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AN INTERVAL KALMAN FILTER ENHANCED BY LOWERING THE COVARIANCE MATRIX UPPER BOUND

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This paper proposes a variance upper bound based interval Kalman filter that enhances the interval Kalman filter based on the same principle proposed by Tran *et al*. (2017) for uncertain discrete time linear models. The systems under consideration are subject to bounded parameter uncertainties not only in the state and observation matrices, but also in the covariance matrices of the Gaussian noises. By using the spectral decomposition of a symmetric matrix and by optimizing the gain matrix of the proposed filter, we lower the minimal upper bound on the state estimation error covariance for all admissible uncertainties. This paper contributes with an improved algorithm that provides a less conservative error covariance upper bound than the approach proposed by Tran *et al*. (2017). The state estimates are determined using interval analysis in order to enclose the set of all possible solutions of the classical Kalman filter consistent with the uncertainties.

Keywords: uncertain linear systems, Kalman filter, interval analysis, estimation, covariance matrix.

1. Introduction

Set-membership (SM) methods have expanded considerably in recent years and they have reached a maturity allowing consideration of many applications (Jaulin *et al.*, 2001a; 2001b; Ribot *et al.*, 2007; Tran, 2017; Cayero *et al.*, 2019). The past years have seen considerable progress in the way of formulating problems in this context as well as optimized handling various types of sets. Interval analysis, introduced by Moore (1966), operates on box-shaped sets and is particularly relevant for dealing with nonlinear systems. It has been used for nonlinear estimation, and several algorithms have been proposed (for more details, see Jaulin *et al.*, 2001b; Ribot *et al.*, 2007; Kieffer *et al.*, 1999). Other estimation approaches dedicated to linear models include ellipsoid shaped methods (Lesecq *et al*., 2003), or parallelotope and zonotope based methods (Ingimundarson *et al*., 2009).

In contrast to stochastic estimation approaches (see, e.g., Chabir *et al*., 2018), SM estimation advantageously provides guaranteed results meaning that the obtained sets are guaranteed to include all the solutions consistent with the bounded uncertainties. However, SM estimation does not bring any information about the probability distribution inside the sets and it is often criticized for the overestimation of the results.

This paper is motivated by the idea that stochastic and SM approaches have specific advantages and that they complement each other more than they compete.

In the stochastic estimation framework, the experimental conditions about noise and disturbances are usually properly modeled through appropriate probability distributions. However, other sources of uncertainty are not well-suited to stochastic modeling and are better represented with bounded uncertainties. This is the case with parameter uncertainties that generally arise from design tolerances and from ageing. Therefore, combining stochastic and bounded uncertainties opens new perspectives for modeling complex systems more accurately.

Motivated by the above facts, we consider the filtering problem for discrete time linear models with bounded parameter uncertainties and Gaussian measurement noise. In the work of Chen *et al*. (1997), the classical Kalman filter (Kalman, 1960) has been extended

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to this type of uncertain systems. The authors propose to bypass a singularity problem by using the upper bound on the interval matrix to be inverted. This method hence leads to a solution that is not guaranteed, i.e., the solution set may not include all the classical Kalman filter solutions consistent with the bounded uncertainties represented in the system. In the work of Xiong *et al.* (2013), an improved interval Kalman filter (iIKF) has been proposed that solves the interval matrix inversion problem with the set inversion algorithm called SIVIA (set inversion via interval analysis) and constraint satisfaction problems (CSPs) (Jaulin *et al.*, 2001b). Nevertheless, this algorithm demands high computational time if there exist large uncertainties affecting the considered system (Tran *et al.*, 2016). The minimum upper bound of variance interval Kalman filter (UBIKF) has been presented by Tran *et al.* (2017) with two main goals: minimizing the upper bound on the estimation error covariance and enclosing the set of possible solutions of the filtering problem for interval linear systems. Since the gain matrix handled by the UBIKF is a point matrix, this approach encloses all the estimates consistent with the parameter uncertainties in a much less conservative manner than the iIKF.

Our contribution consists in proposing an improved minimum upper bound for the variance interval Kalman filter (iUBIKF) using the spectral decomposition of a symmetric matrix that provides a less conservative error covariance upper bound than the UBIKF. The iUBIKF also provides interval estimates that are guaranteed to enclose all the optimal estimates consistent with the parameter uncertainties. In this respect, the iUBIKF differs from the joint zonotopic and Gaussian Kalman filter proposed by Combastel (2015) for discrete linear time varying (LTV) systems simultaneously subject to bounded disturbances and Gaussian noises. This latter indeed uses a criterion combining the minimization of the estimate variance and the minimization of the size of the zonotope bounding the support of the estimate bounded imprecision.

This paper is organized as follows. The problem is formulated in Section 2. Sections 3 and 4 review the main notions of interval analysis and matrix inequalities that are necessary for the development of the new algorithm. Then the new interval Kalman filter is derived in Section 5, followed by the application of the two filters, the UBIKF and the iUBIKF, to an academic numerical example and to a case study of a two-wheeled vehicle in Section 6. In that section a comparative analysis is made. Section 7 concludes the paper.

2. Problem formulation

We consider linear discrete-time dynamic systems represented by a state and an observation equation subject to noises similar to the standard Kalman model (Kalman, 1960):

$$\begin{cases} \mathbf{x}_{k+1} = A_k \mathbf{x}_k + B_k \mathbf{u}_k + \mathbf{w}_k, \\ \mathbf{y}_k = C_k \mathbf{x}_k + \mathbf{v}_k, \quad k \in \mathbb{N}, \end{cases} \tag{1}$$

where $\mathbf{x}_k \in \mathbb{R}^{n_x}$, $\mathbf{y}_k \in \mathbb{R}^{n_y}$ and $\mathbf{u}_k \in \mathbb{R}^{n_u}$ denote state, measurement and input vectors, respectively. The matrices A_k, B_k, C_k are time-varying parameters, while $\{\mathbf{w}_k\}$ and $\{\mathbf{v}_k\}$ are independent centered Gaussian white noise sequences, with positive definite covariance matrices Q_k and R_k:

$$\begin{aligned} E\{\mathbf{w}_k, \mathbf{w}_l\} &= Q_k \delta_{kl}, \\ E\{\mathbf{v}_k, \mathbf{v}_l\} &= R_k \delta_{kl}, \\ E\{\mathbf{w}_k, \mathbf{v}_l\} &= E\{\mathbf{w}_k, \mathbf{x}_0\} \\ &= E\{\mathbf{v}_k, \mathbf{x}_0\} = 0, \end{aligned}$$

$\forall (k,l) \in \mathbb{N}^2$, where δ_{kl} is the Kronecker symbol.

Based on the motivations reported in Introduction, we propose to combine two modeling paradigms: measurement and system noises are modeled in a stochastic framework but parameters are assumed uncertain and this uncertainty is bounded. This is achieved by assuming that the matrices A_k, B_k, C_k, Q_k and R_k of (1) are *interval matrices*, as defined in the following section, containing all possible values of each parameter. Since it is impossible to solve directly the Kalman filtering problem due to parameter uncertainties, our goal is to obtain an upper bound $\mathcal{P}_k^+$ such that

$$E\left[(\hat{\mathbf{x}}_k^+ - \mathbf{x}_k)(\hat{\mathbf{x}}_k^+ - \mathbf{x}_k)^T\right] \preceq \mathcal{P}_k^+ \tag{2}$$

for the set of all models with parameters bounded by the interval matrices. The envelope enclosing the set of state estimates corresponding to the gain K is then computed.

In the next section, the basic concepts of interval analysis are introduced.

3. Interval analysis

Interval analysis operates on intervals instead of real numbers (Moore, 1959; Jaulin *et al.*, 2001b). Obtaining guaranteed results from floating point algorithms was the first motivation. It was then extended to verified numerics (Moore, 1966).

A *guaranteed result* first means that the result set encloses the exact solution. Second, it also means that the algorithm is able to decide on the existence of a solution in a finite time or a finite number of iterations.

3.1. Main concepts. A real interval $[p] = [\underline{p}, \overline{p}]$ is a closed and connected subset of $\mathbb{R}$, where $\underline{p}$ and $\overline{p}$ represent the lower and upper bounds of $[p]$, respectively. The *radius* of an interval $[p]$ is defined by $\mathrm{rad}([p]) = (\overline{p} - \underline{p})/2$, and its *midpoint* by $\mathrm{mid}([p]) = (\overline{p} + \underline{p})/2$. If $w([p]) = 0$, then $[p]$ is degenerated and reduced to a real number. The set of all real intervals of $\mathbb{R}$ is denoted by $\mathbb{IR}$. Real arithmetic operations have been extended to intervals (Moore, 1966):

$$\circ \in \{+, -, *, /\},$$
$$[p_1] \circ [p_2] = \{x \circ y \mid x \in [p_1],\ y \in [p_2]\}.$$

The following property is useful to describe a quantity in terms of its nominal value and a bounded uncertainty.

Property 1. (*More* et al.*, 2009*) Given a real value x belonging to an interval $[x]$, there exists a real value $\alpha \in [-1, 1]$ such that $x = \mathrm{mid}([x]) + \alpha \mathrm{rad}([x])$.

An *interval vector* (or *box*) $[\alpha]$ is a vector with interval components. It may equivalently be seen as a Cartesian product of scalar intervals:

$$[\alpha] = [\alpha_1] \times [\alpha_2] \times \ldots \times [\alpha_n].$$

An *interval matrix* is a matrix with interval components. The set of n-dimensional real interval vectors is denoted by $\mathbb{IR}^n$ and the set of $n \times m$ real interval matrices is denoted by $\mathbb{IR}^{n\times m}$. The midpoint $\mathrm{mid}(\cdot)$ (resp. the radius $\mathrm{rad}(\cdot)$) of an interval vector (resp. an interval matrix) is a vector (resp. a matrix) composed of the midpoints (resp. the radii) of its interval components. Classical operations for interval vectors (resp. interval matrices) are direct extensions of the same operations for real vectors (resp. real matrices) (Moore, 1966). In order to simplify the notation, the midpoint and the radius of a matrix $[M]$ are respectively denoted by M_m and M_r.

Using Property 1 for matrices, the following result is is obtained.

Proposition 1. (Tran, 2017) *Given an $m \times n$ real matrix M belonging to an interval matrix $[M]$, there exist mn real values $\alpha^{ij} \in [-1, 1]$ with $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$, such that*[1]

$$M = M_m + \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha^{ij} M_r^{(i,j)}, \tag{3}$$

where $M_r^{(i,j)}$ is an $m \times n$ matrix whose entry (i, j) is the radius of entry (i, j) of $[M]$, and the other elements are zero.

In the case of symmetric matrices, the following representation should be considered.

Proposition 2. (Tran, 2017) *Given an $n \times n$ real symmetric matrix M belonging to a symmetric interval matrix $[M]$, there exist $n(n+1)/2$ real values $\alpha^{ij} \in [-1, 1]$ such that*

$$\begin{aligned} M =& M_m + \mathrm{diag}\,(M_r)\,\mathrm{diag}\left(\alpha^{ii}\right) \\ &+ \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \alpha^{ij} M_r^{((i,j),(j,i))}, \end{aligned} \tag{4}$$

where $\mathrm{diag}\,(M_r)$ is a diagonal matrix containing the radii of diagonal elements of $[M]$, $M_r^{((i,j),(j,i))}$ is a symmetric matrix whose (i, j) and (j, i) entries are the radii of entries (i, j) and (j, i) of $[M]$, and the other elements are zero. The matrix $\mathrm{diag}\left(\alpha^{ii}\right)$ is diagonal and $\alpha^{ij} \in [-1, 1]$, for all $1 \le i \le j \le n$.

3.2. Inclusion function. Given a box $[\alpha]$ in $\mathbb{IR}^n$ and a function f from $\mathbb{IR}^n$ to $\mathbb{IR}^m$, an *inclusion function* of f aims at getting a box containing the image of $[\alpha]$ by f. The *range* of f over $[\alpha]$ is given by

$$f([\alpha]) = \{f(x) \mid x \in [\alpha]\}.$$

Then, the interval function $[f]$ from $\mathbb{IR}^n$ to $\mathbb{IR}^m$ is an *inclusion function* for f if

$$\forall [\alpha] \in \mathbb{IR}^n,\ f([\alpha]) \subset [f]([\alpha]).$$

An inclusion function of f can be obtained by replacing each occurrence of a real variable by its corresponding interval and by replacing each standard function by its interval evaluation. Such a function is called the *natural inclusion function*. A function f generally has several inclusion functions, which depend on the syntax of f.

4. Upper bounds of matrices

This section introduces two matrix inequalities used in the proposed interval Kalman filter in order to bound the state estimation error covariance.

Proposition 3. (Tran, 2017) *Given two nonnull matrices M, N with the same size and an arbitrary real number $\beta > 0$, the following inequality holds:*

$$MN^T + NM^T \preceq \beta^{-1} MM^T + \beta NN^T. \tag{5}$$

Proposition 4. (Combastel, 2016) *Let M be a symmetric matrix and $M = VDV^T$ be its spectral decomposition, where V is an orthogonal matrix and D is a diagonal matrix. Let $M^+ = V|D|V^T$, where $|\cdot|$ is the element-by-element absolute value operator. Then $M^+ \succeq 0$ and $\forall \alpha \in [-1, 1]$, $\alpha M \preceq M^+$.*

The following proposition can be used to determine an upper bound on the expression MPM^T, where $M \in [M]$, $[M] \in \mathbb{IR}^{m\times n}$ and $P \in \mathbb{R}^n$ is a positive definite symmetric matrix.

[1] This expression is an expanded form of the Hadamard product.

Proposition 5. *Given an $m \times n$ real matrix M belonging to an interval matrix $[M]$ and a positive definite matrix symmetric P of order n, there exists a positive definite symmetric matrix S of order m such that $MPM^T \preceq S$.*

The matrix S can be determined by using Propositions 1 and 4.

Proof. By using Proposition 1 for $M \in [M]$ and then expanding MPM^T, we obtain

$$\begin{aligned} &MPM^T \\ &= M_m P M_m^T + \sum_{i=1}^{m}\sum_{j=1}^{n} \left(\alpha^{ij}\right)^2 M_r^{(i,j)} P \left(M_r^{(i,j)}\right)^T \\ &\quad + \sum_{i=1}^{m}\sum_{j=1}^{n} \alpha^{ij} \left(\mathbf{M}_m^{(i,j)} + (\mathbf{M}_m^{(i,j)})^T\right) \\ &\quad + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{\substack{l=1\\ k\neq i \vee l\neq j}}^{n} \alpha^{ij}\alpha^{kl}\Big(\mathbf{M}_{(i,j)}^{(k,l)} \\ &\quad + (\mathbf{M}_{(i,j)}^{(k,l)})^T\Big), \end{aligned} \tag{6}$$

where

$$\mathbf{M}_m^{(i,j)} = M_m P \left(M_r^{(i,j)}\right)^T,$$

$$\mathbf{M}_{(i,j)}^{(k,l)} = M_r^{(i,j)} P \left(M_r^{(k,l)}\right)^T$$

and $\alpha^{ij} \in [-1, 1]$.

Proposition 4 is then used to determine upper bounds of the two symmetric parts of Eqn. (6), i.e., $S_m^{(i,j)}$ for $\mathbf{M}_m^{(i,j)} + (\mathbf{M}_m^{(i,j)})^T$ and $S_{(i,j)}^{(k,l)}$ for $\mathbf{M}_{(i,j)}^{(k,l)} + (\mathbf{M}_{(i,j)}^{(k,l)})^T$. The upper bound S of MPM^T can be written as follows:

$$\begin{aligned} S =& M_m P M_m^T + \sum_{i=1}^{m}\sum_{j=1}^{n} M_r^{(i,j)} P \left(M_r^{(i,j)}\right)^T \\ &+ \sum_{i=1}^{m}\sum_{j=1}^{n} S_m^{(i,j)} + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{\substack{l=1\\ k\neq i \vee l\neq j}}^{n} S_{(i,j)}^{(k,l)}, \end{aligned} \tag{7}$$

where S is symmetric and positive definite. ∎

Similarly, a positive definite upper bound of an interval symmetric matrix can be computed.

Proposition 6. *Given a symmetric matrix M belonging to an interval symmetric matrix $[M] \in \mathbb{IR}^n$, there exists a positive definite symmetric matrix M^+ of order n such that $M \preceq M^+$. The matrix M^+ can be determined using Propositions 2 and 4.*

Proof. A symmetric matrix $M \in [M]$ can be decomposed using Proposition 2. Proposition 4 is then applied for each term of the double sum in Eqn. (4) to determine an upper bound M^+ on M:

$$\begin{aligned} M^+ &= M_m + \operatorname{diag}(M_r) \\ &\quad + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \left(M_r^{((i,j),(j,i))}\right)^+, \end{aligned} \tag{8}$$

where $\left(M_r^{((i,j),(j,i))}\right)^+$ is the upper bound on $\alpha^{ij} M_r^{((i,j),(j,i))}$ given by Proposition 4. ∎

5. From the Kalman filter to the interval Kalman filter

5.1. Kalman filter. Given the system (1), the conventional Kalman filter (KF) provides the minimum variance estimate $\hat{\mathbf{x}}_{k|k}$ of $\mathbf{x}_k$ and the associated covariance matrix $P_{k|k}$.

Kalman equations (Kalman, 1960) are determined using mathematical curve-fitting functions of data points from a least-squares approximation (Welch and Bishop, 2001) or probabilistic methods such as the likelihood function to maximize the conditional probability of the state estimate from incoming measurements (Masreliez and Martin, 1977). We consider the following notation:

- $\hat{\mathbf{x}}_{k+1|k} \in \mathbb{R}^n$ is the *a priori* state estimate vector at time $k+1$ given a state estimate at time k,
- $\hat{\mathbf{x}}_{k|k} \in \mathbb{R}^n$ is the *a posteriori* state estimate vector at time k given observations at time k,
- $P_{k+1|k} \in \mathbb{R}^{n\times n}$ is the *a priori* error covariance matrix,
- $P_{k|k} \in \mathbb{R}^{n\times n}$ is the *a posteriori* error covariance matrix.

Also, $P_{\cdot|\cdot}$ defines the precision of the state estimate:

$$P_{l|k} = E\left((\hat{\mathbf{x}}_{l|k} - \mathbf{x}_l)(\hat{\mathbf{x}}_{l|k} - \mathbf{x}_l)^T\right), \tag{9}$$

$l = k$ or $k+1$. We assume that $P_{0|0} = P_0 \in \mathbb{IR}^{n\times n}$ and $\mathbf{x}_{0|0} = \mathbf{x}_0 \in \mathbb{IR}^n$.

The Kalman filtering algorithm contains two steps for each iteration: a prediction step and a correction step (Kalman, 1960).

If we consider the algorithm as an operator $\mathcal{K}$, we can write

$$\begin{aligned} &(\hat{\mathbf{x}}_{k|k}, P_{k|k}) \\ &\quad = \mathcal{K}(A_k, B_k, C_k, \mathbf{x}_0, P_0, \mathbf{u}_{[1\ldots k-1]}, \mathbf{y}_{[1\ldots k-1]}). \end{aligned}$$

In the following, the point time-varying matrices A_k, B_k, C_k, Q_k and R_k are constrained to belong to the interval matrices $[A]$, $[B]$, $[C]$, $[Q]$, and $[R]$, respectively. In other words, their parameters can vary within some specified bounds.

5.2. Interval Kalman filter. Bounded uncertainties can occur not only through the interval matrices $[A]$, $[B]$, $[C]$, $[D]$, $[Q]$ and $[R]$, but also through $\mathbf{x}_{0|0}, P_{0|0}, \mathbf{u}_k, \mathbf{y}_k$ due to deterministic measurement errors and instrument imprecisions. Since it is impossible to solve directly the Kalman filtering problem due to parameter uncertainties, our goal is to obtain an upper bound $P^+_{k|k}$ on the state estimation error covariance:

$$E\left[(\hat{\mathbf{x}}_{k|k} - \mathbf{x}_k)(\hat{\mathbf{x}}_{k|k} - \mathbf{x}_k)^T\right] \preceq P^+_{k|k}. \tag{10}$$

In this section, an interval Kalman filter, called the *improved minimum upper bound of variance interval Kalman filter (iUBIKF)*, is proposed. This algorithm is developed from the interval Kalman filter introduced by Tran *et al*. (2017) in order to reduce the overestimation of the state estimation error covariance. The iUBIKF can be designed in two steps: prediction and correction.

5.2.1. Prediction step. In the prediction step, the interval state estimate from the previous time step and the transition model are used to predict the state at the current time step. This step is performed similarly as in the original Kalman algorithm (Kalman, 1960), using the natural interval extension, as follows:

$$\left[\hat{\mathbf{x}}_{k|k-1}\right] = [A]\left[\hat{\mathbf{x}}_{k-1|k-1}\right] + [B]\mathbf{u}_k. \tag{11}$$

For any $A_k \in [A]$ and $Q_k \in [Q]$, the *a priori* covariance matrix $P_{k|k-1}$ is computed as

$$P_{k|k-1} = A_k P^+_{k-1|k-1} A_k^T + Q_k, \tag{12}$$

where $P^+_{k-1|k-1}$ is the upper bound on the *a posteriori* covariance matrix at time $k-1$. In order to determine an upper bound $P^+_{k|k-1}$ on $P_{k|k-1}$, i.e., $P_{k|k-1} \preceq P^+_{k|k-1}$, Proposition 5 and 6 are respectively applied to $A_k P^+_{k-1|k-1} A_k^T$ and Q_k, $A_k \in [A]$ and $Q_k \in [Q]$:

$$P^+_{k|k-1} = P^+_k + Q^+_k, \tag{13}$$

where $A_k P^+_{k-1|k-1} A_k^T \preceq P^+_k$ and $Q_k \preceq Q^+_k$.

5.2.2. Correction step. The state estimate at time step k is computed by the natural interval extension of (Kalman, 1960)

$$\left[\hat{\mathbf{x}}_{k|k}\right] = \left[\hat{\mathbf{x}}_{k|k-1}\right] + K_k\left(y_k - [C_k]\left[\hat{\mathbf{x}}_{k|k-1}\right]\right), \tag{14}$$

given $\hat{\mathbf{x}}^-_k \in [\hat{\mathbf{x}}_k]^-$ and $C_k \in [C_k]$. In order to reduce the effect of the dependency problem ((Jaulin *et al.*, 2001b)), Eqn. (14) is rearranged as follows:

$$\left[\hat{\mathbf{x}}_{k|k}\right] = (I - K_k[C_k])\left[\hat{\mathbf{x}}_{k|k-1}\right] + K_k y_k. \tag{15}$$

The box $\left[\hat{\mathbf{x}}_{k|k}\right]$ encloses all possible values of $\hat{\mathbf{x}}_{k|k}$.

The gain matrix K_k is determined as follows. The expression of the error covariance matrix after the correction step, for any $C_k \in [C_k]$, $R_k \in [R]$, is

$$\begin{aligned} P_{k|k} = {} & (I - K_k C_k)\, P^+_{k|k-1}\,(I - K_k C_k)^T \\ & + K_k R_k K_k^T. \end{aligned} \tag{16}$$

An upper bound on $P_k|k$ can be obtained by using Proposition 1 for the matrix C_k, and then developing Eqn. (16):

$$\begin{aligned} & P_{k|k} \\ & = (I - K_k C_m)\, P^+_{k|k-1}\,(I - K_k C_m)^T \\ & \quad + \sum_{i=1}^{n_y}\sum_{j=1}^{n_x} M^{(i,j)} + (M^{(i,j)})^T \\ & \quad + \sum_{i=1}^{n_y}\sum_{j=1}^{n_x} (\alpha^{ij})^2 K_k C_r^{(i,j)} P^+_{k|k-1}\left(C_r^{(i,j)}\right)^T K_k^T \\ & \quad + \frac{1}{2}\sum_{i=1}^{n_y}\sum_{j=1}^{n_x}\sum_{m=1}^{n_y}\sum_{\substack{l=1\\ m\neq i \wedge l\neq j}}^{n_x} K_k\Big(N^{(i,j)}_{(m,l)} \\ & \quad + (N^{(i,j)}_{(m,l)})^T\Big) K_k^T + K_k R_k K_k^T, \end{aligned} \tag{17}$$

where

$$\begin{aligned} M^{(i,j)} &= \alpha^{ij}\left(K_k C_r^{(i,j)} P^+_{k|k-1}(K_k C_m - I)^T\right), \\ N^{(i,j)}_{(m,l)} &= \alpha^{ij}\alpha^{ml} C_r^{(i,j)} P^+_{k|k-1}\left(C_r^{(m,l)}\right)^T. \end{aligned}$$

The term $M^{(i,j)}$ can be rewritten as

$$M^{(i,j)} = \left(\alpha^{ij} K_k C_r^{(i,j)}\sqrt{P^+_{k|k-1}}\right)\left((K_k C_m - I)\sqrt{P^+_{k|k-1}}\right)^T.$$

Proposition 3 is then applied to $M^{(i,j)} + (M^{(i,j)})^T$ with $\beta = 1$. This yields

$$\begin{aligned} & M^{(i,j)} + (M^{(i,j)})^T \\ & \quad \preceq K_k C_r^{(i,j)} P^+_{k|k-1}\left(C_r^{(i,j)}\right)^T K_k^T \\ & \qquad + (I - K_k C_m)\, P^+_{k|k-1}\,(I - K_k C_m)^T. \end{aligned} \tag{18}$$

Upper bounds $S^{(m,l)}_{(i,j)}$ on the terms $N^{(i,j)}_{(m,l)} + (N^{(i,j)}_{(m,l)})^T$ are given by Proposition 4.

Therefore, the following expression is obtained for an upper bound $P^+_{k|k}$ on $P_{k|k}$:

$$\begin{aligned} &P_{k|k} \\ &\preceq (n_0+1)\left(I - K_k C_m\right) P^+_{k|k-1} \left(I - K_k C_m\right)^T \\ &\quad + 2\sum_{i=1}^{n_y}\sum_{j=1}^{n_x} K_k C_r^{(i,j)} P^+_{k|k-1} \left(C_r^{(i,j)}\right)^T K_k^T \\ &\quad + \frac{1}{2}\sum_{i=1}^{n_y}\sum_{j=1}^{n_x}\sum_{m=1}^{n_y}\sum_{\substack{l=1\\ m\neq i\wedge l\neq j}}^{n_x} K_k S^{(m,l)}_{(i,j)} K_k^T \\ &\quad + K_k R_k^+ K_k^T \\ &= P^+_{k|k}, \end{aligned} \tag{19}$$

where n_0 is the number of interval elements of the matrix $[C]$, i.e., $n_0 = n_y \times n_x$. The matrix $R_k^+ \succeq R_k$ is determined by Proposition 6.

Having the expression of $P^+_{k|k}$ as a function of K_k, we look for K_k that minimizes the trace of $P^+_{k|k}$. The first and second derivatives of $\mathrm{tr}\left(\mathrm{P^+_{k|k}}\right)$ with respect to K_k are

$$\begin{aligned} \frac{\mathrm{d}\,\mathrm{tr}\left(P^+_{k|k}\right)}{\mathrm{d}K_k} &= -2(n_0+1) P^+_{k|k-1} C_m^T \\ &\quad + 2(n_0+1) K_k C_m P^+_{k|k-1} C_m^T \\ &\quad + 4\sum_{i=1}^{n_y}\sum_{j=1}^{n_x} K_k C_r^{(i,j)} P^+_{k|k-1} \left(C_r^{(i,j)}\right)^T \\ &\quad + \sum_{i=1}^{n_y}\sum_{j=1}^{n_x}\sum_{m=1}^{n_0}\sum_{l=1}^{n_x} K_k S^{(m,l)}_{(i,j)} \\ &\quad + 2K_k R_k^+, \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}^2\,\mathrm{tr}\left(P^+_{k|k}\right)}{\mathrm{d}K_k^2} &= 2(n_0+1) C_m P^+_{k|k-1} C_m^T \\ &\quad + 4\sum_{i=1}^{n_y}\sum_{j=1}^{n_x} C_r^{(i,j)} P^+_{k|k-1} \left(C_r^{(i,j)}\right)^T \\ &\quad + \sum_{i=1}^{n_y}\sum_{j=1}^{n_x}\sum_{m=1}^{n_y}\sum_{l=1}^{n_x} S^{(m,l)}_{(i,j)} \\ &\quad + 2R_k^+. \end{aligned}$$

The second derivative is always positive definite, which guarantees the existence of a minimum value for $\mathrm{tr}\left(\mathrm{P^+_{k|k}}\right)$, and K_k is obtained from the first derivative:

$$K_k = (n_0+1) P^+_{k|k-1} C_m^T S_k^{-1}, \tag{20}$$

Algorithm 1. iUBIKF algorithm.

Require: $\left[\hat{\mathbf{x}}_{0|0}\right]$, $P^+_{0|0}$, $[A]$, $[B]$, $[C]$, $[Q]$, $[R]$, $\mathbf{y}_k$, $\mathbf{u}_k$, A_k, Q_k, R_k, $k = 1, 2, \ldots$

Ensure: $\left[\hat{\mathbf{x}}_{k|k}\right]$, $P^+_{k|k}$

1: **for** $k = 1, 2, \ldots$ **do**

2: **Prediction step**:

$$\begin{aligned} \left[\hat{\mathbf{x}}_{k|k-1}\right] &= [A]\left[\hat{\mathbf{x}}_{k-1|k-1}\right] + [B]\,\mathbf{u}_k, \\ P_{k|k-1} &= A_k P^+_{k-1|k-1} A_k^T + Q_k, \\ P^+_{k|k-1} &\succeq P_{k|k-1} \text{ (Props. 5 and 6).} \end{aligned}$$

3: **Correction step**

$$\begin{aligned} R_k^+ &\succeq R_k \text{ (Prop. 5 and 6)}, \\ K_k &= (n_0+1) P^+_{k|k-1} C_m^T S_k^{-1}, \\ S_k &= (n_0+1)\mathrm{mid}\left([C]\right) P^+_{k|k-1} \mathrm{mid}\left([C]\right)^T \\ &\quad + 2\sum_{i=1}^{n_y}\sum_{j=1}^{n_x} C_r^{(i,j)} P^+_{k|k-1} \left(C_r^{(i,j)}\right)^T \\ &\quad + \frac{1}{2}\sum_{i=1}^{n_y}\sum_{j=1}^{n_x}\sum_{m=1}^{n_y}\sum_{l=1}^{n_x} S^{(m,l)}_{(i,j)} \\ &\quad + R_k^+. \\ P^+_{k|k} &= (n_0+1)\left(I - K_k \mathrm{mid}\left([C]\right)\right) P^+_{k|k-1}. \\ \left[\hat{\mathbf{x}}_{k|k}\right] &= \left(I - K_k\left[C_k\right]\right)\left[\hat{\mathbf{x}}_{k|k-1}\right] + K_k y_k. \end{aligned}$$

4: **end for**

where

$$\begin{aligned} S_k &= (n_0+1) C_m P^+_{k|k-1} C_m^T \\ &\quad + 2\sum_{i=1}^{n_y}\sum_{j=1}^{n_x} C_r^{(i,j)} P^+_{k|k-1} \left(C_r^{(i,j)}\right)^T \\ &\quad + \frac{1}{2}\sum_{i=1}^{n_y}\sum_{j=1}^{n_x}\sum_{m=1}^{n_y}\sum_{l=1}^{n_x} S^{(m,l)}_{(i,j)} + R_k^+. \end{aligned} \tag{21}$$

The expression for the covariance matrix bound $P^+_{k|k}$ is obtained from Eqn. (19) using K_k as given in Eqn. (20):

$$P^+_{k|k} = (n_0+1)\left(I - K_k C_m\right) P^+_{k|k-1}. \tag{22}$$

The algorithm steps are summarized in Algorithm 1.

6. Case studies

This section applies the proposed filter (iUBIKF) to two systems. The first one is an academic example that is used to compare the estimation results with the previous filter UBIKF and to show how the upper bound on the covariance matrix is less conservative. The second is a case study of a two-wheeled vehicle used to compare the iUBIKF results with those of the interval observer of Raka and Combastel (2013).

6.1. Academic example.

Consider an uncertain system described by the following equations:

$$\begin{cases} \mathbf{x}_{k+1} = A_k \mathbf{x}_k + \mathbf{w}_k, \\ \mathbf{y}_k = C_k \mathbf{x}_k + \mathbf{v}_k, \quad k \in \mathbb{N}. \end{cases} \tag{23}$$

Both $\{w_k\}$ and $\{v_k\}$ are independent centered Gaussian white noise sequences with covariance matrices Q_k and R_k, respectively.

We assume that A_k, C_k, Q_k and R_k are respectively bounded by the interval matrices $[A]$, $[C]$, $[Q]$ and $[R]$ defined as

$$[A] = \begin{pmatrix} [2.55, 2.65] & [-1.43 - 1.37] & [0.26, 0.28] \\ [6.57, 6.83] & [-3.47, -3.33] & [2.55, 2.65] \\ [-0.77, -0.73] & [0.29, 0.31] & [0.09, 0.11] \end{pmatrix},$$

$$[C] = \begin{pmatrix} [-8.24, -7.76] & [-4.12, -3.88] & [1.94, 2.06] \\ [-2.06, -1.94] & [-2.06, -1.94] & [-6.18, -5.82], \\ [-0.41, -0.39] & [15.52, 16.48] & [6.79, 7.21] \end{pmatrix},$$

$$[Q] = \begin{pmatrix} [8, 12] & [-6, -4] & [3.2, 4.8] \\ [-6, -4] & [8, 12] & [1.6, 2.4] \\ [3.2, 4.8] & [1.6, 2.4] & [8.12] \end{pmatrix},$$

$$[R] = \begin{pmatrix} [8, 12] & [-6, -4] & [3.2, 4.8] \\ [-6, -4] & [8, 12] & [1.6, 2.4] \\ [3.2, 4.8] & [1.6, 2.4] & [8.12] \end{pmatrix}.$$

First, we compare the results provided by the original UBIKF presented by Tran *et al*. (2017) and by our improved filter (iUBIKF). The efficiency of the filtering algorithms is evaluated via a set of criteria. One of them is the upper bound of the root mean square error $\overline{\text{RMSE}}$ defined as

$$\overline{\text{RMSE}} = \sup \Big(\frac{1}{L} \sum_{k=1}^{L} \big(\mathbf{x}_k - [\hat{\mathbf{x}}_{k|k}]\big)^T \cdot \big(\mathbf{x}_k - [\hat{\mathbf{x}}_{k|k}]\big) \Big)^{\frac{1}{2}}. \tag{24}$$

In Eqn. (24), L represents the number of iterations, $[\hat{\mathbf{x}}_{k|k}]$ is the interval estimate. Additionally, we propose to compute the percentage of time steps O where the confidence interval $[I_{c_k}]$, defined as

$$[I_{c_k}] = [\hat{\mathbf{x}}_{k|k}] + \Big[-3\sqrt{\operatorname{diag}(P^+_{k|k})}\,,\, 3\sqrt{\operatorname{diag}(P^+_{k|k})} \Big], \tag{25}$$

contains the actual state, where $\operatorname{diag}(M)$ is the vector of diagonal elements of matrix M. This index allows one to determine the confidence degree of the state envelopes.

The simulations are run on the time stage [0,1000] with the toolbox Intlab of Matlab (Rump, 1999). The comparison of the two filtering algorithms based on the indexes $\overline{\text{RMSE}}$, O and the execution time is shown in Table 1. The 3σ confidence intervals $[I_{c_k}]$ (cf. Eqn. (25)) given by the two filters (Figs. 1–3) enclose the actual states at any time step ($O = 100\%$). However, the confidence intervals on it are tighter since the iUBIKF provides a better upper bound of the state estimation error covariance.

Figures 1–3 provide the three components of the actual state and the 3σ confidence intervals $[I_{c_k}]$ (noted as CI in the figures' internal caption) given by the UBIKF and the iUBIKF.

Let us now compare the estimation error covariance upper bounds given by the two filters with reference to the conventional Kalman filter. To do so, the original Kalman filter (Kalman, 1960) is applied to a set of 1000 models $\{A_k, C_k, Q_k, R_k\}$, where the matrices A_k, C_k, Q_k and R_k are uniformly sampled from the interval matrices $[A_k]$, $[C_k]$, $[Q_k]$ and $[R_k]$, respectively. The maximum diagonal elements of the covariance matrices generated for this set of models by the Kalman filter are compared with the diagonal elements of the covariance upper bound given by the UBIKF and the iUBIKF in Figs. 4–6.

As shown in the three figures, the iUBIKF provides a less conservative upper bound on the covariance matrix than the UBIKF, which is quite close to the maximal value obtained by the conventional Kalman filter run on the set of sampled models.

6.2. Case study from the automotive domain.

The second example comes from the automotive domain. It is based on the continuous-time nonlinear model

Table 1. UBIKF and iUBIKF comparative evaluation.

		UBIKF	iUBIKF
x_1	$\overline{\text{RMSE}}$	3.64	3.55
	$O(\%)$	100	100
x_2	$\overline{\text{RMSE}}$	3.60	3.49
	$O(\%)$	100	100
x_3	$\overline{\text{RMSE}}$	2.88	2.83
	$O(\%)$	100	100
Time (s)		15	30

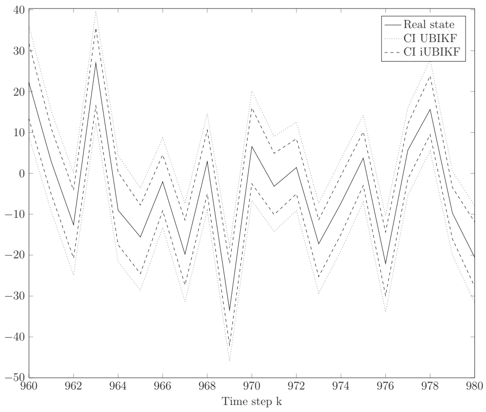

Fig. 1. Actual x_1 state component and the 3σ confidence intervals $[I_{c_k}]$ obtained by the UBIKF and the iUBIKF.

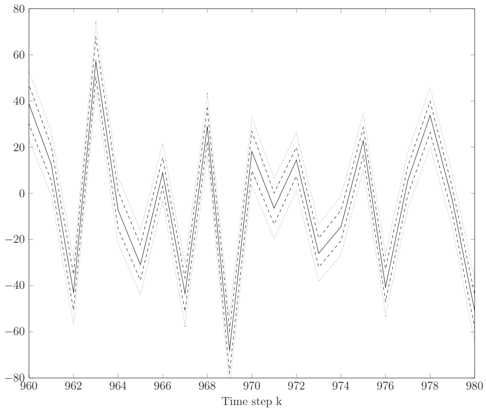

Fig. 2. Actual x_2 state component and the 3σ confidence intervals $[I_{c_k}]$ obtained by the UBIKF and the iUBIKF.

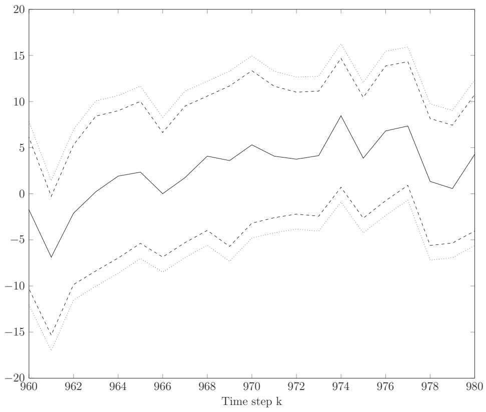

Fig. 3. Actual x_3 state component and the 3σ confidence intervals $[I_{c_k}]$ obtained by the UBIKF and the iUBIKF.

of the dynamics of a two-wheeled vehicle that has been linearized and discretized to be suitable for the UBIKF/iUBIKF. The resulting state space model has two states: x_1 is the angular speed of the slideslip angle and x_2 is the acceleration of the vehicle yaw. We compare the iUBIKF estimates with those of the interval observer proposed by Raka and Combastel (2013).

The interval matrices $[A]$, $[C]$, $[Q]$ and $[R]$ bounding A_k, C_k, Q_k, and R_k, respectively, are the following:

$$[A] = \begin{pmatrix} [0.6439, 1.1814] & [-0.0131, 0.1023] \\ [-0.2393, -0.1006] & [0.8516, 0.9646] \end{pmatrix},$$

$$[C] = \begin{pmatrix} [-2.3594, -1.1150] & [0.0211, 1.9326] \\ [0.0849, 1.9151] & [-0.6333, 0.6333] \\ [-3.7331, -3.7123] & [-2.1423, -1.5307] \\ [-0.0702, 0.0702] & [0.8322, 1.1678] \end{pmatrix},$$

$$[Q] = [0.0000, 0.0501] I_{n_x}, \quad n_x = 2,$$

$$[R] = [0.0000, 0.0501] I_{n_y}, \quad n_y = 4.$$

The performance comparison of the three filtering algorithms UBIKF, iUBIKF and the interval observer of Raka and Combastel (2013) is given in Table 2.

Figure 7 shows the evolution of the state estimates produced by the UBIKF, the iUBIKF, i.e., $[\hat{x}_1]$ and $[\hat{x}_2]$, and the interval observer., i.e., $[x_1]$ and $[x_2]$.

As indicated by the $\overline{\text{RMSE}}$ in Table 2 and by the graphs in Fig. 7, the interval observer bounds are far wider than the estimation bounds of the iUBIKF and the UBIKF, those of the iUBIKF being the tightest. On the other hand, the computation time is higher and may limit some applications.

7. Conclusion

An improved minimum upper bound of variance interval Kalman filter (iUBIKF) which provides a lower error covariance upper bound was proposed. This filter allows bounding the set of all possible state estimates given by the Kalman filter for any admissible parameter uncertainties. Through a number of simulations, the advantages of the iUBIKF with respect to previous versions and to other proposals of the literature are exhibited.

The proposed iUBIKF is intended for systems of moderate dimensions as it has not been optimized for larger ones. For example, square root filtering algorithms

Table 2. Comparative evaluation of the UBIKF, the iUBIKF, and an interval observer.

		UBIKF	iUBIKF	Int. obs
x_1	$\overline{\text{RMSE}}$	0.17585	0.051212	1.1276
x_2	$\overline{\text{RMSE}}$	0.291	0.080989	1.1274
Time (s)		2.3916	7.6362	0.40902

are known as viable alternatives to the conventional Kalman filter that inherently involves unstable numerics. Updating the iUBIKF in this direction can be seen as a nice perspective for future work.

This work shows that the integration of statistical and bounded uncertainties in the same model can be successfully achieved, which opens wide perspectives from a practical point of view.

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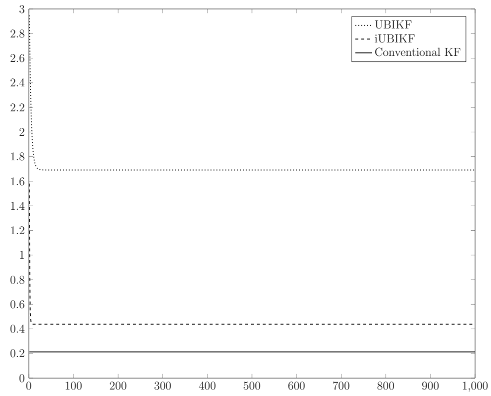

Fig. 4. $P^{11}_{k|k}$ for the UBIKF, the iUBIKF and $\max(P^{11}_{k|k})$ for the conventional Kalman filter.

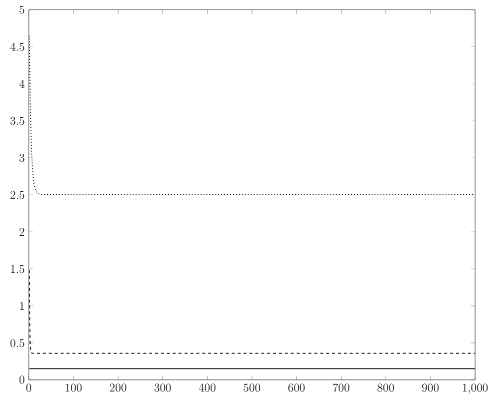

Fig. 5. $P^{22}_{k|k}$ for the UBIKF, the iUBIKF and $\max(P^{22}_{k|k})$ for the conventional Kalman filter.

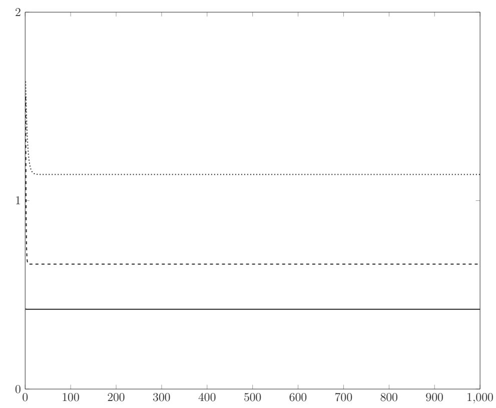

Fig. 6. $P^{33}_{k|k}$ for the UBIKF, the iUBIKF and $\max(P^{33}_{k|k})$ for the conventional Kalman filter.

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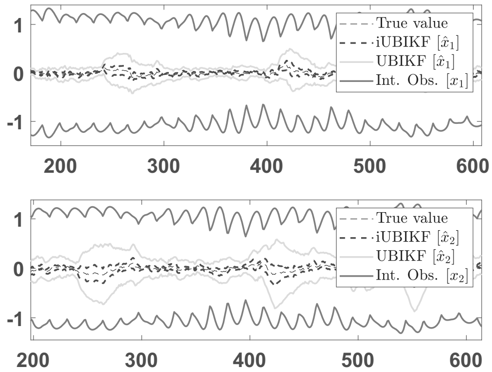

Fig. 7. Estimation results for the UBIKF, the iUBKF and the interval observer for the two-wheeled vehicle model: the angular speed of the slideslip angle x_1 (top) and the acceleration of the vehicle yaw x_2 (bottom).

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