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TIME–OPTIMAL CONTROL OF LINEAR FRACTIONAL SYSTEMS WITH VARIABLE COEFFICIENTS

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Linear systems described by fractional differential equations (FDEs) with variable coefficients involving Riemann–Liouville and Caputo derivatives are examined in the paper. For these systems, a solution of the initial-value problem is derived in terms of the generalized Peano–Baker series and a time-optimal control problem is formulated. The optimal control problem is treated from the convex-analytical viewpoint. Necessary and sufficient conditions for time-optimal control similar to that of Pontryagin's maximum principle are obtained. Theoretical results are supported by examples.

Keywords: fractional calculus, Riemann–Liouville derivative, variable coefficients, optimal control.

1. Introduction

Fractional differential equations (FDEs) provide a powerful tool to describe the memory effect and hereditary properties of various materials and processes (Podlubny, 1998; Sierociuk and Dzieliński, 2006; Li *et al.*, 2010; Luchko, 2009; Datsko and Gafiychuk, 2018; Datsko *et al.*, 2019). While linear systems of FDEs represent a fairly well investigated field of research, relatively few papers deal with linear FDEs involving variable coefficients. Meanwhile, a number of real-life systems and processes can be described by linear FDEs with variable coefficients, e.g., linearized aircraft models, linearized models of population restricted growth, models related to the distribution of parameters in the charge transfer and the diffusion of batteries, etc.

Linear differential equations with variable coefficients arise in a natural way when modeling RLC-circuits with variable capacitance or inductance. With the advent of electronic components like super-capacitors (also called ultracapacitors) and fractances, one should employ fractional differential equations for circuit models (Kaczorek and Rogowski, 2015; Skovranek *et al.*, 2019). Some examples of circuit models involving linear FDEs with variable coefficients can be found in the work of Martínez *et al.* (2018). This provides a motivation for research into FDEs with variable coefficients and related control problems.

Explicit solutions to linear systems of differential equations provide a basis to perform stability analysis and to solve control problems. Analytical solutions of linear systems of fractional differential equations with constant coefficients were derived by Chikrii and Eidelman (2000), Chikrii and Matichin (2008), or Kaczorek (2008), and then applied to solving control problems by Matychyn and Onyshchenko (2015; 2018b; 2018a; 2019), Dzieliński and Czyronis (2013), Balaska *et al.* (2020), and Si *et al.* (2021). Explicit solutions to linear systems of differential equations are usually expressed in terms of the state transition matrix. In the case of FDEs with constant coefficients the state transition matrix can be represented using the matrix Mittag-Leffler function (Chikrii and Eidelman, 2000; Chikrii and Matichin, 2008).

In recent years a number of papers have been devoted

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to solutions of systems of FDEs with variable coefficients and their control. A solution to the initial value problem for a linear system with variable coefficients involving Caputo derivatives was obtained by Kaczorek and Idczak (2017). In the work of Eckert *et al*. (2019) explicit solutions for the linear systems of initialized (Lorenzo and Hartley, 2000) FDEs are obtained in terms of the generalized Peano–Baker series (Baake and Schlägel, 2011). Linear systems of FDEs with variable coefficients and their state-transition matrices are also discussed by Bourdin (2018), Matychyn (2019), Matychyn and Onyshchenko (2020), or Malesza *et al*. (2019).

This paper deals with the initial value problem for linear systems of FDEs with variable coefficients involving Riemann–Liouville and Caputo derivatives. For these systems a solution of the initial-value problem is derived in terms of the generalized Peano–Baker series and a time-optimal control problem is formulated. The optimal control problem is treated from the convex-analytical viewpoint. Necessary and sufficient conditions for time-optimal control similar to that of Pontryagin's maximum principle are obtained. The paper is a further development of the approach consisting in the extension of the Pontryagin maximum principle to fractional-order systems (Kamocki, 2014; Matychyn and Onyshchenko, 2015; 2018b; 2018a; 2019; Bergounioux and Bourdin, 2020). Theoretical results are supported by illustrative examples.

2. Preliminary results

2.1. Fractional integrodifferentiation. Denote by $\mathbb{R}^n$ the n-dimensional Euclidean space and by I some interval of the real line, $I \subset \mathbb{R}$. In what follows we will assume that $I = [t_0, T]$ for some $T > t_0$ and denote $\mathring{I} = (t_0, T)$. Suppose $f : I \to \mathbb{R}^n$ is an absolutely continuous function. Recall that the Riemann–Liouville (left-sided) fractional integral and derivative of order α, $0 < \alpha < 1$, are respectively defined as

$$
{}_{t_0}J_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau,
$$

$$
{}_{t_0}D_t^{\alpha} f(t) = \frac{\mathrm{d}}{\mathrm{d}t}\, {}_{t_0}J_t^{1-\alpha} f(t), \quad t \in \mathring{I}.
$$

Hereafter, $\Gamma(\cdot)$ stands for the Gamma function defined by

$$
\Gamma(\alpha) = \int\limits_0^{\infty} t^{\alpha-1} e^{-t}\,\mathrm{d}t.
$$

The Riemann–Liouville fractional derivative of a constant does not equal zero. Moreover, it becomes infinite as t approaches t_0 and due to this fact FDEs with the Riemann–Liouville derivative require initial conditions of a special form lacking clear physical meaning. That is why the regularized Caputo derivative was introduced, which is free from these shortcomings.

The Caputo (regularized) derivative of a fractional order α, $0 < \alpha < 1$, can be introduced by the following formula:

$$
{}_{t_0}D_t^{(\alpha)} f(t) = {}_{t_0}J_t^{1-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} f(t), \quad t \in \mathring{I}. \tag{1}
$$

The following properties of the fractional integrals and derivatives (Kilbas *et al*., 2006; Podlubny, 1998) will be used in the sequel.

Lemma 1. *If $\alpha, \beta > 0$, and $f(t)$ is such that the derivatives and integrals below exist, the following equalities hold true:*

$$
{}_{t_0}D_t^{\alpha}\, {}_{t_0}J_t^{\alpha} f(t) = f(t), \tag{2}
$$

$$
{}_{t_0}D_t^{(\alpha)}\, {}_{t_0}J_t^{\alpha} f(t) = f(t), \tag{3}
$$

$$
{}_{t_0}J_t^{\alpha}\, {}_{t_0}J_t^{\beta} f(t) = {}_{t_0}J_t^{\alpha+\beta} f(t). \tag{4}
$$

If, moreover, $\alpha < 1$, then

$$
{}_{t_0}D_t^{(\alpha)} f(t) = {}_{t_0}D_t^{\alpha} f(t) - f(t_0) \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}. \tag{5}
$$

Lemma 2. *For $\beta > 0$,*

$$
{}_{t_0}J_t^{\alpha} (t-t_0)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (t-t_0)^{\beta+\alpha-1}, \tag{6}
$$

$$
{}_{t_0}D_t^{\alpha} (t-t_0)^{\beta-1}
= \begin{cases} 0, & \beta \in \{\alpha - m + 1, \dots, \alpha\}, \\ \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (t-t_0)^{\beta-\alpha-1}, & \textit{otherwise}, \end{cases} \tag{7}
$$

$$
{}_{t_0}D_t^{(\alpha)} (t-t_0)^{\beta-1},
= \begin{cases} 0, & \beta \in \{1, 2 \dots, m\}, \\ \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (t-t_0)^{\beta-\alpha-1} & \beta > m, \\ \textit{non-existent,} & \textit{otherwise}, \end{cases} \tag{8}
$$

where $m = \lceil \alpha \rceil$ is the least integer greater than or equal to α.

In particular, from (6)–(8) it follows that

$$
{}_{t_0}D_t^{\alpha} \frac{(t-t_0)^{\alpha-1}}{\Gamma(\alpha)} = 0, \tag{9}
$$

$$
{}_{t_0}D_t^{(\alpha)} 1 = 0, \tag{10}
$$

$$
{}_{t_0}J_t^{1-\alpha} \frac{(t-t_0)^{\alpha-1}}{\Gamma(\alpha)} = 1. \tag{11}
$$

2.2. Time-varying linear systems with Riemann–Liouville fractional derivatives. Consider the following initial value problem:

$$
\begin{aligned}
{}_{t_0}D_t^{\alpha} x(t) &= A(t)x(t), \quad t \in \mathring{I}, \\
{}_{t_0}J_t^{1-\alpha} x(t)\big|_{t=t_0} &= x_0.
\end{aligned} \tag{12}
$$

Hereafter it is assumed that $x(t)$ is a vector function taking values in $\mathbb{R}^n$ and the matrix function $A(t) = (a_{ij}(t))_{i,j=1,\dots,n}$ such that $a_{ij}(t) : I \to \mathbb{R}$, $i = 1,\dots,n$, $j = 1,\dots,n$ are continuous and

$$a_{ij}(t) \geq 0, \quad i = 1,\dots,n, \quad j = 1,\dots,n, \quad t \in I. \tag{13}$$

Definition 1. The state-transition matrix of the system (12) is defined as follows:

$$\Phi(t,t_0) = \sum_{k=0}^{\infty} \Im_k^\alpha(t,t_0), \tag{14}$$

where

$$\Im_0^\alpha(t,t_0) = \frac{(t-t_0)^{\alpha-1}}{\Gamma(\alpha)}\mathbb{I},$$
$$\Im_{k+1}^\alpha(t,t_0) = {}_{t_0}J_t^\alpha(A(t)\Im_k^\alpha(t,t_0)), \quad k = 0,1,\dots.$$

Hereafter $\mathbb{I}$ stands for an identity matrix.

We will refer to the series on the right-hand side of (14) as the generalized Peano–Baker series (Eckert *et al.*, 2019; Baake and Schlägel, 2011).

Assumption 1. The generalized Peano–Baker series on the right-hand side of (14) converges uniformly.

The following lemma was presented by Matychyn (2019), as well as Matychyn and Onyshchenko (2020).

Lemma 3. *Under Assumption 1 the state-transition matrix $\Phi(t,t_0)$ satisfies the following initial value problem:*

$${}_{t_0}D_t^\alpha\Phi(t,t_0) = A(t)\Phi(t,t_0),\ {}_{t_0}J_t^{1-\alpha}\Phi(t,t_0)\big|_{t=t_0} = \mathbb{I}. \tag{15}$$

On the other hand, the following lemma also holds true.

Lemma 4. *Let the matrix function $\Phi(t,t_0)$ be a solution to the initial value problem* (15). *Then $\Phi(t,t_0)$ can be represented in the form of the generalized Peano–Baker series* (14).

Proof. As shown by Diethelm (2010), the initial value problem (15) is equivalent to the following Volterra integral equation:

$$\begin{aligned}\Phi(t,t_0) =& \mathbb{I}\frac{(t-t_0)^{\alpha-1}}{\Gamma(\alpha)} \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-\tau)^{\alpha-1}A(\tau)\Phi(\tau,t_0)\,\mathrm{d}\tau \\ =& \mathbb{I}\frac{(t-t_0)^{\alpha-1}}{\Gamma(\alpha)} + {}_{t_0}J_t^\alpha(A(t)\Phi(t,t_0)).\end{aligned} \tag{16}$$

By means of a formal Picard iteration, this leads to the desired representation in the form of the generalized Peano–Baker series (14). ∎

As shown by Bourdin (2018), there exists a unique solution to the initial value problem (15). Thus, in view of Dini's theorem (Zorich and Paniagua, 2016) as well as Lemmas 3 and 4, taking into account (13), this implies Assumption 1 holds true.

Consider the state transition matrix $\Phi(t,t_0) = (\varphi_{ij}(t,t_0))_{i,j=1,\dots,n}$. The following result was presented by Bourdin (2018).

Lemma 5. *There exists $\Theta \geq 0$ such that*

$$|\varphi_{ij}(t,s)| \leq (t-s)^{\alpha-1}\Theta \tag{17}$$

for almost every $t_0 \leq s < t \leq T$ and for every $i,j \in \{1,\dots,n\}$.

Lemma 3 implies the following result.

Theorem 1. *Under Assumption 1, the solution to the homogeneous initial value problem* (12) *is given by the following expression:*

$$x(t) = \Phi(t,t_0)\,x_0. \tag{18}$$

Remark 1. If $A(t)$ is a constant matrix, i.e., $A(t) \equiv A$, then in view of (6) one gets

$$\Im_k^\alpha(t,t_0) = \frac{(t-t_0)^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)}A^k$$

and

$$\begin{aligned}\Phi(t,t_0) &= e_\alpha^{(t-t_0)A} \\ &= (t-t_0)^{\alpha-1}\sum_{k=0}^{\infty}\frac{A^k(t-t_0)^{\alpha k}}{\Gamma[(k+1)\alpha]} \\ &= (t-t_0)^{\alpha-1}E_{\alpha,\alpha}(A(t-t_0)^\alpha),\end{aligned}$$

where $E_{\alpha,\alpha}(A(t-t_0)^\alpha)$ is a matrix Mittag-Leffler function and $e_\alpha^{(t-t_0)A}$ is the matrix α-exponential function (Kilbas *et al.*, 2006).

Equation (18) takes on the form

$$x(t) = e_\alpha^{(t-t_0)A}x_0,$$

which is consistent with the formulas obtained for systems of fractional differential equations with constant coefficients (Kilbas *et al.*, 2006; Matychyn and Onyshchenko, 2015).

Example 1. Consider a homogeneous system with fractional dynamics described by the equation

$${}_0D_t^\alpha x(t) = A(t)x(t), \quad t \in (0,T), \quad 0 < \alpha < 1, \tag{19}$$

under the initial condition

$${}_0J_t^{1-\alpha}x(t)\big|_{t=0} = x_0, \tag{20}$$

where $x \in \mathbb{R}^2$,

$$A(t) = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}.$$

Direct calculation yields

$$\Phi(t,\tau) = \begin{pmatrix} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} & \frac{\alpha(t-\tau)^{2\alpha-1}(t+\tau)}{\Gamma(2\alpha+1)} \\ 0 & \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \end{pmatrix}.$$

It can be readily seen that

$$\begin{aligned} {}_0D_t^\alpha \Phi(t,0) &= \begin{pmatrix} 0 & \frac{t^\alpha}{\Gamma(\alpha)} \\ 0 & 0 \end{pmatrix} = A(t)\Phi(t,0), \\ {}_0J_t^{1-\alpha} x(t)\big|_{t=0} \Phi(t,0) &= \begin{pmatrix} 1 & \frac{t^{\alpha+1}}{\Gamma(\alpha+1)} \\ 0 & 1 \end{pmatrix}\Bigg|_{t=0} = \mathbb{I}, \end{aligned}$$

hence Lemma 3 holds true.

Suppose that

$$x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Then, the solution of the initial value problem (19), (20) can be written as follows:

$$\begin{aligned} x(t) &= \begin{pmatrix} \frac{t^{\alpha-1}}{\Gamma(\alpha)} & \frac{\alpha}{\Gamma(2\alpha+1)} t^{2\alpha} \\ 0 & \frac{t^{\alpha-1}}{\Gamma(\alpha)} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{\alpha}{\Gamma(2\alpha+1)} t^{2\alpha} \\ \frac{t^{\alpha-1}}{\Gamma(\alpha)} \end{pmatrix}. \end{aligned}$$

♦

Now consider the inhomogeneous linear initial value problem

$$\begin{aligned} {}_{t_0}D_t^\alpha x(t) &= A(t)x(t) + u(t), \quad t \in \mathring{I}, & (21) \\ {}_{t_0}J_t^{1-\alpha} x(t)\big|_{t=t_0} &= x_0. & (22) \end{aligned}$$

We assume $u : I \to U \subset \mathbb{R}^n$ to be measurable on I, taking values from a nonempty compact set $U \subset \mathbb{R}^n$.

The following theorem was proved by Matychyn (2019), as well as Matychyn and Onyshchenko (2020).

Theorem 2. *Provided that Assumption 1 is fulfilled, a solution to the initial value problem* (21), (22) *can be written as follows:*

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^t \Phi(t,\tau)u(\tau)\,\mathrm{d}\tau. \quad (23)$$

Remark 2. Since the Mittag-Leffler function becomes equal to an exponential when $\alpha = 1$, i.e., $E_{1,1}(A(t-t_0)) = e_1^{A(t-t_0)} = e^{A(t-t_0)}$, it should be noted that for $A(t) \equiv A = \text{const}$ and $\alpha = 1$ one gets $\Phi(t,t_0) = e^{A(t-t_0)}$, where $e^{A(t-t_0)}$ is the matrix exponential defined as the sum of the following convergent series:

$$e^{A(t-t_0)} = \sum_{k=0}^\infty \frac{(t-t_0)^k}{k!} A^k,$$

and (23) yields the well-known explicit formula

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^t e^{A(t-\tau)}u(\tau)\,\mathrm{d}\tau$$

for the solution of the integer-order Cauchy problem

$$\begin{aligned} \dot{x} &= Ax + u, \\ x(t_0) &= x_0. \end{aligned}$$

2.3. Time-varying linear systems with Caputo fractional derivatives. We now examine homogeneous linear FDEs with variable coefficients involving Caputo derivatives. Consider the following initial value problem:

$$\begin{aligned} {}_{t_0}D_t^{(\alpha)} x(t) &= A(t)x(t), \quad t \in \mathring{I}, \\ x(t_0) &= \tilde{x}_0, \end{aligned} \quad (24)$$

where the matrix function $A(t)$ is continuous on I.

Definition 2. The state-transition matrix of the system (24) is defined as follows:

$$\Psi(t,t_0) = \sum_{k=0}^\infty \mathfrak{I}_k^\alpha(t,t_0) \quad (25)$$

where

$$\begin{aligned} \mathfrak{I}_0^\alpha(t,t_0) &= \mathbb{I}, \\ \mathfrak{I}_{k+1}^\alpha(t,t_0) &= {}_{t_0}J_t^\alpha(A(t)\mathfrak{I}_k^\alpha(t,t_0)), \quad k = 0, 1, \dots \end{aligned}$$

Again, we will refer to the series on the right-hand side of (25) as the generalized Peano–Baker series (Eckert *et al.*, 2019; Baake and Schlägel, 2011).

Assumption 2. The generalized Peano–Baker series on the right-hand side of (25) converges uniformly.

In view of Lemma 1 as well as (10), (11), the following lemma holds true.

Lemma 6. *Under Assumption 2 the state-transition matrix $\Psi(t,t_0)$ satisfies the following initial value problem:*

$${}_{t_0}D_t^{(\alpha)}\Psi(t,t_0) = A(t)\Psi(t,t_0), \quad \Psi(t_0,t_0) = \mathbb{I}.$$

Lemma 6 implies the following.

Theorem 3. *Under Assumption 2 a solution to the initial value problem* (24) *is given by the following expression:*

$$x(t) = \Psi(t,t_0)\tilde{x}_0. \quad (26)$$

Remark 3. If $A(t)$ is a constant matrix, i.e., $A(t) \equiv A$, then in view of (6) we get

$$\mathfrak{I}_k^\alpha(t,t_0) = \frac{(t-t_0)^{k\alpha}}{\Gamma(k\alpha+1)} A^k$$

and

$$\Psi(t,t_0) = E_\alpha((t-t_0)^\alpha A) = \sum_{k=0}^{\infty} \frac{A^k (t-t_0)^{\alpha k}}{\Gamma[k\alpha+1]},$$

where $E_\alpha(t^\alpha A) = E_{\alpha,1}(t^\alpha A)$.

Equation (26) takes on the form

$$x(t) = E_\alpha((t-t_0)^\alpha A)\tilde{x}_0,$$

which is consistent with the formulas obtained for the systems of fractional differential equations with constant coefficients (Kilbas *et al.*, 2006; Matychyn and Onyshchenko, 2015).

Example 2. Consider the system

$$\begin{aligned} {}_0D_t^{(\alpha)} x(t) &= \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} x(t), \\ x(0) &= \tilde{x}_0. \end{aligned} \tag{27}$$

Direct calculation yields

$$\Psi(t,\tau) = \begin{pmatrix} 1 & \frac{(t-\tau)^\alpha (t+\alpha\tau)}{\Gamma(\alpha+2)} \\ 0 & 1 \end{pmatrix}. \tag{28}$$

It can be readily seen that

$$\begin{aligned} {}_0D_t^{(\alpha)} \Psi(t,0) &= \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} = A(t)\Psi(t,0), \\ \Psi(0,0) &= \mathbb{I}. \end{aligned}$$

Hence Lemma 6 holds true.

Suppose that

$$\tilde{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Then, the solution of the initial value problem (28) can be written as follows:

$$x(t) = \begin{pmatrix} 1 & \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \\ 1 \end{pmatrix}.$$

♦

Consider the inhomogeneous linear initial value problem

$$\begin{aligned} {}_{t_0}D_t^{(\alpha)} x(t) &= A(t)x(t) + u(t), \quad t \in \mathring{I}, \\ x(t_0) &= \tilde{x}_0 \end{aligned} \tag{29}$$

Again, we assume $u : I \to \mathbb{R}^n$ to be continuous on I.

The following theorem was proved by Matychyn (2019).

Theorem 4. *Under Assumption 2 a solution to the initial value problem* (29) *can be written as follows:*

$$x(t) = \Psi(t,t_0)\tilde{x}_0 + \int_{t_0}^{t} \Phi(t,\tau)u(\tau)\,\mathrm{d}\tau. \tag{30}$$

2.4. Convex analysis. Denote by $\operatorname{co} X$ and $\overline{\operatorname{co}}\, X$ the convex hull and the closure of the convex hull of a set $X \subset \mathbb{R}^n$, respectively. Here we recall the definition of the support function. Let $M \subset \mathbb{R}^n$ be a convex closed set, i.e., $M = \overline{\operatorname{co}}\, M$. Then the function

$$\sigma_M(\psi) = \sup_{m \in M} (\psi, m), \quad \psi \in \mathbb{R}^n,$$

where $(\cdot,\cdot)$ stands for the scalar (inner) product, is called the support function of M. An important property of the support function is as follows:

$$m \in M \Leftrightarrow (\psi, m) \le \sigma_M(\psi), \quad \forall \psi \in \mathbb{R}^n. \tag{31}$$

Let us present a useful result of convex analysis (Matychyn and Onyshchenko, 2018a).

Lemma 7. *Let X and M be convex closed sets. Moreover, assume that X is bounded. Then $X \cap M = \emptyset$ if and only if there exist a vector $\psi \in \mathbb{R}^n$ and a number $\varepsilon > 0$ such that*

$$\sigma_X(\psi) + \sigma_M(-\psi) \le -\varepsilon. \tag{32}$$

Corollary 1. *Let $X = \overline{\operatorname{co}}\, X$, $M = \overline{\operatorname{co}}\, M$ and X be bounded. Then $X \cap M \neq \emptyset$ if and only if*

$$\lambda_{X,M} = \min_{\|\psi\|=1} [\sigma_X(\psi) + \sigma_M(-\psi)] \ge 0. \tag{33}$$

Let us recall the definition and some properties of the normal cone (Rockafellar, 1970).

The normal cone of a set $K \subset \mathbb{R}^n$ at the point x_0 is given by

$$N_K(x_0) = \left\{ \psi \in \mathbb{R}^n : \sup_{x \in K} (\psi, x - x_0) \le 0 \right\}.$$

The cone $N_K(x_0)$ is always nonempty since $0 \in N_K(x_0)$. The cone $N_K(x_0)$ is convex and closed. If there exists $\psi \in N_K(x_0)$, $\psi \neq 0$, then the hyperplane $\{x \in \mathbb{R}^n : (\psi, x - x_0) = 0\}$ is called the hyperplane supporting the set K at the point x_0, while x_0 is referred to as a support point. If the set $K \subset \mathbb{R}^n$ is closed and convex, then any boundary point $x_0 \in \partial K$ is a support point, i.e., there exists $\psi \in N_K(x_0)$, $\psi \neq 0$.

2.5. Set valued maps. Consider a set-valued map $G(\tau)$, $G : \mathcal{I} \to K(\mathbb{R}^n)$, where $\mathcal{I} \subset \mathbb{R}$ is some interval and $K(\mathbb{R}^n)$ is the set of all nonempty compacts (closed and bounded subsets of $\mathbb{R}^n$). The following theorems can be found in the work of Blagodatskikh and Filippov (1985) and are useful in integrating set-valued maps. In view of the properties of the Lebesgue integral, the interval $\mathcal{I}$ can be assumed either open or closed, without loss of generality. In what follows for any $A \in K(\mathbb{R}^n)$ we write $|A| = \sup_{a \in A} \|a\|$.

Theorem 5. *Let the set-valued map $G(\tau)$ be measurable and satisfy the inequality $|G(\tau)| \leq k(\tau)$, $\tau \in \mathcal{I}$, where $k(\tau)$ is some scalar valued function integrable over $\mathcal{I}$. Then the following equality holds:*

$$\sigma_{\int_{\mathcal{I}} G(\tau)d\tau}(\psi) = \int_{\mathcal{I}} \sigma_{G(\tau)}(\psi)\,\mathrm{d}\tau. \tag{34}$$

Theorem 6. *Let the set-valued map $G(\tau)$ be measurable and satisfy the inequality $|G(\tau)| \leq k(\tau)$, $\tau \in \mathcal{I}$, where $k(\tau)$ is some scalar valued function integrable over $\mathcal{I}$. Then the integral $\int_{\mathcal{I}} G(\tau)\,\mathrm{d}\tau$ is a convex compact set in $\mathbb{R}^n$.*

The integral $\int_{\mathcal{I}} G(\tau)\,\mathrm{d}\tau$ is to be thought of in the sense of Aumann (1965), i.e., as the set of integrals of all measurable selections of the set-valued map $G(\tau)$.

3. Time optimal control of fractional linear systems with variable coefficients (Riemann–Liouville case)

3.1. Problem statement. A system of linear FDEs with variable coefficients described by (21) and (22) can be considered a control system in state-space form. Since we do not deal with the subjects of controllability or observability in this paper, we assume for simplicity that the input matrix $B(t)$ is an identity matrix. Let us repeat the equations here for the sake of convenience:

$${}_{t_0}D_t^{\alpha} x(t) = A(t)x(t) + u(t), \quad t \in \mathring{I}, \tag{21 revisited}$$

subject to the initial condition

$${}_{t_0}J_t^{1-\alpha} x(t)\big|_{t=t_0} = x_0. \tag{22 revisited}$$

As before, we assume the matrix $A(t)$ to have continuous nonnegative components and $u(t)$ to be measurable on I, taking values from a nonempty compact set $U \subset \mathbb{R}^n$.

Let us fix a point $m \in \mathbb{R}^n$. Here we formulate the optimal control problem: Find a control function $u(\cdot)$, $u : I \to U$, from the class $\mathcal{U}(I)$ of measurable functions taking their values in a nonempty compact set U, $U \subset \mathbb{R}^n$, such that the corresponding trajectory of (21), (22) hits m in the shortest time t_*.

3.2. Reachable set. If we fix some admissible control function $u(\cdot) \in \mathcal{U}(I)$, then the solution to the initial value problem (21), (22) is given by (23). Consider the reachable set

$$\begin{aligned} R(t, x_0) &= \Big\{\Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)u(\tau)\,\mathrm{d}\tau : \\ &\qquad u(\cdot) \in \mathcal{U}(I)\Big\} \qquad (35) \\ &= \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)U\,\mathrm{d}\tau. \end{aligned}$$

Since the set U is bounded, there exists a number K, $0 < K < \infty$, such that $|U| \leq K$. Hence, in view of Lemma 5, $|\Phi(t, \tau)U| < k(\tau)$, $\tau \in (t_0, t)$, where $k(\tau) = (t - \tau)^{\alpha-1}K\Theta$. As can be easily verified, $k(\tau)$ is a scalar valued function integrable over I. Therefore, according to properties of integrals of set-valued maps, in view of Theorem 6, the reachable set $R(t, x_0)$ is closed and convex.

3.3. Necessary conditions of optimal control.

Theorem 7. *If $u^*(\cdot) \in \mathcal{U}(I)$ is a time optimal control with the minimum time $t_* > 0$, then for any $\psi \in N_{R(t_*,x_0)}(x(t_*))$ the following equality holds:*

$$(\Phi^*(t_*, t)\psi, u^*(t)) = \max_{v\in U}(\Phi^*(t_*, t)\psi, v), \tag{36}$$

where $\Phi^*(t_*, t)$ stands for the conjugate transpose of the matrix $\Phi(t_*, t)$, for a.e. $t \in [t_0, t_*]$.

Proof. Let $\psi \neq 0$ be an element of the normal cone of $R(t_*, x_0)$ at the point $x(t_*)$, i.e., $\psi \in N_{R(t_*,x_0)}(x(t_*))$. Then for any $x \in R(t_*, x_0)$ we have

$$(\psi, x - x(t_*)) \leq 0. \tag{37}$$

Let $u^*(\cdot)$ be the time optimal control function and $u(\cdot) \in \mathcal{U}$ be an admissible control that corresponds to $x \in R(t_*, x_0)$. In view of (23), (37) yields

$$\left(\psi, \int_{t_0}^{t_*} \Phi(t_*, \tau)[u(\tau) - u^*(\tau)]\,\mathrm{d}\tau\right) \leq 0.$$

Therefore, for any admissible control $u(\cdot) \in \mathcal{U}(I)$ we have

$$\int_{t_0}^{t_*} (\Phi^*(t_*, \tau)\psi, u^*(\tau) - u(\tau))\,\mathrm{d}\tau \geq 0. \tag{38}$$

Now, let us prove that

$$(\Phi^*(t_*, t)\psi, u^*(t)) = \max_{u\in U}(\Phi^*(t_*, t)\psi, u).$$

Obviously, it suffices to show that $(\Phi^*(t_*, t)\psi, u^*(t)) \geq \max_{u\in U}(\Phi^*(t_*, t)\psi, u)$. For each $k \in \mathbb{N}$ we introduce the set

$$Z_k = \Big\{t \in [t_0, t_*] : \\ (\Phi^*(t_*, t)\psi, u^*(t)) \leq \max_{u\in U}(\Phi^*(t_*, t)\psi, u) - \frac{1}{k}\Big\}.$$

Then we have

$$\begin{aligned} Z_k &= \Big\{t \in [t_0, t_*] : \\ &\qquad (\Phi^*(t_*, t)\psi, u^*(t)) < \max_{u\in U}(\Phi^*(t_*, t)\psi, u)\Big\} \\ &= \bigcup_{k\in\mathbb{N}} Z_k. \end{aligned}$$

Observe that the function $[t_0, t_*] \ni t \mapsto \max_{u\in U}(\Phi^*(t_*, t)\psi, u)$ is measurable. Thus, the set Z_k is measurable for any $k \in \mathbb{N}$. Let for some k the measure of Z_k be positive: $\mu(Z_k) = \varepsilon_k > 0$. Using theorems on measurable selections (Blagodatskikh and Filippov, 1985), one can show that there exists a measurable function $v : Z_k \to U$ such that $(\Phi^*(t_*, t)\psi, v(t)) = \max_{u\in U}(\Phi^*(t_*, t)\psi, u)$. Then, for $t \in Z_k$ we have

$$(\Phi^*(t_*, t)\psi, u^*(t)) \le (\Phi^*(t_*, t)\psi, v(t)) - \frac{1}{k}. \tag{39}$$

Consider the following control function:

$$u(t) = \begin{cases} u^*(t) & \text{for } t \in [t_0, t_*]\backslash Z_k, \\ v(t) & \text{for } t \in Z_k. \end{cases}$$

Then, $u(\cdot) \in \mathcal{U}(I)$ and, in view of (39),

$$\begin{aligned} &\int_{t_0}^{t_*} (\Phi^*(t_*, \tau)\psi, u^*(\tau) - u(\tau))\, \mathrm{d}\tau \\ &= \int_{Z_k} (\Phi^*(t_*, \tau)\psi, u^*(\tau) - v(\tau))\, \mathrm{d}\tau \le -\frac{\varepsilon_k}{k} < 0, \end{aligned}$$

which contradicts (38). ■

3.4. Sufficient condition for optimal control.

Consider the support function of the reachable set (35):

$$\begin{aligned} &\sigma_{R(t,x_0)}(\psi) \\ &= \sup_{x\in R(t,x_0)} (x, \psi) \\ &= \sup_{u(\cdot)\in\mathcal{U}(I)} \left\{(\psi, \Phi(t, t_0)x_0) + \int_{t_0}^{t} (\psi, \Phi(t, \tau)u(\tau))\, \mathrm{d}\tau\right\} \\ &= (\psi, \Phi(t, t_0)x_0) + \int_{t_0}^{t} \sigma_U(\Phi^*(t, \tau)\psi)\, \mathrm{d}\tau. \end{aligned} \tag{40}$$

Here we applied the property of the conjugate transpose matrix with the inner product and Theorem 5.

Introduce the function

$$\lambda(t, x_0) = \min_{\|\psi\|=1} [\sigma_{R(t,x_0)}(\psi) - (m, \psi)] \tag{41}$$

and set

$$T(x_0) = \min\{t > t_0 : \lambda(t, x_0) \ge 0\}. \tag{42}$$

Then the following theorem holds true.

Theorem 8. *The trajectory of the system* (21), (22) *can be brought to the point* m *at the minimal time* $t_* = T(x_0)$, *given by the formula* (42), *with the help of a control function of the form*

$$\hat{u}(\tau) = \arg\max_{v\in U}(\Phi^*(t_*, \tau)\hat{\psi}, v), \tag{43}$$

where

$$\hat{\psi} = \arg\min_{\|\psi\|=1} [\sigma_{R(t_*,x_0)}(\psi) - (\psi, m)].$$

Proof. Observe that the minimal time at which a trajectory of (21), (22) hits the terminal point m is given by $t_* = \min\{t \ge 0 : m \in R(t, x_0)\}$. Here the minimum is attained due to the closedness of $R(t, x_0)$.

Moreover, m is a boundary point of $R(t_*, x_0)$, i.e. $m \in \partial R(t_*, x_0)$. The separation theorem (Rockafellar, 1970) implies that there exists a supporting hyperplane

$$H(\hat{\psi}) = \{x \in \mathbb{R}^n : (\hat{\psi}, x) = \sigma_{R(t_*,x_0)}(\hat{\psi})\} \tag{44}$$

passing through m.

Hence, for some $\hat{\psi}$,

$$(\hat{\psi}, m) = \sigma_{R(t_*,x_0)}(\hat{\psi}). \tag{45}$$

Thus, the control function $\hat{u}(\cdot)$ that ensures bringing the trajectory of (21), (22) to the point m at time t_* is the function at which the maximum in (40) is attained. Therefore, it must satisfy

$$\hat{u}(\tau) = \arg\max_{v\in U}(\Phi^*(t_*, \tau)\psi, v),\ \tau \in [0, t_*].$$

According to Corollary 1, $m \in R(t, x_0)$ if and only if $\lambda(t, x_0) \ge 0$; hence, $t_* = T(x_0) = \min\{t \ge t_0 : \lambda(t, x_0) \ge 0\}$. Since $\lambda(t_*, x_0) \ge 0$, in virtue of (45), $\hat{\psi}$ yields the minimum of the expression $\sigma_{R(t_*,x_0)}(\psi) - (\psi, m)$. ■

Example 3. Let us illustrate the above theoretical results with the following example. Consider a system with fractional dynamics described by the equation

$${}_0D_t^{\alpha}x(t) = A(t)x(t) + u(t), \quad 0 < \alpha < 1, \tag{46}$$

subject to the initial condition

$${}_0J_t^{1-\alpha}x(t)\big|_{t=0} = x_0 = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}, \tag{47}$$

where $x, u \in \mathbb{R}^2$ and

$$A(t) = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}.$$

As shown in Example 1, the state-transition matrix has the form

$$\begin{aligned} \Phi(t, \tau) &= \begin{pmatrix} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} & \frac{\alpha(t-\tau)^{2\alpha-1}(t+\tau)}{\Gamma(2\alpha+1)} \\ 0 & \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \end{pmatrix} \\ &= \begin{pmatrix} \phi_1(t, \tau) & \phi_2(t, \tau) \\ 0 & \phi_1(t, \tau) \end{pmatrix}, \end{aligned}$$

where

$$\phi_1(t, \tau) = \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)},$$

$$\phi_2(t, \tau) = \frac{\alpha(t-\tau)^{2\alpha-1}(t+\tau)}{\Gamma(2\alpha+1)}.$$

Let us assume that $t_0 = 0$, the terminal point is the origin: $m = 0$, and $U = \{(u_1, u_2)^T \in \mathbb{R}^2 : \ u_1 = 0, \ |u_2| \le 1\}$.

Consider a support function of the reachable set (35):

$$\begin{aligned}
&\sigma_{R(t,x_0)}(\psi) \\
&= \sup_{x\in R(t,x_0)} (x,\psi) \\
&= \sup_{u(\cdot)\in\mathcal{U}(I)} \left\{ (\psi, \Phi(t,0)x_0) + \int_0^t (\psi, \Phi(t,\tau)u(\tau))\,\mathrm{d}\tau \right\} \\
&= (\psi, \Phi(t,0)x_0) + \int_0^t \sigma_U(\Phi^*(t,\tau)\psi)\,\mathrm{d}\tau \\
&= (\psi, \Phi(t,0)x_0) + \int_0^t |\phi_2(t,\tau)\psi_1 + \phi_1(t,\tau)\psi_2|\,\mathrm{d}\tau.
\end{aligned}$$

Then

$$\lambda(t,x_0) = \min_{\|\psi\|=1} \sigma_{R(t,x_0)}(\psi)$$

and

$$t_* = T(x_0) = \min\{t > 0 : \ \lambda(t,x_0) \ge 0\}.$$

Suppose that

$$\hat\psi = \arg\min_{\|\psi\|=1} [\sigma_{R(t_*,x_0)}(\psi)].$$

The optimal control is

$$\hat u(\tau) = \begin{pmatrix} 0 \\ \upsilon(\tau) \end{pmatrix},$$

where, according to (43),

$$\upsilon(\tau) = \arg\max_{|\upsilon|\le 1} (\phi_2(t_*,\tau)\hat\psi_1 + \phi_1(t_*,\tau)\hat\psi_2)\upsilon.$$

Thus,

$$\upsilon(\tau) = \begin{cases} -1, & \phi_2(t_*,\tau)\hat\psi_1 + \phi_1(t_*,\tau)\hat\psi_2 < 0 \\ 1, & \phi_2(t_*,\tau)\hat\psi_1 + \phi_1(t_*,\tau)\hat\psi_2 \ge 0 \end{cases}$$

and we have the control of a "bang-bang" type.

Let us assume that the expression $\phi_2(t_*,\tau)\hat\psi_1 + \phi_1(t_*,\tau)\hat\psi_2$ changes its sign only once at $\tau = t_1$. Then

$$\upsilon(\tau) = \begin{cases} -1, & \tau \le t_1, \\ 1, & t_1 < \tau \le t_* \end{cases}$$

and, in view of (23), we have

$$\begin{aligned}
&x(t_*) \\
&= \Phi(t_*,0)x_0 - \int_0^{t_1} \Phi(t_*,\tau)(0,\ 1)^T\,\mathrm{d}\tau \\
&\quad + \int_{t_1}^{t_*} \Phi(t_*,\tau)(0,\ 1)^T\,\mathrm{d}\tau \\
&= \begin{pmatrix} \frac{x_1^0 t_*^{\alpha-1}}{\Gamma(\alpha)} + \frac{\alpha x_2^0 t_*^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2(t_*-t_1)^{2\alpha}(\alpha(t_*+t_1)+t_*)-(\alpha+1)t_*^{2\alpha+1}}{\Gamma(2(\alpha+1))} \\ \frac{x_2^0 t_*^{\alpha-1}}{\Gamma(\alpha)} + \frac{2(t_*-t_1)^{\alpha}-t_*^{\alpha}}{\Gamma(\alpha+1)} \end{pmatrix}
\end{aligned}$$

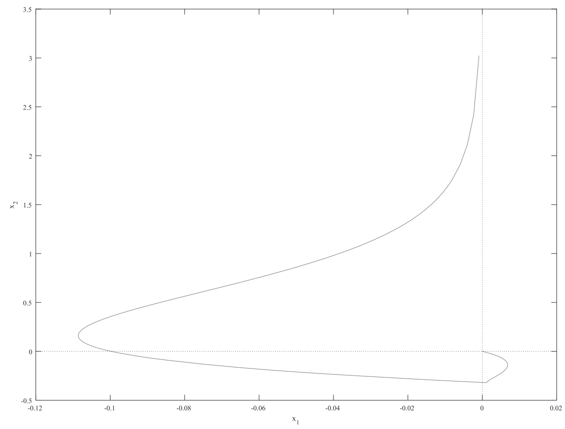

Fig. 1. Time-optimal phase trajectory of (46), (47) for $\alpha = 0.7$, $x_1^0 = 0$, $x_2^0 = 1$.

Since $x(t_*) = 0$, we obtain simultaneous equations:

$$\begin{cases} \frac{x_1^0 t_*^{\alpha-1}}{\Gamma(\alpha)} + \frac{\alpha x_2^0 t_*^{2\alpha}}{\Gamma(2\alpha+1)} \\ \quad + \frac{2(t_*-t_1)^{2\alpha}(\alpha(t_*+t_1)+t_*)-(\alpha+1)t_*^{2\alpha+1}}{\Gamma(2(\alpha+1))} = 0, \\ \frac{x_2^0 t_*^{\alpha-1}}{\Gamma(\alpha)} + \frac{2(t_*-t_1)^{\alpha}-t_*^{\alpha}}{\Gamma(\alpha+1)} = 0, \end{cases} \tag{48}$$

from which t_* and t_1 can be derived.

For example, let $x_1^0 = 0$, $x_2^0 = 1$, and $\alpha = 0.7$. Then, from (48) we find that $t_1 \approx 0.99014$ and $t_* \approx 1.08052$. Figure 1 shows the respective phase trajectory of (46), (47). The axes correspond to the components of the phase vector $x(t)$. The point of non-smoothness indicates control switching at the time instant t_1.

♦

4. Time optimal control of fractional linear systems with variable coefficients (Caputo case)

4.1. Problem statement. Consider a system of linear FDEs with variable coefficients:

$$ {}_{t_0}D_t^{(\alpha)} x(t) = A(t)x(t) + u(t), \quad t \in \mathring{I}, \tag{49}$$

subject to the initial condition

$$x(t_0) = \tilde x_0. \tag{50}$$

As before, we assume the matrix $A(t)$ to have continuous components and $u(t)$ to be measurable on I, taking values from a nonempty compact set $U \subset \mathbb{R}^n$.

Again, we fix a point $m \in \mathbb{R}^n$ and formulate the optimal control problem: Find a control function $u(\cdot)$, $u : I \to U$, from the class $\mathcal{U}(I)$ of measurable functions taking their values in a nonempty compact set $U, U \subset \mathbb{R}^n$, such that the corresponding trajectory of (49), (50) hits m in the shortest time t_*.

4.2. Reachable set. If we fix some admissible control function $u(\cdot) \in \mathcal{U}(I)$, then the solution to the initial value problem (49), (50) is given by (30). Consider the reachable set

$$\tilde{R}(t,\tilde{x}_0) = \left\{ \Psi(t,t_0)\tilde{x}_0 + \int_{t_0}^{t} \Phi(t,\tau)u(\tau)\,\mathrm{d}\tau : \right.$$
$$\left. u(\cdot) \in \mathcal{U}(I) \right\} \tag{51}$$
$$= \Psi(t,t_0)\tilde{x}_0 + \int_{t_0}^{t} \Phi(t,\tau)U\,\mathrm{d}\tau.$$

As before, since the set U is bounded, in view of Lemma 5 and Theorem 6, the reachable set $\tilde{R}(t,\tilde{x}_0)$ is closed and convex.

4.3. Necessary conditions for optimal control.

Theorem 9. *If $u^*(\cdot) \in \mathcal{U}(I)$ is a time optimal control with the minimum time $t_* > 0$, then for any $\psi \in N_{\tilde{R}(t_*,\tilde{x}_0)}(x(t_*))$ the following equality holds:*

$$(\Phi^*(t_*,t)\psi, u^*(t)) = \max_{v\in U}(\Phi^*(t_*,t)\psi, v), \tag{52}$$

where $\Phi^*(t_*,t)$ stands for the conjugate transpose of the matrix $\Phi(t_*,t)$, for a.e. $t \in [t_0,t_*]$.

The proof of this theorem is similar to that of Theorem 7.

4.4. Sufficient condition for optimal control. Consider a support function of the reachable set (51):

$$\begin{aligned}
&\sigma_{\tilde{R}(t,\tilde{x}_0)}(\psi) \\
&= \sup_{x\in\tilde{R}(t,\tilde{x}_0)} (x,\psi) \\
&= \sup_{u(\cdot)\in\mathcal{U}(I)} \left\{ (\psi, \Psi(t,t_0)\tilde{x}_0) + \int_{t_0}^{t} (\psi, \Phi(t,\tau)u(\tau))\,\mathrm{d}\tau \right\} \\
&= (\psi, \Psi(t,t_0)\tilde{x}_0) + \int_{t_0}^{t} \sigma_U(\Phi^*(t,\tau)\psi)\,\mathrm{d}\tau.
\end{aligned} \tag{53}$$

Here we applied the property of the conjugate transpose matrix with inner product and Theorem 5.

Let us introduce the function

$$\tilde{\lambda}(t,\tilde{x}_0) = \min_{\|\psi\|=1}[\sigma_{\tilde{R}(t,\tilde{x}_0)}(\psi) - (m,\psi)] \tag{54}$$

and write

$$\tilde{T}(\tilde{x}_0) = \min\{t > t_0 : \tilde{\lambda}(t,\tilde{x}_0) \geq 0\}. \tag{55}$$

Then the following theorem holds true.

Theorem 10. *The trajectory of the system* (49), (50) *can be brought to the point m at the minimal time $t_* = \tilde{T}(\tilde{x}_0)$, given by the formula* (55)*, with the help of a control function of the form*

$$\hat{u}(\tau) = \arg\max_{v\in U}(\Phi^*(t_*,\tau)\hat{\psi}, v), \tag{56}$$

where

$$\hat{\psi} = \arg\min_{\|\psi\|=1}[\sigma_{\tilde{R}(t_*,\tilde{x}_0)}(\psi) - (\psi,m)].$$

Proof. Let us observe that the minimal time at which a trajectory of (49), (50) hits the terminal point m is given by $t_* = \min\{t \geq 0 : m \in \tilde{R}(t,\tilde{x}_0)\}$. Here the minimum is attained due to the closedness of $\tilde{R}(t,\tilde{x}_0)$.

Moreover, m is a boundary point of $\tilde{R}(t_*,\tilde{x}_0)$, i.e., $m \in \partial\tilde{R}(t_*,\tilde{x}_0)$. The separation theorem (Rockafellar, 1970) implies that there exists a supporting hyperplane

$$H(\hat{\psi}) = \{x \in \mathbb{R}^n : (\hat{\psi}, x) = \sigma_{\tilde{R}(t_*,\tilde{x}_0)}(\hat{\psi})\} \tag{57}$$

passing through m.

Hence, for some $\hat{\psi}$,

$$(\hat{\psi}, m) = \sigma_{\tilde{R}(t_*,\tilde{x}_0)}(\hat{\psi}). \tag{58}$$

Thus, the control function $\hat{u}(\cdot)$ that ensures bringing the trajectory of (49), (50) to the point m at time t_* is the function at which the maximum in (53) is attained. Therefore it must satisfy

$$\hat{u}(\tau) = \arg\max_{v\in U}(\Phi^*(t_*,\tau)\psi, v), \quad \tau \in [0,t_*].$$

According to Corollary 1, $m \in \tilde{R}(t,\tilde{x}_0)$ if and only if $\tilde{\lambda}(t,\tilde{x}_0) \geq 0$, hence $t_* = \tilde{T}(\tilde{x}_0) = \min\{t \geq t_0 : \tilde{\lambda}(t,\tilde{x}_0) \geq 0\}$. Since $\tilde{\lambda}(t_*,\tilde{x}^0) \geq 0$, in virtue of (58), $\hat{\psi}$ yields the minimum of the expression $\sigma_{\tilde{R}(t_*,\tilde{x}_0)}(\psi) - (\psi,m)$. ■

Example 4. Let us illustrate the above theoretical results with the following example. Consider a system with fractional dynamics described by the equation

$${}_0D_t^{(\alpha)}x(t) = A(t)x(t) + u(t), \quad 0 < \alpha < 1, \tag{59}$$

under the initial condition

$$x(t_0) = \tilde{x}_0 = \begin{pmatrix} \tilde{x}_1^0 \\ \tilde{x}_2^0 \end{pmatrix}, \tag{60}$$

where $x, u \in \mathbb{R}^2$,

$$A(t) = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}.$$

As shown in Examples 1 and 2, the matrices $\Psi(t,\tau)$ and $\Phi(t,\tau)$ are of the form

$$\Psi(t,\tau)=\begin{pmatrix}1 & \frac{(t-\tau)^{\alpha}(t+\alpha\tau)}{\Gamma(\alpha+2)}\\ 0 & 1\end{pmatrix},$$

$$\Phi(t,\tau)=\begin{pmatrix}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} & \frac{\alpha(t-\tau)^{2\alpha-1}(t+\tau)}{\Gamma(2\alpha+1)}\\ 0 & \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\end{pmatrix} = \begin{pmatrix}\phi_1(t,\tau) & \phi_2(t,\tau)\\ 0 & \phi_1(t,\tau)\end{pmatrix},$$

where

$$\phi_1(t,\tau)=\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)},$$
$$\phi_2(t,\tau)=\frac{\alpha(t-\tau)^{2\alpha-1}(t+\tau)}{\Gamma(2\alpha+1)}.$$

Let us assume that the terminal point is the origin: $m=0$, and $U=\{(u_1,u_2)^T\in\mathbb{R}^2:\ u_1=0,\ |u_2|\le 1\}$.

Consider a support function of the reachable set (51):

$$\begin{aligned}
&\sigma_{\tilde R(t,\tilde x_0)}(\psi)\\
&=\sup_{x\in\tilde R(t,\tilde x_0)}(x,\psi)\\
&=\sup_{u(\cdot)\in\mathcal{U}(I)}\left\{(\psi,\Psi(t,0)\tilde x_0)+\int_0^t(\psi,\Phi(t,\tau)u(\tau))\,\mathrm{d}\tau\right\}\\
&=(\psi,\Psi(t,0)\tilde x_0)+\int_0^t\sigma_U(\Phi^*(t,\tau)\psi)\,\mathrm{d}\tau\\
&=(\psi,\Psi(t,0)\tilde x_0)+\int_0^t|\phi_2(t,\tau)\psi_1+\phi_1(t,\tau)\psi_2|\,\mathrm{d}\tau.
\end{aligned}$$

Then

$$\tilde\lambda(t,\tilde x_0)=\min_{\|\psi\|=1}\sigma_{\tilde R(t,\tilde x_0)}(\psi)$$

and

$$t_*=\tilde T(\tilde x_0)=\min\{t>0:\ \tilde\lambda(t,\tilde x_0)\ge 0\}.$$

Suppose that

$$\hat\psi=\arg\min_{\|\psi\|=1}[\sigma_{\tilde R(t_*,\tilde x_0)}(\psi)].$$

The optimal control is

$$\hat u(\tau)=\begin{pmatrix}0\\ \upsilon(\tau)\end{pmatrix},$$

where, according to (56),

$$\upsilon(\tau)=\arg\max_{|\upsilon|\le 1}(\phi_2(t_*,\tau)\hat\psi_1+\phi_1(t_*,\tau)\hat\psi_2)\upsilon.$$

Thus,

$$\upsilon(\tau)=\begin{cases}-1, & \phi_2(t_*,\tau)\hat\psi_1+\phi_1(t_*,\tau)\hat\psi_2<0,\\ 1, & \phi_2(t_*,\tau)\hat\psi_1+\phi_1(t_*,\tau)\hat\psi_2\ge 0,\end{cases}$$

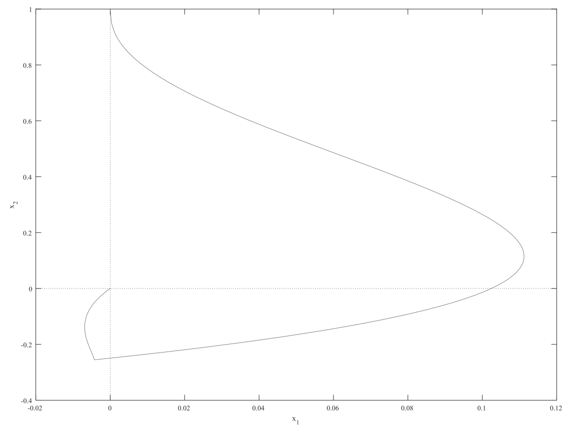

Fig. 2. Time-optimal phase trajectory of (59), (60) for $\alpha=0.7$, $x_1^0=0$, $x_2^0=1$.

and we have a control of the "bang-bang" type.

Let us assume that the expression $\phi_2(t_*,\tau)\hat\psi_1+\phi_1(t_*,\tau)\hat\psi_2$ changes its sign only once at $\tau=t_1$. Then

$$\upsilon(\tau)=\begin{cases}-1, & \tau\le t_1,\\ 1, & t_1<\tau\le t_*\end{cases}$$

and, in view of (30), we have

$$\begin{aligned}
x(t_*)&=\Psi(t_*,0)\tilde x_0-\int_0^{t_1}\Phi(t_*,\tau)(0,\ 1)^T\,\mathrm{d}\tau\\
&\quad+\int_{t_1}^{t_*}\Phi(t_*,\tau)(0,\ 1)^T\,\mathrm{d}\tau\\
&=\begin{pmatrix}\frac{x_1^0t_*^{\alpha-1}}{\Gamma(\alpha)}+\frac{\alpha x_2^0t_*^{2\alpha}}{\Gamma(2\alpha+1)}\\ \quad+\frac{2(t_*-t_1)^{2\alpha}(\alpha(t_*+t_1)+t_*)-(\alpha+1)t_*^{2\alpha+1}}{\Gamma(2(\alpha+1))}\\ \frac{x_2^0t_*^{\alpha-1}}{\Gamma(\alpha)}+\frac{2(t_*-t_1)^{\alpha}-t_*^{\alpha}}{\Gamma(\alpha+1)}\end{pmatrix}
\end{aligned}$$

Since $x(t_*)=0$, we obtain simultaneous equations:

$$\begin{cases}\frac{x_1^0t_*^{\alpha-1}}{\Gamma(\alpha)}+\frac{\alpha x_2^0t_*^{2\alpha}}{\Gamma(2\alpha+1)}\\ \quad+\frac{2(t_*-t_1)^{2\alpha}(\alpha(t_*+t_1)+t_*)-(\alpha+1)t_*^{2\alpha+1}}{\Gamma(2(\alpha+1))}=0,\\ \frac{x_2^0t_*^{\alpha-1}}{\Gamma(\alpha)}+\frac{2(t_*-t_1)^{\alpha}-t_*^{\alpha}}{\Gamma(\alpha+1)}=0,\end{cases}\tag{61}$$

from which t_* and t_1 can be derived.

For example, let $\tilde x_1^0=0$, $\tilde x_2^0=1$, and $\alpha=0.7$. Then, from (61) we find that $t_1\approx 0.99014$ and $t_*\approx 1.08052$. Figure 2 shows the respective phase trajectory of (59), (60). The axes correspond to the components of the phase vector $x(t)$. The point of non-smoothness indicates control switching at the time instant t_1. ♦

5. Conclusions

A time-optimal control problem for linear non-stationary systems of FDEs was investigated. The cases of both Riemann–Liouville and Caputo type derivatives were considered. Necessary and sufficient conditions for an optimal control were derived in terms of convex reachable sets and their support functions. Theoretical results were illustrated by examples, in which optimal control functions of the "bang-bang" type were constructed and optimal trajectories obtained. For future work, the authors intend to extend these results to fractional linear systems involving broader class of matrices than those considered in the examples of the present paper and satisfying the condition (13).

References

Aumann, R.J. (1965). Integrals of set-valued functions, *Journal of Mathematical Analysis and Applications* **12**(1): 1–12.

Baake, M. and Schlägel, U. (2011). The Peano–Baker series, *Proceedings of the Steklov Institute of Mathematics* **275**(1): 155–159.

Balaska, H., Ladaci, S., Djouambi, A., Schulte, H. and Bourouba, B. (2020). Fractional order tube model reference adaptive control for a class of fractional order linear systems, *International Journal of Applied Mathematics and Computer Science* **30**(3): 501–515, DOI: 10.34768/amcs-2020-0037.

Bergounioux, M. and Bourdin, L. (2020). Pontryagin maximum principle for general Caputo fractional optimal control problems with Bolza cost and terminal constraints, *ESAIM: Control, Optimisation and Calculus of Variations* **26**: 35, DOI: 10.1051/cocv/2019021.

Blagodatskikh, V.I. and Filippov, A.F. (1985). Differential inclusions and optimal control, *Trudy Matematicheskogo Instituta Imeni VA Steklova* **169**: 194–252.

Bourdin, L. (2018). Cauchy–Lipschitz theory for fractional multi-order dynamics: State-transition matrices, Duhamel formulas and duality theorems, *Differential and Integral Equations* **31**(7/8): 559–594.

Chikrii, A. and Eidelman, S. (2000). Generalized Mittag-Leffler matrix functions in game problems for evolutionary equations of fractional order, *Cybernetics and System Analysis* **36**(3): 315–338.

Chikrii, A. and Matichin, I. (2008). Presentation of solutions of linear systems with fractional derivatives in the sense of Riemann–Liouville, Caputo, and Miller–Ross, *Journal of Automation and Information Sciences* **40**(6): 1–11.

Datsko, B. and Gafiychuk, V. (2018). Complex spatio-temporal solutions in fractional reaction-diffusion systems near a bifurcation point, *Fractional Calculus and Applied Analysis* **21**(1): 237–253.

Datsko, B., Podlubny, I. and Povstenko, Y. (2019). Time-fractional diffusion-wave equation with mass absorption in a sphere under harmonic impact, *Mathematics* **7**(5): 433.

Diethelm, K. (2010). *The Analysis of Fractional Differential Equations*, Springer, Berlin/Heidelberg.

Dzieliński, A. and Czyronis, P. (2013). Fixed final time and free final state optimal control problem for fractional dynamic systems—Linear quadratic discrete-time case, *Bulletin of the Polish Academy of Sciences: Technical Sciences* **61**(3): 681–690.

Eckert, M., Nagatou, K., Rey, F., Stark, O. and Hohmann, S. (2019). Solution of time-variant fractional differential equations with a generalized Peano–Baker series, *IEEE Control Systems Letters* **3**(1): 79–84.

Kaczorek, T. (2008). Fractional positive continuous-time linear systems and their reachability, *International Journal of Applied Mathematics and Computer Science* **18**(2): 223–228, DOI: 10.2478/v10006-008-0020-0.

Kaczorek, T. and Idczak, D. (2017). Cauchy formula for the time-varying linear systems with Caputo derivative, *Fractional Calculus and Applied Analysis* **20**(2): 494–505.

Kaczorek, T. and Rogowski, K. (2015). *Fractional Linear Systems and Electrical Circuits*, Springer, Cham.

Kamocki, R. (2014). Pontryagin maximum principle for fractional ordinary optimal control problems, *Mathematical Methods in the Applied Sciences* **37**(11): 1668–1686.

Kilbas, A.A., Srivastava, H.M. and Trujillo, J.J. (2006). *Theory and Applications of Fractional Differential Equations*, Elsevier Science Limited, Amsterdam.

Li, Y., Chen, Y. and Podlubny, I. (2010). Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag-Leffler stability, *Computers and Mathematics with Applications* **59**(5): 1810–1821.

Lorenzo, C.F. and Hartley, T.T. (2000). Initialized fractional calculus, *International Journal of Applied Mathematics* **3**(3): 249–265.

Luchko, Y. (2009). Maximum principle for the generalized time-fractional diffusion equation, *Journal of Mathematical Analysis and Applications* **351**(1): 218–223.

Malesza, W., Macias, M. and Sierociuk, D. (2019). Analytical solution of fractional variable order differential equations, *Journal of Computational and Applied Mathematics* **348**: 214–236.

Martínez, L., Rosales, J., Carreño, C. and Lozano, J. (2018). Electrical circuits described by fractional conformable derivative, *International Journal of Circuit Theory and Applications* **46**(5): 1091–1100.

Matychyn, I. (2019). Analytical solution of linear fractional systems with variable coefficients involving Riemann–Liouville and Caputo derivatives, *Symmetry* **11**(11): 1366.

Matychyn, I. and Onyshchenko, V. (2015). Time-optimal control of fractional-order linear systems, *Fractional Calculus and Applied Analysis* **18**(3): 687–696.

Matychyn, I. and Onyshchenko, V. (2018a). On time-optimal control of fractional-order systems, *Journal of Computational and Applied Mathematics* **339**: 245–257.

Matychyn, I. and Onyshchenko, V. (2018b). Optimal control of linear systems with fractional derivatives, *Fractional Calculus and Applied Analysis* **21**(1): 134–150.

Matychyn, I. and Onyshchenko, V. (2019). Optimal control of linear systems of arbitrary fractional order, *Fractional Calculus and Applied Analysis* **22**(1): 170–179.

Matychyn, I. and Onyshchenko, V. (2020). Solution of linear fractional order systems with variable coefficients, *Fractional Calculus and Applied Analysis* **23**(3): 753–763.

Podlubny, I. (1998). *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, Academic Press, San Diego.

Rockafellar, R.T. (1970). *Convex Analysis*, Princeton University Press, Princeton.

Si, X., Yang, H. and Ivanov, I.G. (2021). Conditions and a computation method of the constrained regulation problem for a class of fractional-order nonlinear continuous-time systems, *International Journal of Applied Mathematics and Computer Science* **31**(1): 17–28, DOI: 10.34768/amcs-2021-0002.

Sierociuk, D. and Dzieliński, A. (2006). Fractional Kalman filter algorithm for the states, parameters and order of fractional system estimation, *International Journal of Applied Mathematics and Computer Science* **16**(1): 129–140.

Skovranek, T., Macias, M., Sierociuk, D., Malesza, W., Dzielinski, A., Podlubny, I., Pocsova, J. and Petras, I. (2019). Anomalous diffusion modeling using ultracapacitors in domino ladder circuit, *Microelectronics Journal* **84**: 136–141.

Zorich, V.A. and Paniagua, O. (2016). *Mathematical Analysis II*, Springer, Berlin/Heidelberg.

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