EXISTENCE OF SOLUTIONS OF BOUNDARY VALUE PROBLEMS OF DIFFERENTIAL INCLUSIONS

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We are interested in the necessary and sufficient conditions for the existence of solutions of differential inclusions $\dot{x}(t) \in F(t, x(t))$ for a.e. $t \in [0, T]$ satisfying boundary conditions $x(0) = x_0$ and $x(T) = x_1$.

1. Introduction

The existence of solutions of initial–value problems of differential inclusions can be obtained by means of properties of sets of all integrable selectors of their right–hand sides. In recent years the study of these sets has been done by many authors, e.g. (Bridgland, 1970; Castaing, 1967; Hiai and Umegaki, 1977; Kisielewicz, 1989 and Papageorgiou, 1985 and 1987). Some applications to the theory of neutral–functional differential and stochastic integral inclusions can be found in (Kisielewicz, 1989 and 1992), where these sets are called subtrajectory integrals of set–valued functions. In this paper the properties of subtrajectory integrals of set–valued functions are applied to obtain the existence of solutions of boundary–value problems to differential inclusions of the form $\dot{x}(t) \in F(t, x(t))$.

We shall deal here with set–valued functions taking their values in the space Comp ($\mathbb{R}^n$) of all nonempty compact subsets of the n–dimensional Euclidean space $\mathbb{R}^n$. This space is considered as a metric space with the Hausdorff metric h defined in the usual way by the Hausdorff subdistance $\overline{h}$.

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2. Subtrajectory Integrals of Set–Valued Functions

Given σ–finite measure space $(X, \mathcal{A}, \sigma)$ a set–valued function $R : X \to \text{Comp}\ (\mathbb{R}^n)$ is said to be $\mathcal{A}$–measurable if $R^-(\mathbb{C}) := \{x \in X : R(x) \cap \mathbb{C} \neq \emptyset\} \in \mathcal{A}$ for every closed set $\mathbb{C} \subset \mathbb{R}^n$. Among others it is equivalent to the existence of a sequence (f_n) of $\mathcal{A}$ - measurable functions $f_n : X \to \mathbb{R}^n$ such that $R(x) = \text{cl}\{f_n(x) : n \geq 1\}$ for all $x \in X$. Denote by $\mathcal{M}(X, \mathbb{R}^n)$ the family of all $\mathcal{A}$ - measurable set–valued functions $R : X \to \text{Comp}\ (\mathbb{R}^n)$. For $R \in \mathcal{M}(X, \mathbb{R}^n)$ the subtrajectory integrals $\mathcal{F}(R)$ is defined by $\mathcal{F}(R) = \{f \in L(X, \mathbb{R}^n) : f(x) \in R(x) \text{ a.e.}\}$. Here and later $L(X, \mathbb{R}^n)$ denotes the Banach space $L(X, \mathcal{A}, \sigma, \mathbb{R}^n)$ of (equivalent classes of) $\mathcal{A}$ - measurable functions $f : X \to \mathbb{R}^n$ such that the norm $|f| = \int_x |f(x)| d\mu$ is finite. We call $R \in \mathcal{M}(X, \mathbb{R}^n)$ integrable bounded if a real - valued function $X \ni x \to \|R(x)\|$ with $\|R(x)\| := \sup\{|r| : r \in R(x)\}$ belongs to $L(X, \mathbb{R}^n)$. It follows immediately by the Kuratowski and Ryll–Nardzewski (1965) measurable selection theorem that $\mathcal{F}(R) \neq \emptyset$ for every integrable bounded $R \in \mathcal{M}(X, \mathbb{R}^n)$.

The following results given by Hiai and Umegaki (1977) will be used in the sequel.

Proposition 1. *Let $R \in \mathcal{M}(X, \mathbb{R}^n)$. If $\mathcal{F}(R)$ is nonempty, then there exists a sequence $\{f_n\}$ contained in $\mathcal{F}(R)$ such that $R(x) = \text{cl}\{f_n(x) : n \geq 1\}$ for all $x \in X$.* ■

Proposition 2. *Let $R \in \mathcal{M}(X, \mathbb{R}^n)$ be integrable bounded. Then*

(i) *$\mathcal{F}(R)$ is a nonempty closed subset of $L(X, \mathbb{R}^n)$,*

(ii) *$\mathcal{F}(\overline{co}R)$ is convex and weakly compact in $L(X, \mathbb{R}^n)$, where $\overline{co}R$ is defined by $(\overline{co}R) := \overline{co}R(x)$ for $x \in X$.* ■

Let Z be a locally convex Hausdorff vector space, $\mathcal{D}$ its nonempty subset and I a compact interval of the real line. A set- -valued function $R : I \times \mathcal{D} \to \text{Comp}\ (\mathbb{R}^n)$ will be called measurable with respect to its first variable if $R(\cdot, z)$ is measurable for each fixed $z \in \mathcal{D}$. It is said to be uniformly integrable bounded if there is a function $m_R \in L(I, \mathbb{R}^1)$ such that $\|R(t,z)\| \leq m_R(t)$ a.e. $t \in I$. We call R upper [weakly - weakly sequentially upper] semicontinuous (u.s.c.) [(w–w.s.u.s.c.)] with respect to its last variable if for every $z_0 \in \mathcal{D}$ and every sequence (z_n) of $\mathcal{D}$ converging [weakly converging] to z_0 one has $\lim_{n\to 0} \overline{h}(R(t, z_n), R(t, z_0)) = 0$ for a.e. $t \in I$ $[\lim_{n\to\infty} \overline{h}(\int_E R(t, z_n)dt,\ \int_E R(t, z)dt) = 0$ for every Lebesgue measurable set $E \subset I]$.

In what follows by $\mathcal{F}(R)(z)$ we shall denote a subtrajectory integrals of $R(\cdot,z)$ for fixed $z \in \mathcal{D}$ whereas $\mathcal{F}(R)$ will denote a set–valued mapping $\mathcal{D} \ni z \to \mathcal{F}(R)(z) \subset L(I,\mathbb{R}^n)$. We call $\mathcal{F}(R)$ weakly - weakly [sequentially] upper semicontinuous (w–w.u.s.c) [(w–w.s.u.s.c.)] on $\mathcal{D}$ if for every weakly closed set $\mathbb{C} \subset L(I,\mathbb{R}^n)$, a set $\mathcal{F}(R)^-(\mathbb{C}) := \{z \in \mathcal{D} : \mathcal{F}(R)(z) \cap \mathbb{C} \neq \emptyset\}$ is weakly [sequentially weakly] closed in Z. If $\mathcal{D}$ is weakly compact then immediately from Šmulian's theorem it follows that $\mathcal{F}(R)$ is w.–w.u.s.c on $\mathcal{D}$ if and only if it is w.–w.s.u.s.c there.

3. The Ky Fan Fixed Point Theorem

Let Z be a locally convex Hausdorff vector space and let Z^* be its dual. Denote by $< z,p >$ a duality pairing on $Z^* \times Z$ defined in the usual way by setting $< z,p >:= p(z)$ for all $p \in Z^*$ and $z \in Z$.

In this section, we supply Z with the weak topology $\sigma(Z,Z^*)$. Let $K \subset Z$ be a nonempty set. By $s(K,\cdot) : Z^* \to (-\infty,\infty]$ we denote the support function of K, i.e., $s(K,p) := \sup_{z\in K} < z,p >$. It is well know that for a nonempty closed convex set $K \subset Z$ one has $K = \{z \in Z :< z,p > \leq s(K,p)$ for every $p \in Z^*\}$.

Assume $K \subset Z$ and F is a set–valued map from K to Z. For every $p \in Z^*$ we can consider the function with values in $(-\infty,\infty]$ given by $K \ni z \to s(F(z),p) \in (-\infty,\infty]$.

We say that F is upper hemicontinuous (u.h.c) at $z_0 \in K$ if for every $p \in Z^*$, the function $z \to sF(z),p)$ is u.s.c at z_0. F is called upper hemicontinuous if it is u.h.c. at each $z_0 \in K$.

We have the following known result (see Aubin and Cellina, 1984, p. 80).

Proposition 3. *Any upper semicontinuous map from K to Z supplied with the weak topology is upper hemicontinuous.* ■

Corollary 1. *Let $R : I\times\Lambda \to$ Comp $(\mathbb{R}^n)$ have convex values, be uniformly integrable bounded, measurable with respect to its first variable and w. - w.s.u.s.c with respect to the second one. If Λ is a weakly compact subset of $L(I,\mathbb{R}^n)$ then $\mathcal{F}(R)$ is upper hemicontinuous on Λ.*

Indeed, similarly as in (Kisielewicz, 1989, Lemma 3.5) it can be seen that $\mathcal{F}(R)$ is w.–w.s.u.s.c on Λ which by the weak compactness of Λ implies that $\mathcal{F}(R)$ is w.–w.u.s.c. Now our result follows immediately from Proposition 3. ■

For a given nonempty closed and convex set $X \subset Z$ by $\mathbb{N}_x(z)$ and $\mathbf{T}_x(z)$ we denote respectively "normal" and "tangent" cones to X at $z \in Z$. They are defined by

$$\mathbb{N}_X(z) := \{p \in Z^* :< z, p >= \max_{y \in X} < y, p >\} \tag{1}$$

and

$$\mathbf{T}_X(z) := \{x \in Z :< x, p >\leq 0 \text{ for each } p \in \mathbb{N}_X(z)\} \tag{2}$$

The following fixed point theorem will be useful in this paper (see Aubin, 1979, p. 531).

Theorem (Ky - Fan). *Let X be a nonempty convex weakly compact subset of Z and suppose $\mathbf{S}$ is an upper hemicontinuous map from X into the space $\mathrm{Conv_w}(Z)$ of all nonempty convex weakly compact subsets of Z. If furthermore $\mathbf{S}(z) \cap [z + \mathbf{T}_X(z)] \neq \emptyset$ for every $z \in X$ then $\mathbf{S}$ has in X a fixed point.* ■

Finally, we quote the following result (see Aubin and Ekeland, 1984, p.174).

Proposition 4. *Let A be a linear mapping from $L(I, \mathbb{R}^n)$ into $\mathbb{R}^n$ and assume K and M are nonempty closed and convex subsets of $L(I, R^n)$ and $\mathbb{R}^n$, respectively. If K and M are such that $0 \in \mathrm{Int}(A(K) - M)$ then $\mathbf{T}_{K \cap A^{-1}(M)}(x) = \mathbf{T}_K(x) \cap A^{-1}(\mathbf{T}_M(Ax))$ for every $x \in K \cap A^{-1}(M)$.* ■

4. Existence of Solutions of Boundary Value Problems

We shall deal here with the boundary value problem

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) & \text{for a.e.} \quad t \in I := [0, T] \\ x(0) = x_0, & x(T) \in B, \end{cases} \tag{3}$$

where $F : I \times \mathbb{R}^n \to \mathrm{Comp}\,(\mathbb{R}^n)$, $x_0 \in \mathbb{R}^n$ and $B \subset \mathbb{R}^n$ are given.

Assume that F satisfies the following conditions (H):

(i) F is $\mathcal{L}_I \otimes B(\mathbb{R}^n)$ - measurable where $\mathcal{L}_I$ and $B(\mathbb{R}^n)$ denote Lebesgue and Borel σ - algebra of I and $\mathbb{R}^n$, respectively,

(ii) F is uniformly integrable bounded.

Let us define linear mappings $\mathcal{T}_T : L(I, \mathbb{R}^n) \to \mathbb{R}^n$ and $\mathcal{T} : L(I, \mathbb{R}^n) \to \mathbb{C}(I, \mathbb{R}^n)$ by taking $\mathcal{T}_T u = \int_0^T u(t)dt$ and $\mathcal{T}(u)(t) = \int_0^t u(\tau)d\tau$ for $u \in L(I, \mathbb{R}^n)$ and $t \in I$, respectively. It is clear that for every nonempty convex and weakly compact set $K \subset L(I, \mathbb{R}^n)$ its image $\mathcal{T}_T K$ is a nonempty compact and convex subset of $\mathbb{R}^n$. It is also clear that $\mathcal{T}_T$ maps $L(I,\mathbb{R}^n)$ onto $\mathbb{R}^n$.

Let F be such as above and satisfies conditions (H). By $\mathcal{F}(F\Box\mathcal{T})(u)$ we shall denote subtrajectory integrals of a set–valued function $I \ni t \to F(t, x_0+(\mathcal{T}u)(t)) \in \text{Comp}\,(\mathbb{R}^n)$ defined for each fixed $u \in L(I,\mathbb{R}^n)$. Immediately by the Kuratowski–Ryll–Nardzewski measurable selection theorem it follows that $\mathcal{F}(F\Box\mathcal{T})(u) \neq \emptyset$ for fixed $u \in L(I,\mathbb{R}^n)$. It is also a closed subset of $L(I,\mathbb{R}^n)$. If moreover $F(t,x)$ is convex for $(t,x) \in I \times \mathbb{R}^n$ then $\mathcal{F}(F\Box\mathcal{T})(u)$ is also convex. By (ii) of Proposition 2 it follows that in the last case $\mathcal{F}(F\Box\mathcal{T})(u)$ is also weakly compact.

Given a nonempty convex and weakly compact set $\Lambda \subset L(I,\mathbb{R}^n)$ and $u \in \Lambda$ by $\mathcal{Z}(\Lambda,u)$ we denote the set $\bigcap_{p\in \mathbb{N}_{\mathcal{T}_T\Lambda}(\mathcal{T}_T u)}\{x \in \mathbb{R}^n :< x,p >\leq s(\mathcal{T}_T\Lambda,p)\}$ where $< \cdot,\cdot >$ is an inner product of $\mathbb{R}^n$ and $s(\mathcal{T}_T\Lambda,\cdot)$ denotes the support function of $\mathcal{T}_T\Lambda$. Finally, $\mathbb{N}_{\mathcal{T}_T\Lambda}(\mathcal{T}_T u)$ denotes as usual a normal cone to $\mathcal{T}_T\Lambda$ at $\mathcal{T}_T u \in \mathcal{T}_T\Lambda$. Now we define a set $\mathcal{C}(\Lambda,u) \subset L(I,\mathbb{R}^n)$ by taking $\mathcal{C}(\Lambda,u) := \{\nu \in L(I,\mathbb{R}^n) : \mathcal{T}_T\nu \in \mathcal{Z}(\Lambda,u)\}$.

Lemma 3. *Let Λ be a nonempty convex weakly compact subset of $L(I,\mathbb{R}^n)$ and let $F : I \times \mathbb{R}^n \to \text{Comp}\,(\mathbb{R}^n)$ satisfy conditions (H). Then for every $u \in \Lambda$ the following conditions are equivalent:*

(i) $[u + \mathbf{T}_\Lambda(u)] \cap \mathcal{F}(F\Box\mathcal{T})(u) \neq \emptyset$

(ii) $\mathcal{C}(\Lambda,u) \cap \mathcal{F}(F\Box\mathcal{T})(u) \neq \emptyset$.

Proof. Let us observe first that for $K = L(I,\mathbb{R}^n)$, $M = \mathcal{T}_T\Lambda$ and $A = \mathcal{T}_T$ the assumptions of Proposition 4 are satisfied. Then for every $u \in \Lambda$ one has

$$\mathbf{T}_\Lambda(u) = \mathbf{T}_{K\cap\mathcal{T}_T^{-1}(M)}(u) = \mathbf{T}_K(u) \cap \mathcal{T}_T^{-1}(\mathbf{T}_M(\mathcal{T}_T u)) = \mathcal{T}_T^{-1}(\mathbf{T}_M(\mathcal{T}_T u)).$$

Therefore

$$[u + \mathbf{T}_\Lambda(u)] \cap \mathcal{F}(F\Box\mathcal{T})(u) = [u + \mathcal{T}_T^{-1}(\mathbf{T}_M(\mathcal{T}_T u))] \cap \mathcal{F}(F\Box\mathcal{T})(u).$$

Now, for every fixed $u \in \Lambda$ we have $[u + \mathbf{T}_\Lambda(u)] \cap \mathcal{F}(F\Box\mathcal{T})(u) \neq \emptyset$ if and only if there exists $\nu \in \mathcal{F}(F\Box\mathcal{T})(u)$ such that $\nu - u \in \mathcal{T}_T^{-1}(\mathbf{T}_M(\mathcal{T}u))$ which is equivalent to $\mathcal{T}_T(\nu - u) \in \mathbf{T}_M(u)$. By the definition of $\mathbf{T}_M(\mathcal{T}u)$ it follows that $\mathcal{T}_T(\nu - u) \in \mathbf{T}_M(\mathcal{T}u)$ if and only if for every $p \in \mathbb{N}_M(\mathcal{T}u)$ one has $< \mathcal{T}_T(\nu - u), p >\leq 0$. But $< \mathcal{T}_T(\nu - u), p >=< \mathcal{T}_T\nu, p > - < \mathcal{T}_T u, p >$ and $< \mathcal{T}_T u, p >= s(M,p)$ for every $p \in \mathbb{N}_M(\mathcal{T}u)$. Therefore $\mathcal{T}_T(\nu - u) \in \mathbf{T}_M(u)$

if and only if $< \mathcal{T}_T\nu, p >\leq s(M,p)$ for every $p \in \mathbb{N}_M(\mathcal{T}_T u)$ that is equivalent to $\mathcal{T}_T\nu \in \mathcal{Z}(\Lambda, u)$ or to $\nu \in \mathcal{T}_T^{-1}(\mathcal{Z}(\Lambda, u)) = \mathcal{C}(\Lambda, u)$. ∎

Now, we can prove the following existence theorem.

Theorem 4. *Suppose $F : I \times \mathbb{R}^n \to$ Comp$(\mathbb{R}^n)$ satisfies conditions (H), has convex values and is such that $F(t,\cdot)$ is u.s.c. for fixed $t \in I$. Then for every $x_0 \in \mathbb{R}^n$ and a nonempty set $B \subset \mathbb{R}^n$ a boundary value problem (3) has at least one solution if and only if there exists a nonempty convex and weakly compact set $\Lambda \subset \mathcal{T}_T^{-1}(B - x_0)$ such that $\mathcal{C}(\Lambda, u) \cap \mathcal{F}(F\square\mathcal{T})(u) \neq \emptyset$ for every $u \in \Lambda$.*

Proof. Suppose $x \in A\mathbb{C}(I, \mathbb{R}^n)$ is a solution of (3) and put $u = \dot{x}$. We have $u \in \mathcal{F}(F\square\mathcal{T})(u)$ and $u \in \mathcal{T}_T^{-1}(B - x_0)$. Taking $\Lambda = \{u\}$ we obtain $\mathbf{T}_\Lambda(u) = \{0\}$. Then $u + \mathbf{T}_\Lambda(u) = \{u\}$ and therefore $[u + \mathbf{T}_\Lambda(u)] \cap \mathcal{F}(F\square\mathcal{T})(u) \neq \emptyset$. By this and Lemma 3 it follows $\mathcal{C}(\Lambda, u) \cap \mathcal{F}(F\square\mathcal{T})(u) \neq \emptyset$.

Suppose there is a nonempty convex weakly compact set $\Lambda \subset \mathcal{T}_T^{-1}(B - x_0)$ such that $\mathcal{C}(\Lambda, u) \cap \mathcal{F}(F\square\mathcal{T})(u) \neq \emptyset$ for every $u \in \Lambda$. Put $\mathbf{S}(u) := \mathcal{F}(F\square\mathcal{T})(u)$ for $u \in \Lambda$. By (ii) of Proposition 2, $\mathbf{S}$ has nonempty convex and weakly compact values. Moreover by Corollary 1, $\mathbf{S}$ is upper hemicontinuous on Λ and using Lemma 3 we deduce that $\mathbf{S}$ satisfies the assumptions of Ky Fan fixed point theorem. Therefore there is in Λ a fixed point for $\mathbf{S}$. Let $u \in \Lambda$ be such fixed point. It is easy to see that $x = x_0 + \mathcal{T}u$ is a solution of the boundary value problem (3). ∎

Corollary 2. *Suppose $F : I \times \mathbb{R}^n \to$ Comp$(\mathbb{R}^n)$ satisfies the assumptions of Theorem 4 and let $x_1 \in \mathbb{R}^n$ be given. The boundary value problem (3) with $B = \{x_1\}$ has at least one solution if and only if there exists a nonempty convex weakly compact set $\Lambda \subset \mathcal{T}_T^{-1}(x_1 - x_0)$ such that $x_1 - x_0 \in \int_0^T F(t, x_0 + (\mathcal{T}u)(t))dt$ for every $u \in \Lambda$.*

Indeed, for every nonempty set $\Lambda \subset \mathcal{T}_T^{-1}(x_1 - x_0)$ we have $\mathbb{N}_{\mathcal{T}_T}(\mathcal{T}u) := \{p \in \mathbb{R}^n :< \mathcal{T}_T u, p >=< x_1 - x_0, p >\} = \mathbb{R}^n$ for every $u \in \Lambda$ because $s(x_1 - x_0, p) =< x_1 - x_0, p >$ and $\mathcal{T}_T u = x_1 - x_0$. Therefore, $\mathcal{Z}(\Lambda, u) = (x_1 - x_0)$ and $\mathcal{C}(\Lambda, u) = \mathcal{T}_T^{-1}(x_1 - x_0)$ for every nonempty set $\Lambda \subset \mathcal{T}_T^{-1}(x_1 - x_0)$ and $u \in \Lambda$.

Now suppose there exists a nonempty convex weakly compact set $\Lambda \subset \mathcal{T}_T^{-1}(x_1 - x_0)$ such that $x_1 - x_0 \in \int_0^T F(t, x_0 + (\mathcal{T}u)(t))dt$ for every $u \in \Lambda$. Then, for every $u \in \Lambda$ there exists $\nu \in \mathcal{F}(F\square\mathcal{T})(u)$ such that $\int_0^T \nu(t)dt = x_1 - x_0$, i.e. $\nu \in \mathcal{T}_T^{-1}(x_1 - x_0) = \mathcal{C}(\Lambda, u)$. Thus $\mathcal{C}(\Lambda, u) \cap \mathcal{F}(F\square\mathcal{T})(u) \neq \emptyset$ for every $u \in \Lambda$ and the existence of solutions of (3) follows immediately from Theorem 4. If (3) has at least one solution x then taking $\Lambda = \{\dot{x}\}$ we have $\dot{x} \in \mathcal{F}(F\Box\mathcal{T})(\dot{x})$ and $\int_0^T \dot{x}(t)dt = x_1 - x_0$. Thus for $u \in \Lambda$ one has $x_1 - x_0 \in \int_0^T F(t, x_0 + (\mathcal{T}u)(t))dt$. ■

Corollary 3. *Let U be a nonempty subset of $\mathbb{R}^m$ and suppose $f : I \times \mathbb{R}^n \times U \to \mathbb{R}^n$ satisfies the Carathéodory conditions and let $x(\cdot, x_0, u)$ be a solution of an initial value problem*

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)) \text{ for a.e. } t \in [0,T] \\ x(0) = x_0 \end{cases}$$

for fixed measurable function $u : I \to U$. Then for every measurable function $\widetilde{u} : I \to U$ such that $\int_0^T f(t, x(t, x_0, \widetilde{u}), \widetilde{u}(t))dt = x_1 - x_0$ the boundary value problem

$$\begin{cases} \dot{x}(t) = f(t, x(t), \widetilde{u}(t)) \text{ for a.e. } t \in [0,T] \\ x(0) = x_0, \ \ x(T) = x_1 \end{cases}$$

has at least one solution $x(\cdot, x_0, \widetilde{u})$. ■

Example 1. *Given real constant matrices A and B of dimensions $n \times n$ and $n \times m$, respectively and $x_0, x_1 \in \mathbb{R}^n$ find a control function $u \in L([0,T], \mathbb{R}^m)$ to a boundary value problem*

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \text{ for a.e. } t \in [0,T] \\ x(0) = x_0, \ \ x(T) = x_1. \end{cases}$$

According to Corollary 3 a desired control function $u \in L([0,T]), \mathbb{R}^m)$ must satisfy an integral equation

$$\int_0^T [Ae^{-At}x_0 + Ae^{-At} \int_0^t e^{-As}Bu(s)ds + Bu(t)]dt = x_1 - x_0$$

which is equivalent to $\int_0^T e^{-At}Bu(t)dt = e^{-AT}x_1 - x_0$. Thus a desired control function $u \in L([0,T], \mathbb{R}^m)$ must be such that $\Phi(\cdot)Bu(\cdot) \in \mathcal{T}_T^{-1}(e^{-AT}x_1 - x_0)$, where Φ is a fundamental matrix of $\dot{x} = Ax$. In this particular case we can take $u = \sum_{i=1}^k \mathbb{C}_i\chi_i$ with $\mathbb{C}_i \in \mathbb{R}^m$; $i = 1, 2, ..., k$ satisfying $\sum_{i=1}^k \Delta_i B\,\mathbb{C}_i = x_0 - e^{-AT}x_1$, where χ_i denotes the characteristic function of $I_i = [t_{i-1}, t_i]$ with $0 = t_0 < t_1 < ... < t_{k-1} < t_k = T$ and $\Delta_i = \Phi(t_i) - \Phi(t_{i-1})$ for $i = 1, 2, ..., k$. ■

Now, immediately from Theorem 4 the following existence theorem is obtained.

Theorem 5. *Let $x_0 \in \mathbb{R}^n$ and a nonempty subset B of $\mathbb{R}^n$ be given. Assume F satisfies conditions (H), has convex values and is such that $F(t,\cdot)$ is u.s.c. for fixed $t \in I$. The boundary value problem (3) has at least one solution if and only if there are a nonempty convex weakly compact set $\Lambda \subset \mathcal{T}_T^{-1}(B - x_0)$ and a family $\{K_u\}_{u\in\Lambda}$ of measurable set-valued functions $K_u : [0,T] \to Cl(\mathbb{R}^n)$ such that*

(i) $\int_0^T K_u(t)dt \subset \mathcal{Z}(\Lambda, u)$ *for* $u \in \Lambda$

(ii) $F(t, x_0 + (\mathcal{T}u)(t)) \cap K_u(t) \neq \emptyset$

for each fixed $u \in \Lambda$ and a.e. $t \in [0,T]$.

Proof. Let $\Phi_u(t) := F(t, x_0 + \mathcal{T}u)(t)) \cap K_u(t)$ for fixed $u \in \Lambda$ and a.e. $t \in [0,T]$. It is clear that for every fixed $u \in \Lambda$, Φ_u has an integrable selector. Suppose $w \in L([0,T], \mathbb{R}^n)$ is such that $w(t) \in \Phi_u(t)$ for a.e. $t \in [0,T]$ and fixed $u \in \Lambda$. Then $w \in \mathcal{F}(F\square\mathcal{T})(u)$ and $\int_0^T w dt \in \mathcal{Z}(\Lambda, u)$ for fixed $u \in \Lambda$ or equivalently $\mathcal{F}(F\square\mathcal{T})(u) \cap \mathcal{C}(\Lambda, u) \neq \emptyset$ for each fixed $u \in \Lambda$. Now our result follows immediately from Theorem 4. ■

Corollary 4. *Let $x_0 \in \mathbb{R}^n$ and a nonempty subset B of $\mathbb{R}^n$ be given and suppose F satisfies conditions (H), has convex values and is such that $F(t,\cdot)$ is u.s.c. for fixed $t \in I$. The boundary value problem (3) has at least one solution if and only if there exists a nonempty convex weakly compact set $\Lambda \subset \mathcal{T}_T^{-1}(B - x_0)$ such that $\frac{e^T-1}{e^t} F(t, x_0 + (\mathcal{T}u)(t)) \cap \mathcal{Z}(\Lambda, u) \neq \emptyset$ for $u \in \Lambda$ and a.e. $t \in [0,T]$.*

The result follows immediately from Theorem 5 with $K_u(t) := \frac{e^t}{e^T-1}\mathcal{Z}$ (Λ, u) for $t \in [0,T]$ and fixed $u \in \Lambda$. ■

Example 2. *Given real numbers x_0 and x_1 find a control function $\nu \in L([0,T], \mathbb{R}^n)$ to a boundary value problem*

$$\begin{cases} \dot{x} = x + \nu \text{ for a.e. } t \in [0,T] \\ x(0) = x_0, \quad x(T) = x_1 \end{cases} \tag{4}$$

According to Corollary 4 a control function $\nu \in L([0,T], \mathbb{R}^n)$ is such that a function $u_\nu := \dot{x}_\nu$ with $x_\nu(t) = e^t(x_0 + \int_0^t e^{-\tau}\nu(\tau)dt)$ satisfies conditions: $\int_0^T u_\nu(t)dt = x_1 - x_0$ and $x_0 + \int_0^t u_\nu(\tau)dt + \nu(t) = \frac{e^t}{e^T-1}(x_1 - x_0)$ for $t \in [0,T]$. In such case the assumptions of Corollary 4 will be satisfied with $\Lambda = \{u_\nu\}$. Denoting $w(t) = \int_0^t e^{-\tau}\nu(\tau)d\tau$ we can define a desired control function ν by the formula $\nu(t) = e^t\dot{w}(t)$ for a.e. $t \in [0,T]$ where w is a solution to the initial value problem:

$$\begin{cases} \dot{w} + w = \frac{x_1 - x_0}{e^T - 1} \\ w(0) = 0. \end{cases}$$

It is easy to see that $w(t) = \frac{e^t - 1}{e^t}\left(\frac{x_1 - x_0}{e^T - 1} - x_0\right)$ and then $\nu(t) = \frac{x_1 - x_0}{e^T - 1} - x_0$. Finally, we can easily check that a function $x(t) = x_0 + (e^t - 1)\frac{x_1 - x_0}{e^T - 1}$ is a solution of a boundary value problem (4) with $\nu = \frac{x_1 - x_0}{e^T - 1} - x_0$. ■

Example 3. *Given real numbers x_0 and x_1 find a control function $\nu \in L([0,T],\mathbb{R}^n)$ to a boundary value problem*

$$\begin{cases} \dot{x} = x\nu \quad \text{for a.e.} \quad t \in [0,T] \\ x(0) = x_0, \quad x(T) = x_1 \end{cases} \tag{5}$$

Similarly as above, we can take $\Lambda = \{u_\nu\}$ with $u_\nu = \dot{x}_\nu$ where $x_\nu(t) = x_0 \exp \int_0^t \nu d\tau$ and $\nu \in L([0,T],\mathbb{R}^1)$ is such that $\int_0^T u_\nu d\tau = x_1 - x_0$ and $(x_0 + \int_0^t u_\nu d\tau)\nu(t) = \frac{e^t}{e^T - 1}(x_1 - x_0)$ for $t \in [0,T]$. Hence it follows that the above function ν satisfies a functional equation $d/dt\left(x_0 \exp \int_0^t \nu d\tau\right) = \frac{e^t}{e^T - 1}(x_1 - x_0)$ which can be written in the form $x_0 \exp \int_0^t \nu d\tau - x_0 = \frac{e^t - 1}{e^T - 1}(x_1 - x_0)$. By such control function ν the solution of a boundary value problem (5) is defined by $x(t) = \frac{e^t - 1}{e^T - 1}(x_1 - x_0) + x_0$. Indeed, we have

$$\dot{x}(t) = \frac{e^t}{e^T - 1}(x_1 - x_0) \quad \text{and} \quad x(t)\nu(t) =$$

$$= \left[\frac{e^t - 1}{e^T - 1}(x_1 - x_0) + x_0\right]\left[\frac{(e^T - 1)x_0 \frac{e^t}{(e^T - 1)x_0}(x_1 - x_0)}{(e^t - 1)(x_1 - x_0) + x_0(e^T - 1)}\right] =$$

$$= \frac{(e^t - 1)(x_1 - x_0) + x_0(e^T - 1)}{E^T - 1} \cdot \frac{e^t(x_1 - x_0)}{(e^t - 1)(x_1 - x_0) + x_0(e^T - 1)} =$$

$$= \frac{e^t}{e^T - 1}(x_1 - x_0).$$

Then $\dot{x}(t) - x(t)\nu(t)$ for a.e. $t \in [0,T]$. It is also easily seen that $x(0) = x_0$ and $x(T) = x_1$.

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