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QUEUEING SYSTEMS WITH RANDOM VOLUME CUSTOMERS AND A SECTORIZED UNLIMITED MEMORY BUFFER

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In the present paper, we concentrate on basic concepts connected with the theory of queueing systems with random volume customers and a sectorized unlimited memory buffer. In such systems, the arriving customers are additionally characterized by a non-negative random volume vector. The vector's indications can be understood as the sizes of portions of information of a different type that are located in the sectors of memory space of the system during customers' sojourn in it. This information does not change while a customer is present in the system. After service termination, information immediately leaves the buffer, releasing its resources. In analyzed models, the service time of a customer is assumed to be dependent on his volume vector characteristics, which has influence on the total volume vector distribution. We investigate three types of such queueing systems: the Erlang queueing system, the single-server queueing system with unlimited queue and the egalitarian processor sharing system. For these models, we obtain a joint distribution function of the total volume vector in terms of Laplace (or Laplace–Stieltjes) transforms and formulae for steady-state initial mixed moments of the analyzed random vector, in the case when the memory buffer is composed of two sectors. We also calculate these characteristics for some practical case in which the service time of a customer is proportional to the customer's length (understood as the sum of the volume vector's indications). Moreover, we present some numerical computations illustrating theoretical results.

Keywords: queueing systems with random volume customers, sectorized memory buffer, total volume vector, Laplace and Laplace–Stieltjes transforms, multi-variate L'Hospital rule.

1. Introduction

In the classical queueing theory, we usually assume that arriving customers are homogeneous, which means that they differ only in arrival times. Their basic characteristics having influence on system behavior are the same. For example, each customer's service time has the same distribution. This assumption is present in the analysis of the classical well-known queueing systems: $M/M/n/m$, $M/G/n/0$, $M/G/1/\infty$ and the single-server queueing system with processor sharing $M/G/1/\infty$–*EPS* (Bocharov *et al.*, 2004; Yashkov and Yashkova, 2007). For that reason, investigating such models is less complicated, but on the other hand they often cannot be applied in real systems.

In many modern telecommunication or computer systems, customers must be considered non-homogeneous. They may come from different

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sources, have different service time characteristics and different priorities, need more than one server to service or own other random requirements, which makes analysis more complex but lets us efficiently use introduced models in practice. In recent years, many researchers from different countries have analyzed analogous models. Non-homogenity of the arriving customers appeared, e.g., in the works of Shun-Chen (1980), Boxma (1989), Guo and Zipkin (2007), Kim and Ward (2013), Dudin *et al.* (2013), or Fiems and De Vuyst (2018).

In some papers, authors assume that customers have some random volume (size), which is connected with the fact that they deliver some portion of information that is integrally stored in a system's memory buffer until their service termination. These models (in which customers can also be treated as non-homogeneous) are called queueing systems with random volume customers. Analysis of such queueing systems is rather a novel direction in applied mathematics.

In real systems, a customer's service time usually depends on his volume (the size of the portion of information he delivers). The purpose of research in this case is much wider than in the classical one—we want to obtain not only characteristics of the number of customers present in the system at fixed time moment t (or in steady state, if it exists), but also characteristics of the total volume of customers (the sum of the volumes of all customers present in the system), as well as loss characteristics in the case of a memory buffer size limitation (in this case, an arriving customer is accepted for service if there are free servers or waiting positions in the queue and his volume is not too big; otherwise, it is lost). Analogous models were previously investigated by the tools of classical queueing theory (Schwarz, 1977; 1987), but the results of simulations showed that the obtained theoretical results (connected mainly with total volume characteristics) were not correct because they did not take into account a possible dependence between a customer's volume and his service time.

In the last decades of the twentieth century first papers appeared that investigated models with random volume customers using some extended methods (Alexandrov and Kaz, 1973; Sengupta, 1984). In the last years many articles have been written that analyze models with non-homogeneous customers. Their popularity and number of applications increase mainly because of the headway in computer science, which is a basic discipline of their existence. Interesting analyses are presented, e.g., by Juneja *et al.* (2012), Naumov *et al.* (2015; 2016), Naumov and Samuilov (2018), Samouylov *et al.* (2015; 2017; 2018), Lisovskaya *et al.* (2017; 2018), Zhernovyi and Kopytko (2016), Matalytsky and Zając (2019), Cascone *et al.* (2010), Rumyantsev and Morozov (2017), Kerobyan *et al.* (2018), or Nowak *et al.* (2020). It is worth highlighting that, in the cited papers, the service time of a customer is still often treated as independent with regard to his volume so the obtained results can be applied in real systems in a limited range. In addition, many of the them contain only some approximate analysis and do not deal with calculating the total volume distribution function.

In the theory of queueing systems with random volume customers we should take into account two important aspects: (i) a possible limitation of the total volume (a limited memory buffer); (ii) the character of the dependency between the customer's volume and his service time (independent or dependent). Therefore, we can investigate models with a limited or unlimited total volume and models in which service time of a customer is dependent or independent with regard to his volume.

It can be easily proven that models with an unlimited total volume in which the service time is independent of the customer's volume can be analyzed without extending classical methods. Analogous models, but with a limited total volume, are usually easy to analyze because they need just small modifications in classical ones. Unfortunately, they have little practical importance (e.g., in computer networks, the service time of a data packet is dependent on its size measured in bytes, usually proportionally).

The most interesting (but more complicated) are models in which these random variables are dependent. They really need introducing a new (compared with classical queueing theory) approach and extended methods. In the case when the total volume is unlimited, the obtained results let us calculate some approximations of loss characteristics in analogous practical models with a limited total volume (Tikhonenko and Ziółkowski, 2018) whereas models with limited total volume are difficult to analyze (but the most practical) and exact results have been obtained only for systems without waiting places (Tikhonenko, 2005).

Some of investigated models can be successfully used to calculate required sizes of memory buffers in the nodes of telecommunication or computer networks (packets of data are a real representation of random volume customers). Important results from this research area (also analyzing systems with non-identical servers and mechanism of packet dropping) can be found, e.g., in the works of Tikhonenko and Kempa (2016) as well as Tikhonenko *et al.* (2019).

Moreover, in last years investigations have also been concentrated on systems in which customers are characterized by some random vectors. This practical assumption is connected with the fact that, in real computer networks, packets are composed of some parts storing data of different types (text parts, attachment parts, info parts). We also have an analogous situation if we analyze multimedia packets that are usually divided into audio and video parts. The different pieces of information are located in different limited sectors of buffer memory of the system. This approach can be found in technical reports (patents) by Kim (2002) and Chen *et al.* (2009), while the first papers analyzing simple models with random volume customers and sectorized memory space are those by Ziółkowski and Tikhonenko (2018) as well as Tikhonenko and Ziółkowski (2019).

This paper presents an analysis of some queueing systems with random volume customers and sectorized unlimited memory space. The rest of the paper is organized as follows. In Section 2, we introduce the necessary notation and show the mathematical background of our investigations. We also present here some technical theorems known from the theory of queueing systems with non-homogeneous customers generalized to the case when customers are characterized by some random vector. They are used in obtaining new results in this area of research in the next sections. In Sections 3–5, we present new results connected with the analysis of some practical models of queueing systems of different types working under the above described policy. In particular, Section 3 contains analysis of the $M/G/n/0$ queueing system with sectorized memory. Section 4 analyzes the single-server queueing system $M/G/1/\infty$ and Section 5—the $M/G/1/\infty$–*EPS* queueing system with egalitarian processor sharing. For all the models analyzed in these sections, we obtain general characteristics of the total volume vector in terms of Laplace or Laplace–Stieltjes transforms and formulae for steady-state initial moments of analyzed random vectors in the case when the memory buffer is divided into two sectors. We also discuss some special practical cases of the analyzed models and present numerical computations. Section 6 contains conclusions and final remarks.

2. Main notation and mathematical background

We assume that each arriving customer is characterized by some random vector $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_k)$, where $k = 1, 2, \ldots$. The components ζ_i ($i = \overline{1,k}$) are non-negative random variables (RVs). RVs ζ_i, $i = \overline{1,k}$ can be practically understood as the sizes of portions of information of a different type that are located in different sectors of the memory buffer. Let $\sigma_i(t)$ be the sum of the i-th components of all customers present in the system at time instant t, $i = \overline{1,k}$. Our purpose is to derive characteristics of the vector $\boldsymbol{\sigma}(t) = (\sigma_1(t), \ldots, \sigma_k(t))$.

The system behavior can be described in the following way. If at some time instant t the customer having volume vector $\mathbf{x} = (x_1, \ldots, x_k)$ is accepted by the system (there are free servers or waiting positions in the queue), then the number of customers present in the system at this time instant $\eta(t)$ increases by one ($\eta(t) = \eta(t^-) + 1$) and the total volume vector increases by the value of $\mathbf{x}$ ($\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}(t^-) + \mathbf{x}$). This means that every indication x_i, $i = \overline{1,k}$, of the vector $\mathbf{x}$ is located in the proper sector of memory buffer and stays there until a customer ends his service. If τ is the time instant in which the same customer finishes his service, then the number of customers decreases by one and the customer releases memory buffer resources. Thus, then we have that $\eta(\tau) = \eta(\tau^-) - 1$ and $\boldsymbol{\sigma}(\tau) = \boldsymbol{\sigma}(\tau^-) - \mathbf{x}$. The analyzed model is schematically presented in Fig. 1.

We also assume that the customer's service time ξ generally depends on his indication vector $\boldsymbol{\zeta}$. This dependence is determined by the joint distribution function (DF):

$$\begin{aligned} F(\mathbf{x}, t) &= F(x_1, \ldots, x_k, t) \\ &= \mathbf{P}\{\zeta_1 < x_1, \ldots, \zeta_k < x_k, \xi < t\} \\ &= \mathbf{P}\{\boldsymbol{\zeta} < \mathbf{x}, \xi < t\}, \end{aligned}$$

where $\mathbf{x} = (x_1, \ldots, x_k)$. Let $L(\mathbf{x}) = F(\mathbf{x}, \infty)$ be the joint DF of a customer's components, $B(t) = F(\vec{\infty}, t)$ be the DF of his service time (here $\vec{\infty} = (\infty, \ldots, \infty)$). Note that we can consider a marginal DF of separate component $L_i(x) = L(\infty, \ldots, x, \ldots, \infty)$, or a joint DF of some separate component and service time: $F_i(x, t) = F(\infty, \ldots, x, \ldots, \infty, t)$. Evidently, we have $B(t) = F_i(\infty, t)$ for all $i = \overline{1,k}$.

Let

$$\begin{aligned} \alpha(\mathbf{s}, q, t) &= \int_0^\infty \cdots \int_0^\infty \int_0^t e^{-(\mathbf{s},\mathbf{x}) - qu} \, \mathrm{d}F(\mathbf{x}, u) \\ &= \int_{\mathbf{0}}^{\vec{\infty}} \int_0^t e^{-(\mathbf{s},\mathbf{x}) - qu} \, \mathrm{d}F(\mathbf{x}, u), \end{aligned}$$

where $\mathbf{0} = (0, \ldots, 0)$ is a k-dimensional vector, $(\mathbf{s}, \mathbf{x}) = s_1 x_1 + \ldots + s_k x_k$. From the probability sense of the

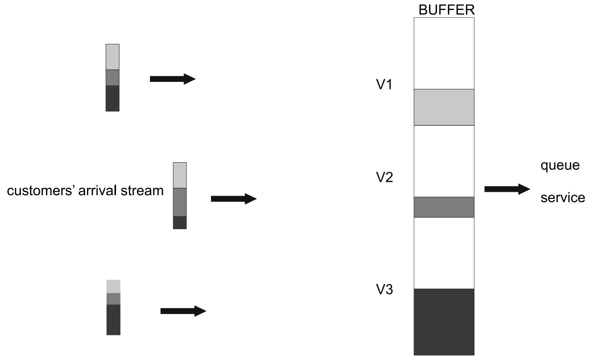

Fig. 1. Scheme of a queueing system with random volume customers and a sectorized unlimited memory buffer.

DF, we can write

$$\begin{aligned}&\mathrm{d}F(\mathbf{x},u)\\&\quad=\mathbf{P}\{\zeta_1\in[x_1;x_1+\mathrm{d}x_1),\dots,\zeta_k\in[x_k;x_k+\mathrm{d}x_k),\\&\qquad\xi\in[u;u+\mathrm{d}u)\}\\&\quad=\mathbf{P}\{\boldsymbol{\zeta}\in[\mathbf{x};\mathbf{x}+\mathrm{d}\mathbf{x}),\xi\in[u;u+\mathrm{d}u)\}.\end{aligned}$$

Denote by $\alpha(\mathbf{s},q)=\alpha(\mathbf{s},q,\infty)$ the $(k+1)$-th Laplace–Stieltjes transform (LST) of the function $F(\mathbf{x},t)$; $\varphi(\mathbf{s})=\alpha(\mathbf{s},0)$ is the k-th LST of function $L(\mathbf{x})$, $\beta(q)=\alpha(\mathbf{0},q)$ is the LST of function $B(t)$. Let us introduce the k-dimensional vector $\mathbf{i}=(i_1,\dots,i_k)$, where $i_l=1,2,\dots$; $l=\overline{1,k}$. Denote by $\Delta(\mathbf{i},j)$ the following differential operator:

$$\Delta(\mathbf{i},j)=(-1)^{i_1+\dots+i_k+j}\frac{\partial^{i_1+\dots+i_k+j}}{\partial s_1^{i_1}\dots\partial s_k^{i_k}\partial q^j},$$

and, analogously,

$$\Delta(\mathbf{i})=(-1)^{i_1+\dots+i_k}\frac{\partial^{i_1+\dots+i_k}}{\partial s_1^{i_1}\dots\partial s_k^{i_k}}.$$

Let $\alpha_{\mathbf{i}j}(t)=\Delta(\mathbf{i},j)\alpha(\mathbf{s},q,t)|_{\mathbf{s}=\mathbf{0},q=0}$. Then we have evidently that $a_{\mathbf{i}j}=\alpha_{\mathbf{i}j}(\infty)$ is the mixed $(i_1+\dots+i_k+j)$-th moment of the DF $F(\mathbf{x},t)$ (if exists). Denote by $\varphi_{\mathbf{i}}$ the mixed $(i_1+\ldots+i_k)$-th moment of the DF $L(\mathbf{x})$ and by β_i the i-th moment of the DF $B(t)$. Introduce the notation

$$\beta_1(t)=\int_0^t u\,\mathrm{d}B(u),\quad \beta_1^*(t)=\int_0^t[1-B(u)]\,\mathrm{d}u.$$

It is clear that $\beta_1=\beta_1(\infty)=\beta_1^*(\infty)$.

We assume that the process of customers' arrivals is a stationary Poisson process with parameter a.

Assume that the service discipline does not depend on the indication vector $\boldsymbol{\zeta}$, and the system is empty at the initial moment $t=0$, i.e., $\boldsymbol{\sigma}(0)=\mathbf{0}$. Introduce the notation $D(\mathbf{x},t)=\mathbf{P}\{\boldsymbol{\sigma}(t)<\mathbf{x}\}=\mathbf{P}\{\sigma_1(t)<x_1,\dots,\sigma_k(t)<x_k\}$. Let

$$\overline{\delta}(\mathbf{s},t)=\mathbf{E}e^{-(\mathbf{s},\boldsymbol{\sigma}(t))}=\int_{\mathbf{0}}^{\vec{\infty}}e^{-(\mathbf{s},\mathbf{x})}\,\mathrm{d}_{\mathbf{x}}D(\mathbf{x},t)$$

be the LST of the function $D(\mathbf{x},t)$ with respect to $\mathbf{x}$. In this case, t is a parameter. The probability sense of $\mathrm{d}_{\mathbf{x}}D(\mathbf{x},t)$ can be written as

$$\begin{aligned}\mathrm{d}_{\mathbf{x}}D(\mathbf{x},t)=\mathbf{P}\{&\sigma_1(t)\in[x_1;x_1+\mathrm{d}x_1),\dots,\\&\sigma_k(t)\in[x_k;x_k+\mathrm{d}x_k)\}.\end{aligned}$$

Denote by $\delta(\mathbf{s},q)=\int_0^\infty e^{-qt}\overline{\delta}(\mathbf{s},t)\,\mathrm{d}t$ the Laplace transform of the function $\overline{\delta}(\mathbf{s},t)$ with respect to t.

Let $\boldsymbol{\chi}(t)=(\chi_1(t),\dots,\chi_k(t))$ be the indication vector of a customer that is served at time instant t and $\xi^*(t)$ be his elapsed service time (the time from service beginning to the moment t).

Lemma 1. *Let $E_y(\mathbf{x})=\mathbf{P}\{\boldsymbol{\chi}(t)<\mathbf{x}|\xi^*(t)=y\}$ be the conditional DF of the random vector $\boldsymbol{\chi}(t)$ under condition $\xi^*(t)=y$. Then,*

$$\mathrm{d}\,E_y(\mathbf{x})=[1-B(y)]^{-1}\int_{u=y}^\infty\mathrm{d}F(\mathbf{x},u).$$

Proof. We have

$$\begin{aligned}\mathrm{d}E_y(\mathbf{x})&=\mathbf{P}\{\boldsymbol{\chi}(t)\in[\mathbf{x};\mathbf{x}+\mathrm{d}\mathbf{x})|\xi^*(t)=y\}\\&=\mathbf{P}\{\boldsymbol{\zeta}\in[\mathbf{x};\mathbf{x}+\mathrm{d}\mathbf{x})|\xi\ge y\}\\&=\frac{\mathbf{P}\{\boldsymbol{\zeta}\in[\mathbf{x};\mathbf{x}+\mathrm{d}\mathbf{x}),\xi\ge y\}}{\mathbf{P}\{\xi\ge y\}}\\&=[1-B(y)]^{-1}\int_{u=y}^\infty\mathrm{d}F(\mathbf{x},u).\end{aligned}$$

■

Note that function $E_y(\mathbf{x})$ takes the form

$$\begin{aligned}E_y(\mathbf{x})&=\int_{\mathbf{0}}^{\mathbf{x}}\mathrm{d}E_y(\mathbf{u})=\mathbf{P}\{\boldsymbol{\zeta}<\mathbf{x}|\xi\ge y\}\\&=\frac{\mathbf{P}\{\boldsymbol{\zeta}<\mathbf{x},\xi\ge y\}}{\mathbf{P}\{\xi\ge y\}}=\frac{L(\mathbf{x})-F(\mathbf{x},y)}{1-B(y)},\end{aligned}$$

where $\mathbf{u}=(u_1,\dots,u_k)$.

Corollary 1. *The LST of the random vector $\boldsymbol{\chi}(t)$ has the form*

$$\begin{aligned}e_y(\mathbf{s})&=\int_{\mathbf{x}=\mathbf{0}}^{\vec{\infty}}e^{-(\mathbf{s},\mathbf{x})}\,\mathrm{d}E_y(\mathbf{x})\\&=[1-B(y)]^{-1}\int_{\mathbf{x}=\mathbf{0}}^{\vec{\infty}}e^{-(\mathbf{s},\mathbf{x})}\int_{u=y}^\infty\mathrm{d}F(\mathbf{x},u).\end{aligned}$$

3. System $M/G/n/0$

Consider a queueing system $M/G/n/0$. Let a be the parameter of the Poisson arrival process. Denote by $\eta(t)$ the number of customers present in the system at time instant t. Set $y=a\beta_1$. Assume that $y<\infty$. This condition guarantees (in the classical approach) steady state existence because, in this system, we have a limited number of servers and no waiting places in the queue, and all customers that come to the system when there are no free servers are lost. Introducing a volume vector for every customer does not change this situation. We shall analyze this system in steady state (as $t\to\infty$). Let η be a stationary number of customers present in it ($\eta(t)\Rightarrow\eta$ in the sense of a weak convergence).

Let

$$\begin{aligned}&P_i(t,y_1,\dots,y_i)\,\mathrm{d}y_1\dots\mathrm{d}y_i\\&\qquad=\mathbf{P}\{\eta(t)=i,\xi_1^*\in[y_1;y_1+\mathrm{d}y_1),\\&\qquad\qquad\dots,\xi_i^*(t)\in[y_i;y_i+\mathrm{d}y_i)\},\end{aligned}$$

where $\xi_j^*(t)$ is the elapsed service time of the j-th customer, $j=1,2,\dots,i$. Write $P_0(t)=\mathbf{P}\{\eta(t)=0\}$

as the probability that the system is empty at time instant t. As $t \to \infty$, we obtain

$$p_i(y_1, \ldots, y_i) = \lim_{t\to\infty} P_i(t, y_1, \ldots, y_i),$$
$$p_0 = \lim_{t\to\infty} P_0(t).$$

From the classical queueing theory (Bocharov *et al.*, 2004), we have

$$p_i(y_1, \ldots, y_i) = \frac{a^i}{i!} p_0 \prod_{j=1}^{i} [1 - B(y_j)], \tag{1}$$

where $p_0 = \left[\sum_{i=0}^{n} y^i/i!\right]^{-1}$.

From the existence of steady state (when $y < \infty$) it follows that $\boldsymbol{\sigma}(t) \Rightarrow \boldsymbol{\sigma}$ in the sense of weak convergence, where the distribution of $\boldsymbol{\sigma}$ does not depend on $\boldsymbol{\sigma}(0)$.

Theorem 1. *For the $M/G/n/0$ queueing system in steady state, the LST of the joint DF of vector $\boldsymbol{\sigma}$ components has the form*

$$\delta(\mathbf{s}) = \frac{\sum_{i=0}^{n} \left[-a\alpha'_q(\mathbf{s}, q)|_{q=0}\right]^i / i!}{\sum_{i=0}^{n} y^i / i!}. \tag{2}$$

Proof. Introduce the notation

$$\begin{aligned} &\mathrm{d}D_i(\mathbf{x}, y_1, \ldots, y_i) \\ &= \mathbf{P}\{\boldsymbol{\sigma} \in [\mathbf{x}; \mathbf{x} + \mathrm{d}\mathbf{x}) | \eta = i, \xi_1^* = y_1, \ldots, \xi_i^* = y_i\}. \end{aligned}$$

This is the conditional probability that the k-th component of vector $\boldsymbol{\sigma}$ lies in the interval $[x_k; x_k + \mathrm{d}x_k)$, $k = \overline{1, i}$, under the condition that there are i customers in the system and their elapsed service times equal $y_1, \ldots, y_i$, respectively. Note that, for $i \geq 1$, the components of the indication vector $\boldsymbol{\sigma}^j = (\sigma_1^j, \ldots, \sigma_k^j)$ of the j-th customer in the system depend on ξ_j^* only ($j = \overline{1, i}$). Then, we have from Lemma 1 that

$$\begin{aligned} &\mathbf{P}\{\boldsymbol{\sigma}^j \in [\mathbf{x}; \mathbf{x} + \mathrm{d}\mathbf{x}) | \xi_j^* = y_j\} \\ &\qquad = [1 - B(y_j)]^{-1} \int_{u=y_j}^{\infty} \mathrm{d}F(\mathbf{x}, u). \end{aligned}$$

The LST of the random vector $\boldsymbol{\sigma}^j$, under condition $\xi_j^* = y_j$, has the form (see Corollary 1)

$$e_{y_j}(\mathbf{s}) = [1 - B(y_j)]^{-1} \int_{\mathbf{x}=\mathbf{0}}^{\vec{\infty}} e^{-(\mathbf{s}, \mathbf{x})} \int_{u=y_j}^{\infty} \mathrm{d}F(\mathbf{x}, u).$$

It is clear that, for $\eta = i$, we have $\sigma_m = \sum_{j=1}^{i} \sigma_m^j$, $m = \overline{1, k}$, and random vectors $\boldsymbol{\sigma}^j$ are independent under condition $\xi_1^* = y_1, \ldots, \xi_i^* = y_i$. Then the LST of the function $D_i(\mathbf{x}, y_1, \ldots, y_i)$ has the form of a product:

$$\begin{aligned} &\delta(\mathbf{s}, y_1, \ldots, y_i) \\ &= \int_{\mathbf{x}=\mathbf{0}}^{\vec{\infty}} e^{-(\mathbf{s},\mathbf{x})} \mathrm{d}D_i(\mathbf{x}, y_1, \ldots, y_i) \\ &= \prod_{j=1}^{i} e_{y_j}(\mathbf{s}) \\ &= \prod_{j=1}^{i} [1 - B(y_j)]^{-1} \int_{\mathbf{x}=\mathbf{0}}^{\vec{\infty}} e^{-(\mathbf{s},\mathbf{x})} \int_{u=y_j}^{\infty} \mathrm{d}F(\mathbf{x}, u). \end{aligned}$$

Using Lemma 1 again and the relation (1), we obtain

$$\begin{aligned} &\delta(\mathbf{s}) \\ &= p_0 + \sum_{i=1}^{n} \int_0^{\infty} \ldots \int_0^{\infty} \delta(\mathbf{s}, y_1, \ldots, y_i) \\ &\quad \times p_i(y_1, \ldots, y_i) \, \mathrm{d}y_1 \ldots \mathrm{d}y_i \\ &= \sum_{i=0}^{n} \frac{a^i}{i!} p_0 \prod_{j=1}^{i} \int_{\mathbf{0}}^{\vec{\infty}} e^{-(\mathbf{s},\mathbf{x})} \int_0^{\infty} \mathrm{d}y_j \int_{u=y_j}^{\infty} \mathrm{d}F(\mathbf{x}, u), \end{aligned}$$

where

$$\begin{aligned} &\int_{\mathbf{0}}^{\vec{\infty}} e^{-(\mathbf{s},\mathbf{x})} \int_{z=0}^{\infty} \mathrm{d}z \int_{u=z}^{\infty} \mathrm{d}F(\mathbf{x}, u) \\ &= \int_{\mathbf{x}=\mathbf{0}}^{\vec{\infty}} \int_{u=0}^{\infty} e^{-(\mathbf{s},\mathbf{x})} \, \mathrm{d}F(\mathbf{x}, u) \int_{z=0}^{u} \mathrm{d}z \\ &= \int_{\mathbf{x}=\mathbf{0}}^{\vec{\infty}} \int_{u=0}^{\infty} u e^{-(\mathbf{s},\mathbf{x})} \, \mathrm{d}F(\mathbf{x}, u) = -\alpha'_q(\mathbf{s}, q)|_{q=0}, \end{aligned}$$

whence, taking into consideration that $p_0 = \left(\sum_{i=0}^{n} y^i/i!\right)^{-1}$, we obtain the statement of the theorem. ∎

The formula (2) shows that the character of the dependence between a customer's volume vector and his service time (which is determined by the function $F(\mathbf{x}, t)$) has substantial influence on steady-state total volume vector characteristics (e.g., its multidimensional LST $\delta(\mathbf{s})$) in the $M/G/n/0$ queueing system. Indeed, in this relation there is a multidimensional LST $\alpha(\mathbf{s}, q)$ of the function $F(\mathbf{x}, t)$. The same is true for queueing models of a different type (see the formulae (8) and (14)).

Using the relation (2), we can determine mixed moments (if they exist) of the random vector $\boldsymbol{\sigma}$:

$$\delta_{\mathbf{i}} = \Delta_{\mathbf{i}} \delta(\mathbf{s})|_{\mathbf{s}=\mathbf{0}}, \tag{3}$$

where $\mathbf{i} = (i_1, \ldots, i_k)$, i_j determines the order of the moment with respect to the j-th component of the random vector $\boldsymbol{\sigma}$.

Let, e.g., $n = 2$. Then we have

$$\delta(s_1, s_2) = \frac{\sum_{i=0}^{n} \left[-a\alpha'_q(s_1, s_2, q)|_{q=0}\right]^i / i!}{\sum_{i=0}^{n} y^i / i!}.$$

Determine the following moments of random vector (σ_1, σ_2):

$$\begin{aligned}
\delta_1^{(1)} &= \mathbf{E}\sigma_1 = -\delta'(s,0)|_{s=0},\\
\delta_1^{(2)} &= \mathbf{E}\sigma_2 = -\delta'(0,s)|_{s=0},\\
\delta_2^{(1)} &= \mathbf{E}\sigma_1^2 = \delta''(s,0)|_{s=0},\\
\delta_2^{(2)} &= \mathbf{E}\sigma_2^2 = \delta''(0,s)|_{s=0},\\
\delta_{11} &= \mathbf{E}(\sigma_1\sigma_2) = \frac{\partial^2\delta(s_1,s_2)}{\partial s_1 \partial s_2}\Big|_{s_1=0,s_2=0}.
\end{aligned}$$

Let

$$\begin{aligned}
\alpha_{ijk} &= \mathbf{E}(\zeta_1^i \zeta_2^j \xi^k)\\
&= \int_0^\infty \int_0^\infty \int_0^\infty x_1^i x_2^j t^k \,\mathrm{d}F(x_1,x_2,t)
\end{aligned}$$

be the mixed moment of the $(i+j+k)$-th order of random vector $(\zeta_1,\zeta_2,\xi), i,j,k = 0,1,\dots$. For example, $\alpha_{000} = 1, \alpha_{001} = \beta_1, \alpha_{020} = \varphi_2^{(2)}$, etc. Then, we obtain, after some calculation,

$$\delta_1^{(1)} = p_0 a \alpha_{101} \sum_{i=0}^{n-1} \frac{y^i}{i!},$$

$$\delta_1^{(2)} = p_0 a \alpha_{011} \sum_{i=0}^{n-1} \frac{y^i}{i!},$$

$$\delta_2^{(1)} = p_0 a \left(\alpha_{201} \sum_{i=0}^{n-1} \frac{y^i}{i!} + a\alpha_{101}^2 \sum_{i=0}^{n-2} \frac{y^i}{i!} \right),$$

$$\delta_2^{(2)} = p_0 a \left(\alpha_{021} \sum_{i=0}^{n-1} \frac{y^i}{i!} + a\alpha_{011}^2 \sum_{i=0}^{n-2} \frac{y^i}{i!} \right),$$

$$\begin{aligned}
\delta_{11} = p_0 a \Big[&\alpha_{111} \sum_{i=0}^{n} \frac{y^{i-1}}{(i-1)!}\\
&+ a\alpha_{101}\alpha_{011} \sum_{i=2}^{n} \frac{y^i}{(i-2)!} \Big].
\end{aligned}$$

If we are interested in a mixed moment with respect to some (not all) components of $\boldsymbol{\sigma}$, we have to take value 0 for all "unnecessary" components of it. Thus, we obtain a new function $\delta(\mathbf{s}')$, where vector $\mathbf{s}'$ consists of the components of our interest. Further determination of the moment is carried out analogously using the relation (3).

Note that, if $n \to \infty$, we obtain from (2) the relation for the function $\delta(\mathbf{s})$ characterizing a steady-state system $M/G/\infty$:

$$\delta(\mathbf{s}) = \exp\left[-y - a\alpha_q'(\mathbf{s},q)|_{q=0} \right].$$

For this system, we obtain, if $n = 2$

$$\delta(s_1,s_2) = \exp\left[-y - a\alpha_q'(s_1,s_2,q)|_{q=0} \right],$$

$$\delta_1^{(1)} = a\alpha_{101}, \quad \delta_1^{(2)} = a\alpha_{011},$$

$$\delta_2^{(1)} - \left(\delta_1^{(1)}\right)^2 = a\alpha_{201}, \quad \delta_2^{(2)} - \left(\delta_1^{(2)}\right)^2 = a\alpha_{021},$$

$$\delta_{11} = a\left(\alpha_{111} + a\alpha_{101}\alpha_{011}\right).$$

Example 1. Consider the system $M/G/n/0$, where a is a parameter of the arrival process. Each customer is characterized by a two-dimensional random volume vector $\boldsymbol{\zeta} = (\zeta_1, \zeta_2)$, where RVs ζ_1 and ζ_2 are independent and their DFs are denoted by $L_1(x)$ and $L_2(x)$, respectively. The RV $\zeta = \zeta_1 + \zeta_2$ will be called a customer's length. Assume that the service time of the customer is proportional to his length: $\xi = c(\zeta_1 + \zeta_2)$, $c > 0$.

Let us determine

$$\begin{aligned}
\alpha(\mathbf{s},q) &= \alpha(s_1,s_2,q)\\
&= \int_{x_1=0}^\infty \int_{x_2=0}^\infty \int_{t=0}^\infty e^{-s_1x_1-s_2x_2-qt}\,\mathrm{d}F(x_1,x_2,t),
\end{aligned}$$

where $F(x_1,x_2,t) = \mathbf{P}\{\zeta_1 < x_1, \zeta_2 < x_2, \xi < t\}$. In this case, we can write

$$\begin{aligned}
\alpha(s_1,s_2,q) = &\int_0^\infty \int_0^\infty e^{-s_1x_1-s_2x_2}\,\mathrm{d}L(x_1,x_2)\\
&\times \int_0^\infty e^{-qt}\,\mathrm{d}B(t|\zeta_1 = x_1, \zeta_2 = x_2),
\end{aligned}$$

where

$$\begin{aligned}
B(t|\zeta_1 = x_1, \zeta_2 = x_2)&\\
&= \mathbf{P}\{\xi < t | \zeta_1 = x_1, \zeta_2 = x_2\},\\
L(x_1,x_2) &= \mathbf{P}\{\zeta_1 < x_1, \zeta_2 < x_2\}\\
&= L_1(x_1)L_2(x_2).
\end{aligned}$$

It is clear that

$$B(t|\zeta_1 = x_1, \zeta_2 = x_2) = \begin{cases} 1, & t > c(x_1+x_2),\\ 0, & t \le c(x_1+x_2). \end{cases}$$

Hence, we obtain, using the Kronecker delta function,

$$\begin{aligned}
&\int_0^\infty e^{-qt}\,\mathrm{d}B(t|\zeta_1 = x_1, \zeta_2 = x_2)\\
&\quad = \int_0^\infty e^{-qt}\delta\left(t - c(x_1+x_2)\right)\mathrm{d}t = e^{-cq(x_1+x_2)}.
\end{aligned}$$

Then we have

$$\begin{aligned}\alpha(s_1, s_2, q) &= \int_0^\infty e^{-s_1x_1-cqx_1}\,\mathrm{d}L_1(x_1)\\ &\quad\times\int_0^\infty e^{-s_2x_2-cqx_2}\,\mathrm{d}L_2(x_2)\\ &= \varphi^{(1)}(s_1+cq)\varphi^{(2)}(s_2+cq), \qquad (4)\end{aligned}$$

where $\varphi^{(1)}(s), \varphi^{(2)}(s)$ are the LSTs of the functions $L_1(x)$ and $L_2(x)$, respectively. If we substitute $\alpha(s_1, s_2, q)$ from (4) to the relation (2), we get

$$\begin{aligned}\delta(\mathbf{s}) &= \delta(s_1, s_2)\\ &= \frac{1}{\sum\limits_{i=0}^{n} y^i/i!}\sum_{i=0}^{n}\frac{(-1)^i}{i!}\Big\{ac[\varphi^{(1)\prime}(s_1)\varphi^{(2)}(s_2)\\ &\quad+\varphi^{(1)}(s_1)\varphi^{(2)\prime}(s_2)]\Big\}^i,\end{aligned}$$

where $y = ac\big(\varphi_1^{(1)}+\varphi_1^{(2)}\big)$. Assume now that the buffer memory is divided into two sectors of an infinite volume. The initial part of a customer's volume ζ_1 is placed in the first sector and the rest ζ_2 in the second one. Let us determine the mean total customer volumes $\delta_1^{(1)}$ and $\delta_1^{(2)}$ present in the first and second sector, respectively. To determine $\delta_1^{(1)}$, we first obtain

$$\begin{aligned}\delta^{(1)}(s) &= \delta(s, 0)\\ &= \frac{\sum_{i=0}^{n}(-1)^i\{ac[\varphi^{(1)\prime}(s)-\varphi_1^{(2)}\varphi^{(1)}(s)]\}^i/i!}{\sum_{i=0}^{n} y^i/i!},\end{aligned}$$

where $\varphi_1^{(2)}$ is the first moment of DF $L_2(x)$. Finally, we have

$$\begin{aligned}\delta_1^{(1)} &= -\delta^{(1)\prime}(0)\\ &= \frac{\sum_{i=1}^{n} acy^{i-1}(\varphi_2^{(1)}+\varphi_1^{(1)}\varphi_1^{(2)})/(i-1)!}{\sum_{i=0}^{n} y^i/i!}.\end{aligned}$$

Analogously, we obtain

$$\begin{aligned}\delta_1^{(2)} &= -\delta^{(2)\prime}(0)\\ &= \frac{\sum_{i=1}^{n} acy^{i-1}(\varphi_2^{(2)}+\varphi_1^{(1)}\varphi_1^{(2)})/(i-1)!}{\sum_{i=0}^{n} y^i/i!}.\end{aligned}$$

Now, let us determine the mixed moment δ_{11} of the order $1+1$: $\frac{\partial^2\delta(s_1,s_2)}{\partial s_1\partial s_2}|_{s_1=0,s_2=0}$, whence we have

$$\begin{aligned}\delta_{11} &= p_0ac\sum_{i=1}^{n}\frac{1}{(i-1)!}y^{i-2}[(i-1)ac\\ &\quad\times(\varphi_2^{(1)}+A)(\varphi_2^{(2)}+A)+y(B+C)],\end{aligned}$$

where $p_0 = (\sum_{i=0}^{n} y^i/i!)^{-1}$, $A = \varphi_1^{(1)}\varphi_1^{(2)}$, $B = \varphi_2^{(1)}\varphi_1^{(2)}$, $C = \varphi_1^{(1)}\varphi_2^{(2)}$.

Consider now some special case of the investigated model in which ζ_1, ζ_2 are exponentially distributed with parameters f and g, respectively. We easily obtain the following formulae:

$$\delta_1^{(1)} = \frac{ac}{2f}\left(\frac{2}{f}+\frac{1}{g}\right)\left(2-y^2p_0\right),$$

$$\delta_1^{(2)} = \frac{ac}{2g}\left(\frac{2}{g}+\frac{1}{f}\right)\left(2-y^2p_0\right),$$

$$\begin{aligned}\delta_{11} &= \frac{acp_0}{fg}\Big[2(1+y)\left(\frac{1}{f}+\frac{1}{g}\right)\\ &\quad+ac\left(\frac{2}{f}+\frac{1}{g}\right)\left(\frac{2}{g}+\frac{1}{f}\right)\Big],\end{aligned}$$

where

$$p_0 = \frac{2}{2+y+y^2}, \quad y = ac\left(\frac{1}{f}+\frac{1}{g}\right).$$

In Tables 1 and 2 we present results of some numerical computations. Table 1 shows results for the following fixed parameters: $f = 1$, $g = 2$, $c = 1$, whereas the value of a is increasing from 1 to 10. In

Table 1. Total volume vector characteristics for the $M/G/2/0$ system, $f = 1$, $g = 2$, $c = 1$.

a	y	$\delta_1^{(1)}$	$\delta_1^{(2)}$	δ_{11}
1	1.5	1.5217	0.6087	2.1739
2	3.0	1.7857	0.7143	3.1429
3	4.5	1.8224	0.7290	3.5327
4	6.0	1.8182	0.7273	3.7273
5	7.5	1.8061	0.7224	3.8403
6	9.0	1.7935	0.7174	3.9130
7	10.5	1.7821	0.7128	3.9633
8	12.0	1.7722	0.7089	4.0000
9	13.5	1.7636	0.7054	4.0278
10	15.0	1.7562	0.7025	4.0496

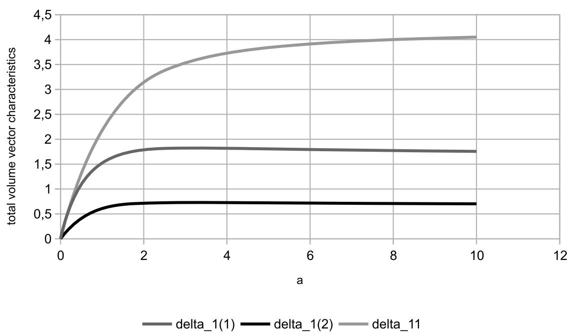

Fig. 2. Total volume vector characteristics for the $M/G/2/0$ system, $f = 1$, $g = 2$, $c = 1$.

this case we can observe the influence of system loading on the total volume vector characteristics $\delta_1^{(1)}$, $\delta_1^{(2)}$ and δ_{11}. In Table 2, we present results of calculations in the case when $f = 1$, $c = 1$, $a = 1$ and this time the value of g is changing. Here we can see the influence of the mean value of second volume vector indication $(1/g)$ on the same characteristics. In both situations the calculated characteristics increase together with the increasing value of system loading y, but the system remains stable because we have here a limited number of servers and, in the case when the value of y becomes bigger, we have many customer losses and the system cannot be overloaded. We can see this fact also in Figs. 2 and 3.

♦

Table 2. Total volume vector characteristics for the $M/G/2/0$ system, $f = 1$, $c = 1$, $a = 1$.

g	y	$\delta_1^{(1)}$	$\delta_1^{(2)}$	δ_{11}
0.1	11.00	1.1642	20.3731	77.0149
0.2	6.00	1.2727	10.0000	36.5909
0.3	4.33	1.3451	6.4454	23.1268
0.4	3.50	1.3944	4.6479	16.4789
0.5	3.00	1.4286	3.5714	12.5714
0.6	2.67	1.4528	2.8616	10.0314
0.7	2.43	1.4704	2.3631	8.2665
0.8	2.25	1.4832	1.9966	6.9799
0.9	2.11	1.4928	1.7179	6.0070
1.0	2.00	1.5000	1.5000	5.2500
1.1	1.91	1.5055	1.3258	4.6469
1.2	1.83	1.5097	1.1840	4.1570
1.3	1.77	1.5129	1.0668	3.7525
1.4	1.71	1.5153	0.9684	3.4137
1.5	1.67	1.5172	0.8851	3.1264
1.6	1.62	1.5187	0.8136	2.8803
1.7	1.59	1.5198	0.7518	2.6674
1.8	1.56	1.5207	0.6979	2.4816
1.9	1.53	1.5213	0.6506	2.3184

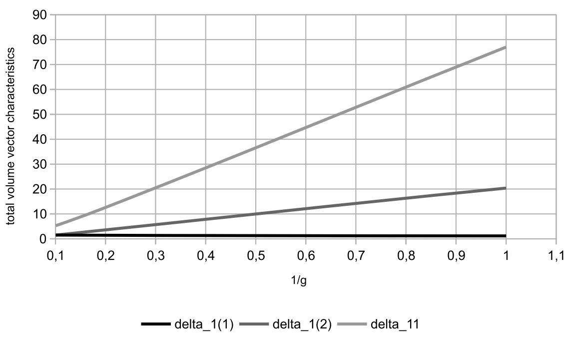

Fig. 3. Total volume vector characteristics for the $M/G/2/0$ system, $f = 1$, $c = 1$, $a = 1$.

Example 2. Consider the system $M/G/\infty$ with the same notation as in Example 1. For this system, we obtain analogously

$$\delta_1^{(1)} = ac(\varphi_2^{(1)} + \varphi_1^{(1)}\varphi_1^{(2)}),$$

$$\delta_1^{(2)} = ac(\varphi_2^{(2)} + \varphi_1^{(1)}\varphi_1^{(2)}),$$

$$\delta_{11} = ac(\varphi_2^{(1)}\varphi_1^{(2)} + \varphi_1^{(1)}\varphi_2^{(2)}) + a^2c^2(\varphi_1^{(1)}\varphi_1^{(2)} + \varphi_2^{(2)})(\varphi_1^{(1)}\varphi_1^{(2)} + \varphi_2^{(1)}).$$

♦

4. System $M/G/1/\infty$

Assume that customers arrive to the system with intensity a. The functions $F(\mathbf{x}, t)$, $L(\mathbf{x})$, $B(t)$, $D(\mathbf{x}, t)$ and their transforms $\alpha(\mathbf{s}, q)$, $\varphi(\mathbf{s})$, $\beta(q)$, $\delta(\mathbf{s}, q)$ have the same meanings as presented in the earlier sections. We additionally introduce some new notation: $\Pi(t)$ is the DF of the busy period of the system under consideration and $\pi(q)$ is its LST.

In the classical queueing theory (Matveev and Ushakov, 1984), we have the following theorem, which will be used in our later investigations.

Theorem 2.
(a) The function $\pi(q)$ is a unique solution of the following functional equation:

$$\pi(q) = \beta(q + a - a\pi(q)),$$

which is analytical in domain $\operatorname{Re} q > 0$*;*

(b) If $\rho = a\beta_1 \le 1$, then $\pi(+0) = \Pi(\infty) = 1$, otherwise $\pi(+0) < 1$, $\Pi(\infty) < 1$;

(c) If $\rho \ge 1$, then the first moment of the busy period is $\pi_1 = \infty$, otherwise the first two moments can be calculated as

$$\pi_1 = \frac{\beta_1}{1-\rho}, \quad \pi_2 = \frac{\beta_2}{(1-\rho)^3}.$$

Let $\eta(t)$ be the number of customers present in the system at time instant t, $P_n(t) = \mathbf{P}\{\eta(t) = n\}, n = 0, 1, \ldots$. Denote by $\xi^*(t)$ the elapsed service time of a customer. Introduce the following notation for $n \ge 1$:

$$\theta_n(y, t)\,\mathrm{d}y = \mathbf{P}\{\eta(t) = n, \xi^*(t) \in [y; y + \mathrm{d}y)\},$$

$$\psi(z, q) = \int_0^\infty e^{-qt}\mathbf{E}z^{\eta(t)}\,\mathrm{d}t,$$

$$\omega(z, y, q) = \frac{\partial}{\partial y}\int_0^\infty e^{-qt}\mathbf{E}\left[z^{\eta(t)} I(\xi^*(t) < y)\right]\mathrm{d}t,$$

$$\chi_n(q) = \int_0^\infty e^{-qt} P_n(t)\,\mathrm{d}t,$$

where $I(A)$ is the indicator function of an event A. The following statement also takes place in the classical queueing theory (Matveev and Ushakov, 1984).

Lemma 2.
(a) For the system under consideration, the function $\omega(z, y, q)$ has the form

$$\omega(z, y, q) = [1 - B(y)]\, e^{-(q+a-az)y} \omega(z, 0, q),$$

where

$$\omega(z, 0, q) = (q + a - a\pi(q))^{-1} \frac{a\,(z - \pi(q))}{1 - z^{-1}\beta(q + a - az)};$$

(b) The function $\chi_0(q)$ has the form

$$\chi_0(q) = (q + a - a\pi(q))^{-1}.$$

Introduce the notation $D(\mathbf{x}, t|A) = \mathbf{P}\{\boldsymbol{\sigma} < \mathbf{x}|A\}$, where A is an event. Similar notation will be used also for other DFs. For the analyzed queueing system we obtain the following result.

Theorem 3. *Function $\delta(\mathbf{s}, q)$ for the system $M/G/1/\infty$ subject to the zero initial condition ($\boldsymbol{\sigma}(0) = \mathbf{0}$) is determined by the following relation:*

$$\delta(\mathbf{s}, q) = (q + a - a\pi(q))^{-1} \Bigg\{ 1 + \frac{a\,[\varphi(\mathbf{s}) - \pi(q)]\,[\varphi(\mathbf{s}) - \alpha\,(\mathbf{s}, q + a - a\varphi(\mathbf{s}))]}{[q + a - a\varphi(\mathbf{s})]\,[\varphi(\mathbf{s}) - \beta\,(q + a - a\varphi(\mathbf{s}))]} \Bigg\},$$

where $\pi(q)$ is determined by Theorem 2.

Proof. If the system is empty at time instant t, we have $\boldsymbol{\sigma}(t) = \mathbf{0}$. If there are customers present in the system, we obtain, for j-th component of the total volume vector,

$$\sigma_j(t) = \sigma_j^1(t) + \sigma_j^2(t), \quad j = \overline{1, l},$$

where $\sigma_j^1(t)$ is the value of the j-th component of the total volume vector of waiting customers at time instant t and $\sigma_j^2(t)$ is the value of the j-th component of the customer in service at time instant t. Let $D^1(\mathbf{x}, t) = \mathbf{P}\{\boldsymbol{\sigma}^1(t) < \mathbf{x}\}, D^2(\mathbf{x}, t) = \mathbf{P}\{\boldsymbol{\sigma}^2(t) < \mathbf{x}\}$.

Random vectors $\boldsymbol{\sigma}^1(t) = \big(\sigma_1^1(t), \ldots, \sigma_l^1(t)\big)$ and $\boldsymbol{\sigma}^2(t) = \big(\sigma_1^2(t), \ldots, \sigma_l^2(t)\big)$ are generally dependent. But, under the condition of fixed $\eta(t)$ ($\eta(t) = 1, 2, \ldots$) and $\xi^*(t)$ ($\xi^*(t) = y$) these random vectors are independent, and we can write

$$D(\mathbf{x}, t|\eta(t) = n, \xi^*(t) = y) = \int_{\mathbf{0}}^{\mathbf{x}} D^1(\mathbf{x} - \mathbf{u}, t|\eta(t) = n)\, \mathrm{d}_{\mathbf{u}} D^2(\mathbf{u}, t|\xi^*(t) = y),$$

because the distribution of $\boldsymbol{\sigma}^1(t)$ does not depend on $\xi^*(t)$ if the number of waiting customers is known, and the distribution of $\boldsymbol{\sigma}^2(t)$ does not depend on $\eta(t)$ if the value of $\xi^*(t)$ is known.

It is clear that total volume vectors of waiting customers are independent if their number is known. Hence, we have

$$D^1(\mathbf{x}, t|\eta(t) = n) = L_*^{(n-1)}(\mathbf{x}).$$

Obviously,

$$D^2(\mathbf{x}, t|\xi^*(t) = y) = E_y(\mathbf{x}),$$

where $E_y(\mathbf{x})$ can be determined from Lemma 1. Finally, we obtain

$$D(\mathbf{x}, t|\eta(t) = n, \xi^*(t) = y) = E_y * L_*^{(n-1)}(\mathbf{x}).$$

Then, on the basis of the total probability theorem, we can write

$$D(\mathbf{x}, t) = P_0(t) + \sum_{n=1}^{\infty} \int_0^t \theta_n(y, t) \left[E_y * L_*^{(n-1)}(\mathbf{x})\right] \mathrm{d}y.$$

Passing to the LST with respect to $\mathbf{x}$, we obtain

$$\overline{\delta}(\mathbf{s}, q) = P_0(t) + \sum_{n=1}^{\infty} (\varphi(\mathbf{s}))^{n-1} \int_0^t \theta_n(y, t) e_y(\mathbf{s})\, \mathrm{d}y, \tag{5}$$

where $e_y(\mathbf{s})$ is determined from Corollary 1. Passing in (5) to the Laplace transform with respect to t, we get

$$\begin{aligned} \delta(\mathbf{s}, q) &= \chi_0(q) + \sum_{n=1}^{\infty} (\varphi(\mathbf{s}))^{n-1} \int_0^{\infty} e_y(\mathbf{s})\, \mathrm{d}y \times \int_y^{\infty} e^{-qt} \theta(y, t)\, \mathrm{d}t \\ &= \chi_0(q) + \int_0^{\infty} \omega\,(\varphi(\mathbf{s}), y, q) \frac{e_y(\mathbf{s})}{\varphi(\mathbf{s})}\, \mathrm{d}y. \end{aligned} \tag{6}$$

Taking into consideration Lemma 2, we obtain

$$\omega\,(\varphi(\mathbf{s}), y, q) = [1 - B(y)]\, e^{-(q+a-a\varphi(\mathbf{s})y} \times (q + a - a\pi(q))^{-1} \frac{a\,(\varphi(\mathbf{s}) - \pi(q))}{1 - z^{-1}\beta\,(q + a - a\varphi(\mathbf{s}))}.$$

If we substitute this result to the relation (6), we get

$$\delta(\mathbf{s}, q) = \frac{1}{q + a - a\pi(q)} + \frac{a\,[\varphi(\mathbf{s}) - \pi(q)]}{[q + a - a\pi(q)]\,[\varphi(\mathbf{s}) - \beta(q + a - a\varphi(\mathbf{s})]} \times \int_0^{\infty} [1 - B(y)]\, e^{-(q+a-a\varphi(\mathbf{s}))y} e_y(\mathbf{s})\, \mathrm{d}y. \tag{7}$$

It is clear that $\mathrm{d}F(\mathbf{x},u) = \mathrm{d}B(u|\boldsymbol{\zeta} = \mathbf{x})\,\mathrm{d}L(\mathbf{x})$. Then, from Corollary 1, we have

$$\begin{aligned}
&\int_0^\infty [1-B(y)]\, e^{-(q+a-a\varphi(\mathbf{s}))y} e_y(\mathbf{s})\mathrm{d}y \\
&= \int_{\mathbf{x}=\mathbf{0}}^{\vec{\infty}} e^{-(\mathbf{s},\mathbf{x})}\mathrm{d}L(\mathbf{x}) \int_{u=0}^{\infty} \mathrm{d}B(u|\boldsymbol{\zeta}=\mathbf{x}) \\
&\quad \times \int_{y=0}^{u} e^{-(q+a-a\varphi(\mathbf{s}))y}\,\mathrm{d}y \\
&= \frac{\varphi(\mathbf{s}) - \alpha\left(\mathbf{s}, q+a-a\varphi(\mathbf{s})\right)}{q+a-a\varphi(\mathbf{s})}.
\end{aligned}$$

If we substitute this result to the relation (7), we obtain the statement of the theorem. ■

Corollary 2. *Let $\rho = a\beta_1 < 1$. Then, the limit $\overline{\delta}(\mathbf{s},t) \to \delta(\mathbf{s})$ exists as $t \to \infty$, where*

$$\delta(\mathbf{s}) = \int_{\mathbf{0}}^{\vec{\infty}} e^{-(\mathbf{s},\mathbf{x})}\, dD(\mathbf{x});$$

$D(\mathbf{x}) = \mathbf{P}\{\boldsymbol{\sigma} < \mathbf{x}\}$ denotes here the DF of the total random indication vector when $t \to \infty$ and the function $\delta(\mathbf{s})$ is determined by the following relation:

$$\delta(\mathbf{s}) = (1-\rho)\left[1 + \frac{\varphi(\mathbf{s}) - \alpha\left(\mathbf{s}, a-a\varphi(\mathbf{s})\right)}{\beta\left(a-a\varphi(\mathbf{s})\right) - \varphi(\mathbf{s})}\right]. \quad (8)$$

Proof. If $\rho < 1$, processes $\boldsymbol{\sigma}(t)$ and $\eta(t)$ are regenerative with points of regeneration coinciding with moments of busy period termination. The regeneration cycle is a sum of two independent RVs: the time to a next customer's arrival (this RV has an exponential distribution with parameter a) and a busy period. It follows from the theory of regenerative processes (Asmussen, 2003) that, in this case, the limit $\boldsymbol{\sigma}(t) \Rightarrow \boldsymbol{\sigma}$ exists in the sense of weak convergence as $t \to \infty$ and does not depend on the initial condition (the distribution of $\boldsymbol{\sigma}(0)$). From this existence, on the basis of the connection between the LST and the Laplace transform, the existence of the limit below follows:

$$\delta(\mathbf{s}) = \mathbf{E}e^{-(\mathbf{s},\boldsymbol{\delta})} = \lim_{q\to 0} q\delta(\mathbf{s},q).$$

Calculation of this limit yields (8). ■

Now, we can obtain steady-state initial moments of the total volume if $l=2$, i.e., $\boldsymbol{\zeta} = (\zeta_1,\zeta_2)$ and

$$\begin{aligned}
\delta(s_1,s_2) = (1-\rho)\Bigg[&1 \\
&+ \frac{\varphi(s_1,s_2) - \alpha\left(s_1,s_2,a-a\varphi(s_1,s_2)\right)}{\beta\left(a-a\varphi(s_1,s_2)\right) - \varphi(s_1,s_2)}\Bigg].
\end{aligned}$$

Then we have

$$\begin{aligned}
\delta_1^{(1)} &= a\alpha_{101} + \frac{a^2\beta_2\varphi_1^{(1)}}{2(1-\rho)}, \\
\delta_1^{(2)} &= a\alpha_{011} + \frac{a^2\beta_2\varphi_1^{(2)}}{2(1-\rho)}, \\
\delta_2^{(1)} &= a\left(\alpha_{201} + a\varphi_1^{(1)}\alpha_{102}\right) \\
&\quad + \frac{a^3\beta_2\varphi_1^{(1)}\alpha_{101}}{1-\rho} + \frac{a^2\beta_2\varphi_2^{(1)}}{2(1-\rho)} \\
&\quad + \frac{a^3\beta_3\left(\varphi_1^{(1)}\right)^2}{3(1-\rho)} + \frac{a^4\beta_2^2\left(\varphi_1^{(1)}\right)^2}{2(1-\rho)^2}, \\
\delta_2^{(2)} &= a\left(\alpha_{021} + a\varphi_1^{(2)}\alpha_{012}\right) \\
&\quad + \frac{a^3\beta_2\varphi_1^{(2)}\alpha_{011}}{1-\rho} + \frac{a^2\beta_2\varphi_2^{(2)}}{2(1-\rho)} \\
&\quad + \frac{a^3\beta_3\left(\varphi_1^{(2)}\right)^2}{3(1-\rho)} + \frac{a^4\beta_2^2\left(\varphi_1^{(2)}\right)^2}{2(1-\rho)^2}, \\
\delta_{11} &= a\alpha_{111} + \frac{a^2}{2}\left(\alpha_{012}\varphi_1^{(1)} + \alpha_{102}\varphi_1^{(2)}\right) \\
&\quad + \frac{a^2\beta_2\varphi_{11} + a^3\beta_2\left(\alpha_{011}\varphi_1^{(1)} + \alpha_{101}\varphi_1^{(2)}\right)}{2(1-\rho)} \\
&\quad + \frac{a^3\beta_3\varphi_1^{(1)}\varphi_1^{(2)}}{3(1-\rho)} + \frac{a^4\beta_2^2\varphi_1^{(1)}\varphi_1^{(2)}}{2(1-\rho)^2},
\end{aligned}$$

where $\varphi_{11} = \alpha_{110}$ is the mixed moment of the $(1+1)$-th order of a random vector (ζ_1,ζ_2)

To calculate δ_{11}, we used L'Hospital's rule for evaluation of indeterminate forms of the function of many variables (Ivlev, 2002; 2003) and the *Mathematica* environment.

Example 3. Consider the system $M/G/1/\infty$, where a is a parameter of the entrance flow. Each customer is characterized by a two-dimensional random volume vector $\boldsymbol{\zeta} = (\zeta_1,\zeta_2)$, where RVs ζ_1 and ζ_2 are independent and their DFs are denoted by $L_1(x)$ and $L_2(x)$, respectively. The RV $\zeta = \zeta_1 + \zeta_2$ will be called a customer's length. Assume that the service time of the customer is proportional to his length: $\xi = c(\zeta_1+\zeta_2)$, $c > 0$. Let $\varphi^{(1)}(s)$, $\varphi^{(2)}(s)$ be the LSTs of the RVs ζ_1 and ζ_2, respectively.

Then we have

$$\alpha(\mathbf{s},q) = \alpha(s_1,s_2,q) = \varphi^{(1)}(s_1+cq)\varphi^{(2)}(s_2+cq),$$

$$\beta(q) = \varphi^{(1)}(cq)\varphi^{(2)}(cq),$$

$$\varphi(\mathbf{s}) = \varphi(s_1,s_2) = \varphi^{(1)}(s_1)\varphi^{(2)}(s_2).$$

In this case the relation (9) takes the form

$$\delta(\mathbf{s}) = \delta(s_1, s_2) = (1-\rho)\Big[1 + \frac{\varphi^{(1)}(s_1)\varphi^{(2)}(s_2) - \varphi^{(1)}(s_1+cq)\varphi^{(2)}(s_2+cq)}{\varphi^{(1)}(cq)\varphi^{(2)}(cq) - \varphi^{(1)}(s_1)\varphi^{(2)}(s_2)}\Big],$$

where $q = a\left[1 - \varphi^{(1)}(s_1)\varphi^{(2)}(s_2)\right]$.

Let $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$. Let us also calculate $\delta_1^{(1)} = \mathbf{E}\sigma_1$, $\delta_1^{(2)} = \mathbf{E}\sigma_2$ and the mixed moment $\delta_{11} = \mathbf{E}(\sigma_1\sigma_2)$. Evidently, we can use general formulae, where

$$\alpha_{101} = c\left(\varphi_2^{(1)} + \varphi_1^{(1)}\varphi_1^{(2)}\right),$$

$$\alpha_{011} = c\left(\varphi_2^{(2)} + \varphi_1^{(1)}\varphi_1^{(2)}\right),$$

$$\alpha_{201} = c\left(\varphi_3^{(1)} + \varphi_2^{(1)}\varphi_1^{(2)}\right),$$

$$\alpha_{021} = c\left(\varphi_3^{(2)} + \varphi_1^{(1)}\varphi_2^{(2)}\right),$$

$$\alpha_{111} = c\left(\varphi_2^{(1)}\varphi_1^{(2)} + \varphi_1^{(1)}\varphi_2^{(2)}\right),$$

$$\alpha_{102} = c^2\left(\varphi_3^{(1)} + 2\varphi_2^{(1)}\varphi_1^{(2)} + \varphi_1^{(1)}\varphi_2^{(2)}\right),$$

$$\alpha_{012} = c^2\left(\varphi_3^{(2)} + 2\varphi_2^{(2)}\varphi_1^{(1)} + \varphi_1^{(2)}\varphi_2^{(1)}\right).$$

Now we consider a special case of this model. Assume that ζ_1 and ζ_2 are exponentially distributed with parameters f and g, respectively. Then, we obtain

$$\alpha_{101} = \frac{c}{f}\left(\frac{2}{f} + \frac{1}{g}\right), \quad \alpha_{011} = \frac{c}{g}\left(\frac{2}{g} + \frac{1}{f}\right),$$

$$\alpha_{201} = \frac{2c}{f^2}\left(\frac{3}{f} + \frac{1}{g}\right), \quad \alpha_{021} = \frac{2c}{g^2}\left(\frac{3}{g} + \frac{1}{f}\right),$$

$$\alpha_{111} = \frac{2c}{fg}\left(\frac{1}{f} + \frac{1}{g}\right),$$

$$\alpha_{102} = \frac{2c^2}{f}\left(\frac{3}{f^2} + \frac{2}{fg} + \frac{1}{g^2}\right),$$

$$\alpha_{012} = \frac{2c^2}{g}\left(\frac{3}{g^2} + \frac{2}{fg} + \frac{1}{f^2}\right).$$

♦

In addition, in this case we can also calculate, the Pearson correlation coefficient

$$R = \frac{\delta_{11} - \delta_1^{(1)}\delta_1^{(2)}}{s_1 s_2},$$

where

$$s_1 = \sqrt{\delta_2^{(1)} - \left(\delta_1^{(1)}\right)^2},$$

$$s_2 = \sqrt{\delta_2^{(2)} - \left(\delta_1^{(2)}\right)^2}.$$

In Tables 3 and 4 we present numerical results of our analyses. Table 3 shows computations for fixed parameters $f = 1$, $g = 2$, $c = 1$; the value of a is changing. In Table 4 we present results for parameters $a = 0.1$, $c = 1$, $g = 2$, and a changing value of f. Results are also presented in Figs. 4 and 5. It can be easily noticed that increasing the values of a or $1/f$ has influence on the total volume vector characteristics. If the value of ρ is close to one, then the system becomes overloaded and the calculated characteristics are strongly increasing. Analyzing the values of the Pearson coefficient R, we also see that sectors of the total volume vector are dependent, because the customer's service time depends on both the volume vector indications and the mean values of the total volume vector indications are also dependent on the customer's service time.

Remark 1. Theorem 3 can be generalized to the case of bulk arrivals of customers (customers arrive to the system in groups forming a Poisson entrance flow with parameter a). Analogously to the proof of the theorem, calculations

Table 3. Total volume vector characteristics for the $M/G/1/\infty$ system, $f = 1$, $g = 2$, $c = 1$.

a	$\delta_1^{(1)}$	$\delta_1^{(2)}$	δ_{11}	R
0.05	0.1297	0.0524	0.0869	0.5159
0.10	0.2706	0.1103	0.2026	0.5262
0.15	0.4258	0.1754	0.3570	0.5373
0.20	0.6000	0.2500	0.5664	0.5506
0.25	0.8000	0.3375	0.8572	0.5664
0.30	1.0364	0.4432	1.2755	0.5856
0.35	1.3263	0.5757	1.9077	0.6096
0.40	1.7000	0.7500	2.9300	0.6403
0.45	2.2154	0.9952	4.7476	0.6797
0.50	3.0000	1.3750	8.4687	0.7306
0.55	4.4000	2.0625	18.0729	0.7955
0.60	7.8000	3.7515	57.5550	0.8755
0.65	31.2000	15.4375	955.7722	0.9679

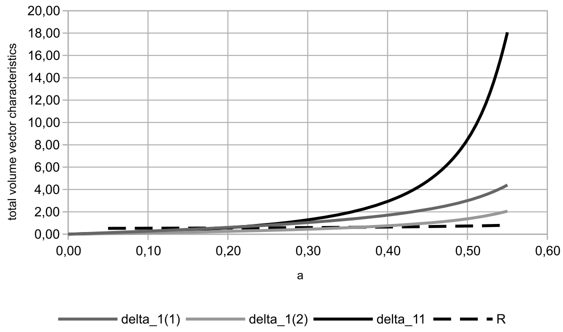

Fig. 4. Total volume vector characteristics for the $M/G/1/\infty$ system, $f = 1$, $g = 2$, $c = 1$.

lead to the following result:

$$\delta(\mathbf{s},q) = (q+a-a\pi(q))^{-1} \times \left\{1+\frac{a\left[G\left(\varphi(\mathbf{s})\right)-\pi(q)\right]}{\left[q+a-aG\left(\varphi(\mathbf{s})\right)\right]} \times \frac{\left[\varphi(\mathbf{s})-\alpha\left(\mathbf{s},q+a-aG\left(\varphi(\mathbf{s})\right)\right)\right]}{\left[\varphi(\mathbf{s})-\beta\left(q+a-aG\left(\varphi(\mathbf{s})\right)\right)\right]}\right\},$$

where $G(z) = \sum_{k=0}^{\infty} q_k z^k$ is the generating function of the number of customers in the arriving group.

Table 4. Total volume vector characteristics for the $M/G/1/\infty$ system, $a = 0.1$, $c = 1$, $g = 2$.

f	$\delta_1^{(1)}$	$\delta_1^{(2)}$	δ_{11}	R
0.15	20.5229	1.2309	55.6724	0.7619
0.25	4.7273	0.4159	5.7918	0.6412
0.35	2.1988	0.2669	2.1306	0.5999
0.45	1.2911	0.2044	1.1194	0.5762
0.55	0.8577	0.1700	0.6991	0.5602
0.65	0.6157	0.1482	0.4836	0.5487
0.75	0.4662	0.1332	0.3579	0.5401
0.85	0.3671	0.1222	0.2778	0.5334
0.95	0.2977	0.1138	0.2235	0.5284
1.05	0.2472	0.1072	0.1847	0.5240
1.15	0.2092	0.1010	0.1560	0.5208
1.25	0.1799	0.0974	0.1341	0.5181
1.35	0.1567	0.0937	0.1170	0.5163
1.45	0.1380	0.0906	0.1033	0.5147
1.55	0.1227	0.0878	0.0921	0.5131
1.65	0.1100	0.0855	0.0829	0.5121
1.75	0.0994	0.0834	0.0752	0.5116
1.85	0.0904	0.0816	0.0686	0.5105
1.95	0.0826	0.0799	0.0630	0.5102
2.05	0.0759	0.0785	0.0582	0.5102

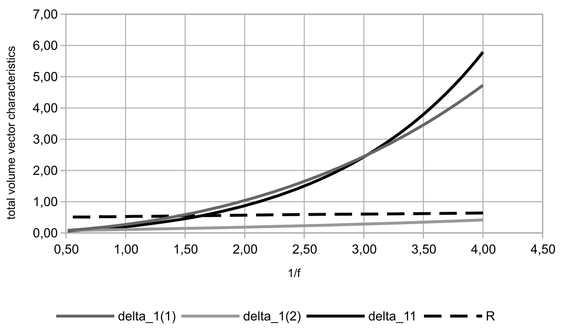

Fig. 5. Total volume vector characteristics for the $M/G/1/\infty$ system, $a = 0.1$, $c = 1$, $g = 2$.

5. Egalitarian processor sharing system

Consider now the classical queueing model, in which all customers present in the system are served simultaneously but the service speed (and, in consequence, the time remaining to service termination) depends on the number of customers present in the system (remaining time increases together with an increasing number of customers present in the system and decreases otherwise). We call such a system an egalitarian processor sharing one and denote it by $M/G/1/\infty$–*EPS*.

By a customer's length, in the analyzed system, we mean the amount of work required for his service, that is, the customer's sojourn time in the system at hand, provided that there are no other customers in the system during this time. By the residual length of the customer we mean the amount of work required to complete his service after some time instant, that is, the residual customer sojourn time, provided that there are no other customers in the system during this time.

We also introduce the following notation for vectors:

$$Y_k = (y_1, \ldots, y_k).$$

It is known (Yashkov and Yashkova, 2007) that the behavior of the classical processor sharing system can be described by the following Markov process:

$$\left(\eta(t), \xi_1^*(t), \ldots, \xi_{\eta(t)}^*(t)\right), \tag{9}$$

where $\xi_j^*(t)$ is the residual length of the j-th customer present in the system at time instant t, $j = \overline{1,\eta(t)}$. Note that, in the case of $\eta(t) = 0$, the components $\xi_j^*(t)$ are absent in (9).

We assume that the system is empty at time instant $t = 0$, i.e., $\eta(0) = 0$ and $\boldsymbol{\sigma}(0) = \mathbf{0}$ (zero initial condition). We also introduce some needed notation: $P_k(t) = \mathbf{P}\{\eta(t) = k\}$, $k = 0, 1, \ldots$, $\theta_k(t, Y_k) = \mathbf{P}\{\eta(t) = k, \xi_j^*(t) < y_j, j = \overline{1,k}\}$. It is clear that, for $k \geq 1$, we have $P_k(t) = \theta_k(t, \infty_k)$, where $\infty_k = (\infty, \ldots, \infty)$ is the k-component vector. Introduce the Laplace transform

$$\widehat{\upsilon}_k(q, Y_k) = \int_0^{\infty} e^{-qt}\, \mathrm{d}_t\theta_k(t, Y_k).$$

Then, for $k \geq 1$, we have

$$\widehat{p}_k(q) = \int_0^{\infty} e^{-qt} P_k(t)\, \mathrm{d}t = \widehat{\upsilon}_k(q, \infty_k).$$

Tikhonenko (2015) proved that

$$\widehat{\upsilon}_k(q, Y_k) = \frac{(q+a)^k}{q+a-a\pi(q)} \prod_{j=1}^{k} \int_0^{y_j} \left(1 - \frac{a}{q+a} B(t)\right) \mathrm{d}t, \quad k = 1, 2, \ldots, \tag{10}$$

where $\pi(q)$ is the LST of a busy period of the system. For $\widehat{p}_k(q)$, we obtain

$$\widehat{p}_k(q) = \widehat{v}_k(q, \infty_k) = \frac{a^k\,(1-\pi(q))^k}{(q+a)\,(q+a-a\pi(q))^{k+1}}. \tag{11}$$

Let $F(\mathbf{x}, t) = \mathbf{P}\{\boldsymbol{\zeta} < \mathbf{x}, \xi < t\}$, where, in this case, ξ is a customer's length, $L(\mathbf{x}) = F(\mathbf{x}, \infty)$, $B(t) = F(\vec{\infty}, t)$. Further, we shall use the notation from Section 4.

Let $\boldsymbol{\chi}(t) = (\chi_1(t), \dots, \chi_k(t))$ be the indication vector of a customer that is served at time instant t and $\xi^*(t)$ be his residual length. Now, we can introduce analogously the function $E_y(\mathbf{x}) = \mathbf{P}\{\boldsymbol{\chi}(t) < \mathbf{x} | \xi^*(t) = y\}$ and its LST,

$$\begin{aligned} e_y(\mathbf{s}) &= \int_{\mathbf{x}=\mathbf{0}}^{\vec{\infty}} e^{-(\mathbf{s},\mathbf{x})}\, \mathrm{d}E_y(\mathbf{x}) \\ &= [1-B(y)]^{-1} \int_{\mathbf{x}=\mathbf{0}}^{\vec{\infty}} e^{-(\mathbf{s},\mathbf{x})} \int_{u=y}^{\infty} \mathrm{d}F(\mathbf{x}, u). \end{aligned}$$

Let

$$\begin{aligned} D(\mathbf{x}, t) &= \mathbf{P}\{\boldsymbol{\sigma}(t) < \mathbf{x}\} \\ &= \mathbf{P}\{\sigma_1(t) < x_1, \dots, \sigma_k(t) < x_k\}, \end{aligned}$$

$$\overline{\delta}(\mathbf{s}, t) = \mathbf{E}e^{-(\mathbf{s}, \boldsymbol{\sigma}(t))} = \int_{\mathbf{0}}^{\vec{\infty}} e^{-(\mathbf{s},\mathbf{x})}\, \mathrm{d}_{\mathbf{x}} D(\mathbf{x}, t),$$

$$\delta(\mathbf{s}, q) = \int_0^{\infty} e^{-qt}\, \overline{\delta}(\mathbf{s}, t)\, \mathrm{d}t.$$

Theorem 4. *For a zero initial condition, the function $\delta(\mathbf{s}, q)$ can be presented as*

$$\delta(\mathbf{s}, q) = \{[q + a - a\pi(q)][1 - I(\mathbf{s}, q)]\}^{-1},$$

where

$$I(s, q) = (q+a) \int_0^{\infty} \left(1 - \frac{a}{q+a} B(y)\right) e_y(\mathbf{s})\, dy.$$

Proof. The DF $D(\mathbf{x}, t)$ can be expressed in the following form:

$$D(\mathbf{x}, t) = P_0(t) + \sum_{k=1}^{\infty} D_k(\mathbf{x}, t | Y_k)\, \mathrm{d}_{Y_k} \theta_k(t, Y_k), \tag{12}$$

where $D_k(\mathbf{x}, t | Y_k) = \mathbf{P}\{\boldsymbol{\sigma}(t) < \mathbf{x} | \eta(t) = k, \xi_j^*(t) = y_j, j = \overline{1,k}\}$ and

$$\mathrm{d}_{Y_k} \theta_k(t, Y_k) = \frac{\partial^k \theta_k(t, Y_k)}{\partial y_1 \dots \partial y_k}\, \mathrm{d}y_1 \dots \mathrm{d}y_k.$$

It is clear that the function $D_k(\mathbf{x}, t | Y_k)$ can be expressed by the Stieltjes convolution $D_k(\mathbf{x}, t | Y_k) = E_{y_1} * \dots * E_{y_k}(\mathbf{x})$. Then, passing to the LST with respect to $\mathbf{x}$ in (12), we obtain

$$\begin{aligned} \overline{\delta}(\mathbf{s}, t) &= P_0(t) \\ &\quad + \sum_{k=1}^{\infty} \int_0^{\infty} \dots \int_0^{\infty} \prod_{j=1}^{k} e_{y_j}(\mathbf{s})\, \mathrm{d}_{Y_k} \theta_k(t, Y_k). \end{aligned}$$

Passing to the Laplace transform with respect to t, we have

$$\begin{aligned} \delta(\mathbf{s}, q) &= \widehat{p}_0(q) \\ &\quad + \sum_{k=1}^{\infty} \int_0^{\infty} \dots \int_0^{\infty} \prod_{j=1}^{k} e_{y_j}(\mathbf{s}) \mathrm{d}_{Y_k} \widehat{v}_k(q, Y_k). \end{aligned}$$

From the last relation, using (10) and (11), we finally obtain

$$\begin{aligned} \delta(\mathbf{s}, q) &= \frac{1}{q + a - a\pi(q)} \left\{ 1 + \sum_{k=1}^{\infty} (q+a) \right. \\ &\quad \left. \times \left[\int_0^{\infty} \left(1 - \frac{a}{q+a} B(y)\right) e_y(\mathbf{s})\, \mathrm{d}y \right]^k \right\}. \end{aligned}$$

This relation is equivalent to the statement of the theorem. ∎

Corollary 3. *Let $\rho = a\beta_1 < 1$. The steady state exists for the system under consideration, and LST $\delta(\mathbf{s})$ of the steady-state customer total volume has the form*

$$\delta(\mathbf{s}) = \frac{1-\rho}{1 + \alpha_q'(\mathbf{s}, q)|_{q=0}}. \tag{13}$$

Proof. The existence of the limit

$$\delta(\mathbf{s}) = \lim_{q \to 0^+} q\delta(\mathbf{s}, q)$$

follows from the theory of regenerative processes (Asmussen, 2003). From this theory, we also obtain

$$\delta(\mathbf{s}) = \lim_{q \to 0^+} q\delta(\mathbf{s}, q) = (1-\rho) \lim_{q \to 0^+} [1 - I(\mathbf{s}, q)]^{-1},$$

where

$$\begin{aligned} \lim_{q \to 0^+} I(\mathbf{s}, q) &= a \int_0^{\infty} [1 - B(y)] e_y(\mathbf{s})\, \mathrm{d}y \\ &= a \int_{\mathbf{x}=\mathbf{0}}^{\vec{\infty}} \int_{u=0}^{\infty} u e^{-(\mathbf{s},\mathbf{x})}\, \mathrm{d}F(\mathbf{x}, u) \\ &= -a\alpha_q'(\mathbf{s}, q)|_{q=0}, \end{aligned}$$

which proves the statement of the corollary. ∎

From (14), we can obtain formulae for mixed moments of the random vector $\boldsymbol{\sigma}$:

$$\begin{aligned} \delta_{i_1 \dots i_l} &= \mathbf{E}\left(\delta_1^{i_1} \dots \delta_l^{i_l}\right) \\ &= (-1)^{i_1 + \dots + i_l} \frac{\partial^{i_1 + \dots + i_l}}{\partial s_1^{i_1} \dots \partial s_l^{i_l}} \delta(\mathbf{s})|_{\mathbf{s}=\mathbf{0}}. \end{aligned}$$

Example 4. Assume that a customer is characterized by the two-dimensional vector $\boldsymbol{\zeta} = (\zeta_1, \zeta_2)$, i.e, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$, and we have

$$\delta(s_1, s_2) = \frac{1-\rho}{1+\alpha_q'(s_1, s_2, q)|_{q=0}}.$$

Then we obtain

$$\delta_1^{(1)} = \mathbf{E}\delta_1 = \frac{a\alpha_{101}}{1-\rho}, \quad \delta_1^{(2)} = \mathbf{E}\delta_2 = \frac{a\alpha_{011}}{1-\rho},$$

$$\delta_2^{(1)} = \mathbf{E}\sigma_1^2 = \frac{a\alpha_{201}}{1-\rho} + \frac{2a(\alpha_{101})^2}{(1-\rho)^2},$$

$$\delta_2^{(2)} = \mathbf{E}\sigma_2^2 = \frac{a\alpha_{021}}{1-\rho} + \frac{2a(\alpha_{011})^2}{(1-\rho)^2},$$

$$\delta_{11} = \mathbf{E}(\delta_1\delta_2) = \frac{a\alpha_{111}}{1-\rho} + \frac{2a^2\alpha_{101}\alpha_{011}}{(1-\rho)^2}.$$

♦

6. Conclusions and final remarks

In the paper, we presented basic concepts connected with the theory of queueing systems with random volume customers and unlimited sectorized memory space. We analyzed three models of queueing systems important from the practical point of view: the $M/G/n/0$ Erlang queueing system, the $M/G/1/\infty$ single-server queueing system and the $M/G/1/\infty$–*EPS* egalitarian processor sharing one. For these models, we obtained general formulae characterizing total volume vectors distributions in terms of Laplace–Stieltjes transforms. We also calculated initial steady–state moments of investigated total volume vectors in the case when the memory buffer contains two sectors, and analyzed special cases of the models in which customers' service time depends on the sum of the customers' volume vector indications. We also presented some numerical examples together with graphs, and discussed the obtained results and noted dependencies.

Our research can be used in the process of designing communication or computer networks to calculate the required size of memory buffer sectors, or during studying the efficiency of real computer or telecommunication networks, especially in a state that is close to an overloaded one.

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