf{u}}_\mathbf{d}^T(\mathbf{t}, \mathbf{h})\mathbf{R}(t)\tilde{\mathbf{u}}_\mathbf{d}(\mathbf{t}, \mathbf{h})\big)\ \mathrm{d}t. \end{aligned} \tag{34}$$

Step 8. Minimize the function $J_d(\mathbf{h})$ on the set H_v, i.e., solve the problem

$$J_d = J_d(\mathbf{h}) \to \min_{\mathbf{h} \in H_v}, \quad \mathbf{h}_d := \arg\min_{\mathbf{h} \in H_v} J_d(\mathbf{h}), \tag{35}$$
$$J_{d0} := J_d(\mathbf{h_d}),$$

repeating Steps 1–7 of this scheme.

The solution $\mathbf{h} = \mathbf{h}_d$ of the problem (35) allows us to construct the following approximation of the Bellman function:

$$V^*(t, \mathbf{x}) = \mathbf{x}^T\mathbf{P}(t, \mathbf{h}_d)\mathbf{x}. \tag{36}$$

Correspondingly, the control law

$$\begin{aligned} \mathbf{u} &= \mathbf{u}_d(t, \mathbf{x}, \mathbf{h}_d) \\ &= -\mathbf{R}^{-1}(t)\mathbf{g}^T(t, \mathbf{x})\mathbf{P}(t, \mathbf{h_d})\mathbf{x} \end{aligned} \tag{37}$$

represents the approximate optimal controller for the initial AQO-problem.

If the optimal value J_0 is known, one can estimate the following measure of the functional J degradation due to the approximate solution

$$\Delta J = \frac{J_{d0} - J_0}{J_0} \le \varepsilon_J. \tag{38}$$

Let us formulate certain recommendations for a practical solution of the optimization problem (35). If the dimension of the vector $\mathbf{h}$ is not large, it is possible to enumerate possibilities on the finite net $H_{fv} \subset H_v$. However, in any case, any modern method can be applied to provide finite-dimensional minimization.

To consider the computational advantages of the presented scheme in comparison with the existing methods, note that dynamic programming and the model predictive control (MPC) can be interpreted as nearest alternative directions. Both of them are quite popular when considering the AOC-problem.

As noted above, the exact implementation of the first direction is supposed to be either unacceptable or not convenient for practical use. To solve the HJB-equation (11), it is necessary to use approximate numerical methods with a significant amount of computations. The proposed computational scheme is free from these drawbacks. The most complicated computational operation here is to solve the finite-dimensional problem (35). The situation is simplified cardinally if tunable vectors $\mathbf{h}$ are of small dimensions, where we can use an enumeration of possibilities on a finite net.

As for the MPC-approach, its most significant disadvantage is the large dimension of the minimization problem that is solved at each step of the control process. The larger the prediction horizon and the higher the accuracy, the greater this dimension. There is no aforementioned difficulty in the framework of the OD-approach. This determines the advantage of the proposed scheme.

Publications using approximate methods to solve the HJB-equation hold a special position. For example, Lukes (1969) uses a power series representation of a solution. Nevertheless, the approximation, which is directly based on the value of the functional, seems to be more natural and effective in comparison with using the direct approximate solution of the HJB-equation (11).

5. Practical example of approximate synthesis

Based on the results of the approximate synthesis presented above, consider a specification of the proposed approach to design control laws of a convey-crane, which transports a load suspended from a cart while minimizing the oscillations of the load. The mathematical model of the crane and the examples of the traditional control laws are taken after Fantoni and Lozano (2002) as well as Collado *et al*. (2000). A more detailed model of the convey-crane system is presented by Aguilar-Ibanez and Suarez-Castanon (2019).

As a controlled plant, consider the convey-crane system with the simplified scheme presented in Fig. 1. This system consists of the cart with mass M of the pendulum and the load of the crane with mass m. The angle between the pendulum and the vertical axis will be denoted by θ, while the parameter l represents the length of the rod.

Using the notation

$$\begin{aligned}
\mathbf{q} &:= \begin{pmatrix} x \\ \theta \end{pmatrix}, \\
\mathbf{M}(\mathbf{q}) &:= \begin{pmatrix} M+m & -ml\cos\theta \\ -ml\cos\theta & ml^2 \end{pmatrix}, \\
\mathbf{C}(\mathbf{q},\dot{\mathbf{q}}) &:= \begin{pmatrix} 0 & ml\sin\theta\dot{\theta} \\ 0 & 0 \end{pmatrix}, \\
\mathbf{G}(\mathbf{q}) &:= \begin{pmatrix} 0 \\ mgl\sin\theta \end{pmatrix}, \\
\tau &:= \begin{pmatrix} u \\ 0 \end{pmatrix},
\end{aligned}$$

we can present the system dynamics with the equation

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q}) = \tau, \tag{39}$$

where x is the longitudinal displacement of the cart, g is the acceleration due to gravity, u is the control action, defined by the longitudinal force applied to the cart. Here matrix $\mathbf{M}(\mathbf{q})$ is symmetric and positive definite. It is obvious that Eqn. (39) can be converted to

$$\ddot{\mathbf{q}} = \mathbf{M}^{-1}(\mathbf{q})\left[-\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} - \mathbf{G}(\mathbf{q}) + \tau\right], \tag{40}$$

where

$$\mathbf{M}^{-1}(\mathbf{q}) = \frac{1}{\Delta_M(\mathbf{q})}\begin{pmatrix} ml^2 & ml\cos\theta \\ ml\cos\theta & M+m \end{pmatrix},$$

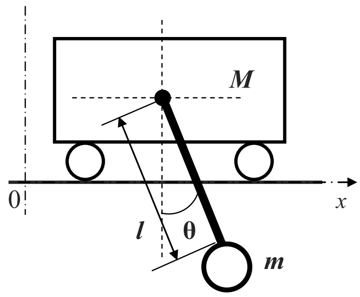

Fig. 1. Convey-crane simplified scheme.

$$\begin{aligned}
\Delta_M(\mathbf{q}) &:= \det(\mathbf{M}^{-1}(\mathbf{q})) \\
&= Mml^2 + m^2l^2\sin^2\theta > 0.
\end{aligned}$$

Next, Eqn. (40) can be rewritten in the following scalar form:

$$\ddot{x} = \frac{1}{M+m\sin^2\theta}\left[u - m\sin\theta(l\dot{\theta}^2 + g\cos\theta)\right], \tag{41}$$

$$\begin{aligned}
\ddot{\theta} = &\frac{1}{l(M+m\sin^2\theta)} \\
&\times\Big[-\sin\theta\left((M+m)g + ml\cos\theta\dot{\theta}^2\right) \\
&+ \cos\theta u\Big].
\end{aligned}$$

Introducing the new notation

$$x_1 := \dot{x}, \quad x_2 := x, \quad x_3 := \dot{\theta}, \quad x_4 = \theta,$$

$$\mathbf{x} = (x_1 \quad x_2 \quad x_3 \quad x_4)^T \in E^4,$$

we can present (41) as a mathematical model of the following autonomous affine control system:

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) + \mathbf{f}_0(\mathbf{x})u, \tag{42}$$

where

$$\mathbf{f}_0(\mathbf{x}) = \begin{pmatrix} \dfrac{-m\sin x_4\left(lx_3^2 + g\cos x_4\right)}{M+m\sin^2 x_4} \\ x_1 \\ \dfrac{-\sin x_4\left((M+m)g + ml\cos x_4 x_3^2\right)}{l(M+m\sin^2 x_4)} \\ x_3 \end{pmatrix},$$

$$\mathbf{g}_0(\mathbf{x}) = \begin{pmatrix} \dfrac{1}{M+m\sin^2 x_4} \\ 0 \\ \dfrac{1}{l(M+m\sin^2 x_4)} \\ 0 \end{pmatrix}.$$

For a numerical example, set the following values of the parameters:

$$M = 1 \text{ kg}, \quad m = 1 \text{ kg}, \quad l = 1 \text{ m}, \quad g = 9.8 \text{ m/s}^2.$$

Next, let us introduce the integral quadratic functional

$$J = \int_0^\infty \left(\mathbf{x}^T \mathbf{Q}_0 \mathbf{x} + r_0^2 u^2\right) \, \mathrm{d}t, \tag{43}$$

which is given on the trajectories of the controlled plant (42), where

$$\mathbf{Q}_0 := \begin{pmatrix} 0.350 & 0 & 0 & 0 \\ 0 & 0.200 & 0 & 0 \\ 0 & 0 & 1.65 & 0 \\ 0 & 0 & 0 & 1.90 \end{pmatrix}, \quad r_0^2 = 0.01.$$

Given Eqn. (42) and the functional (43), it is possible to formulate the following affine-quadratic optimization problem:

$$J(u(\cdot)) \to \min_{u \in E^1}, \quad u_{c0}(\mathbf{x}) = \arg\min_{u \in E^1} J(u(\cdot)), \tag{44}$$
$$J_0 := J(u_{c0}(\cdot))$$

to synthesize the nonlinear autonomous controller

$$u = u_{c0}(\mathbf{x}), \tag{45}$$

which stabilizes the plant (42) and minimizes the functional (43).

From the physical point of view, this AQO-problem corresponds to the synthesis of the controller returning the cart and the pendulum into a zero equilibrium position with the maximum suppression of oscillations.

Believing that a direct solution to this problem is too complicated, resort to OD-theory in accordance with the proposed approach.

The OD-problem associated with (44) can be posed as follows:

$$W = W(\mathbf{x}, u) \to \min_{u \in E^1}, \tag{46}$$
$$u = u_d(\mathbf{x}) := \arg\min_{u \in E^1} W(\mathbf{x}, u),$$

$$W = W(\mathbf{x}, u) := \left.\frac{\mathrm{d}L}{\mathrm{d}t}\right|_{(42)}, \tag{47}$$

$$L = L(t, \mathbf{x}, u) = V(\mathbf{x}) + \int_0^t \left(\mathbf{x}^T \mathbf{Q}_0 \mathbf{x} + r_0^2 u^2\right) \, \mathrm{d}\tau, \tag{48}$$

$$V = V(\mathbf{x}, \mathbf{h}) = \mathbf{x}^T \mathbf{P}(\mathbf{h}) \mathbf{x}. \tag{49}$$

Here, the matrix $P(\mathbf{h})$ is positive definite for any vector $\mathbf{h} \in H_v \subset E^p$, where H_v is a compact set.

It is a matter of simple calculations to verify that the OD-controller for problem (46) has the following form:

$$u = u_d(\mathbf{x}, \mathbf{h}) = -\frac{1}{r_0^2} \mathbf{g}_0^T(\mathbf{x}) \mathbf{P}(\mathbf{h}) \mathbf{x}. \tag{50}$$

To specify the matrix $P(\mathbf{h})$, apply a special procedure.

First, it is possible to linearize the system (42) in the neighborhood of the origin. As a result, we obtain

$$\dot{\mathbf{x}} = \mathbf{A}_m \mathbf{x} + \mathbf{b}_m u, \tag{51}$$

where

$$\mathbf{f}_0(\mathbf{x}) = \begin{pmatrix} 0 & 0 & 0 & -mg/M \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(M+m)g/(lM) \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

$$\mathbf{b}_m = \begin{pmatrix} 1/M \\ 0 \\ 1/(lM) \\ 0 \end{pmatrix}.$$

Based on the system (51), consider the following algebraic Riccati equation:

$$\mathbf{S}\mathbf{A}_m + \mathbf{A}_m^T \mathbf{S} - \mathbf{S}\mathbf{b}_m r_m^{-1} \mathbf{b}_m^T \mathbf{S} + \mathbf{Q}_m = 0, \tag{52}$$

where

$$\mathbf{Q}_m := \begin{pmatrix} 0.350 & 0 & 0 & 0 \\ 0 & 0.200 & 0 & 0 \\ 0 & 0 & h_2^2 & 0 \\ 0 & 0 & 0 & 1.90 \end{pmatrix}, \quad r_m = h_1^2.$$

Note that the selection of the parameter values was realized by the trial and error approach. In this case, the parameter h_1 affects the rate of the angle theta change, and the parameter h_2 affects the intensity of the control action.

The solution $\mathbf{S} = \mathbf{S}(\mathbf{h})$ of this equation is a function of the vector $\mathbf{h} = (h_1 \quad h_2)^T \in E^2$. Note that the matrix $\mathbf{S}(\mathbf{h})$ is symmetric and positive definite: this allows us to a set $\mathbf{P}(\mathbf{h}) \equiv \mathbf{S}(\mathbf{h})$, i.e., the OD-controller (50) takes the form

$$u = u_d(\mathbf{x}, \mathbf{h}) = -\frac{1}{r_0^2} \mathbf{g}_0^T(\mathbf{x}) \mathbf{S}(\mathbf{h}) \mathbf{x}. \tag{53}$$

Let us also introduce an admissible box

$$H_v = \big\{\mathbf{h} = (h_1 \quad h_2)^T \in E^2 : \\ 0.100 \le h_1 \le 0.140, \\ 1.00 \le h_2 \le 1.80\big\}, \tag{54}$$

for the vector $\mathbf{h}$ of tunable parameters.

Next, it is possible to specify the function

$$J_d = J_d(\mathbf{h}) = \int_0^\infty \left(\mathbf{x}^T(t,\mathbf{h})\mathbf{Q}_0\mathbf{x}(t,\mathbf{h}) + r_0^2 u^2(t,\mathbf{h})\right)\,\mathrm{d}t, \tag{55}$$

where $\mathbf{x}(t,\mathbf{h})$ and $u(t,\mathbf{h})$ represent the motion of the closed-loop connection (42), (53) under the initial conditions $\mathbf{x}(0) = \mathbf{x}_0 = (0 \quad -5 \quad 0 \quad -\pi/4)^T$. Note that these conditions are taken after Fantoni and Lozano (2002).

Solving the following optimization problem:

$$J_d = J_d(\mathbf{h}) \to \min_{\mathbf{h}\in H_v}, \quad \mathbf{h}_d := \arg\min_{\mathbf{h}\in H_v} J_d(\mathbf{h}), \quad J_{d0} := J_d(\mathbf{h_d}), \tag{56}$$

we obtain the best vector $\mathbf{h}_d = (0.130 \quad 1.55)^T$ of tunable parameters. A solution to the problem (56) is found by minimizing on a finite net covering the admissible box H_v. The obtained vector determines the approximate optimal controller for initial AQO-problem as follows:

$$u = u_{d0}(\mathbf{x}) \tag{57}$$
$$:= u_d(\mathbf{x},\mathbf{h}_d) \tag{58}$$
$$= -\frac{1}{r_0^2}\mathbf{g}_0^T(\mathbf{x})\mathbf{S}(\mathbf{h}_d)\mathbf{x}$$

with the matrix

$$\mathbf{S}(\mathbf{h}_d) = \begin{pmatrix} 0.625 & 0.499 & -0.473 & 1.31 \\ 0.499 & 0.971 & -0.390 & 0.665 \\ -0.473 & -0.390 & 0.577 & -0.884 \\ 1.31 & 0.665 & -0.884 & 9.28 \end{pmatrix}.$$

Figure 2 represents the transient process for the closed-loop system (42), (57) under the aforementioned initial conditions $\mathbf{x}_0$, providing the value $J_{d0} := J_d(\mathbf{h_d}) = 28.0$ of the minimized functional (43).

For comparison, Fig. 3 shows a similar process provided by the controller

$$u = u_f(\mathbf{x}) = (-4.30 \quad -3.00 \quad 0 \quad 0)\mathbf{x}, \tag{59}$$

which was proposed by Fantoni and Lozano (2002): this controller gives the value $J_f = 42.9$ for the functional (43). Note that for the presented example the energetic costs for both controllers are almost identical.

One can easily see that the OD-controller (57) provides a much higher quality for the control process than the PD-controller (59). This can be explained by the fact that the energy approach used by Fantoni and Lozano (2002) is initially focused on ensuring stability, but not on control quality.

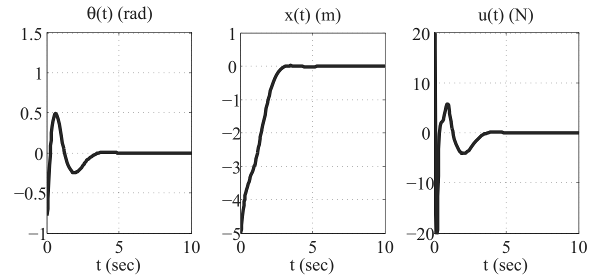

Fig. 2. Transient process for the controller (57).

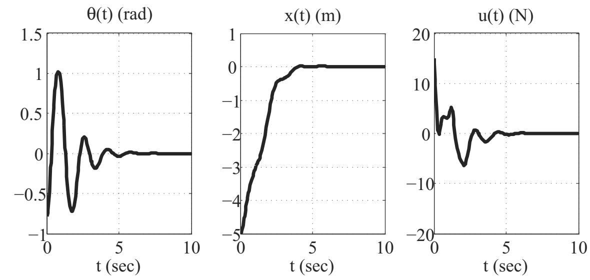

Fig. 3. Transient process for the controller (59).

6. Conclusions

This work aimed at discussing some questions connected with stabilizing controller synthesis for affine-control nonlinear and nonautonomous plants. Such dynamic plants are widely used in numerous areas of human activities: as a rule, these plants have controlled inputs and require the use of automatic controllers.

One of the commonly used approaches to design control laws for the affine plant is based on the AQO-problem, which consists of integral-quadratic functional minimization. The well-known method to find the corresponding optimal controller is based on Bellman's principle. Nevertheless, its application can be essentially hampered by the large computational load connected with a solution of the HJB-equations and with a practical realization of the optimal controllers.

As an alternative, it is proposed to employ a different approach based on Zubov's (1962; 1966; 1978) concept of optimal damping. It is shown that the exact solutions to the aforementioned problems coincide under certain conditions. This coincidence allows us to construct the method of an approximate solution to the AQO-problem, which can be implemented using OD-theory, which reduces the computational load and simplifies the control law.

In the framework of the proposed method, an approximate optimal controller is designed as the OD-controller with respect to the specific functional to be damped. The choice of this functional can be realized as a solution to the special auxiliary optimization problem.

Finally, applicability and effectiveness of the proposed method is illustrated with a practical example of a controller design for a convey-crane system, which transports a load suspended from a cart minimizing the oscillations of the load.

The results of the above research could be expanded to include robust features of the optimal damping controller, and to take into account transport delays in both the input and the output of a controlled plant. It is intended to apply the obtained results for studies which are intensively carried out in the range of multipurpose control of marine vehicles (Vermey, 2019; 2017; Sotnikova and Veremey, 2013; Vermey and Sotnikova, 2019). Special attention is supposed to be paid to reinforcement learning methods.

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