FINITE ELEMENT DYNAMIC MODEL OF MULTILINK FLEXIBLE MANIPULATORS

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Lagrange's finite element approach is used to formulate the dynamic model of a flexible manipulator system in the three-dimensional space. All manipulator links are considered as slim beams connected in one d.o.f. frictionless revolute joints. Beam-type finite elements, with interpolating polynomials of degree three and six generalized perturbation coordinates per finite element nodal point, are used for description of the link small displacement field. The Denavit-Hartenberg 4×4 matrix method is used to describe the flexible manipulator kinematics. Joint and link transformation matrices, describing the transformation of coordinate systems between the ends of the deformed manipulator link, have been applied. The resulting equations of motion take into account a coupling between the rigid body motion and elastic deformations of links.

1. Introduction

One of the main criteria space systems must satisfy is the demand for a minimum weight of the structure, due to the limitations in launch capabilities. That is why all the space structures are relatively flexible, including robot manipulators. In addition to that, the application of non-conventional robot structures made of aluminum alloys or composite materials, with long lightweight links, enable us to achieve a good ratio of payload capacity to manipulator weight, but the resulting robot structures are much more flexible. As the robot operational speed increases, the inertia forces also increase and link deformations cannot be neglected in the robot modelling, simulation and design processes. These three factors cause that the robot structure flexibility should be taken into account for robot dynamics and control calculations, especially if the positioning and force control accuracy, as well as the small total weight of the robot are of importance, as is in space applications. Particularly, designing sophisticated robot control system algorithms and choosing proper control devices call for more accurate mathematical models of controlled flexible robot systems. All these factors require creating and testing effective methods of modelling of flexible robot structures.

Mathematical modelling of flexible robot systems is a difficult problem, mainly due to a coupling between the rigid body motion of links and their elastic deformations. The rigid body motion of links causes changes in the geometrical parameters

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of the system, which affect the small perturbation deformations. Similarly, small perturbation deformations affect the rigid body motion of the manipulator.

In the rigid body modelling approach, robot links are treated as rigid bodies connected in robot joints and the stresses in links are assumed to originate from external load forces and inertia forces only. In the flexible body modelling method, robot links are treated as flexible bodies that are able to store and release potential energy and also undergo finite deformations.

General reviews of the methods of modelling and control of flexible multibody structures are given by Huston (1990; 1991), Meirovitch (1989), Shabana (1989) and Rao et al. (1990), Amirouche and Huston (1988), Silver (1982), Vincent et al. (1990). There are many still unsolved problems regarding modelling methods of flexible multibody structures. The advantages and disadvantages of different modelling methods, pertaining to their accuracy and efficiency, are still widely investigated.

Two methods, the assumed mode method, also called the modal expansion one and the finite element method are commonly used to approximate the deformed shapes of manipulator links. Modelling techniques using the first method consider a flexible manipulator link as an elastic body with an infinite number of degrees of freedom. A deformed shape of link is expanded using the assumed modes. Various modal shapes and boundary conditions have been considered. The assumed modes approach was shown by Baruh and Tadikonda (1989), Benati and Moro (1988), Book and Majette (1983), Book (1984), Castelazo and Lee (1990), Chang and Gannon (1990), Chiou and Shahinpoor (1990), Chonan and Aoshima (1990), Chonan and Umeno (1989), Judd and Falkenburg (1985), Low and Vidyasagar (1988), Morris and Vidyasagar (1990), Sasiadek (1985), Sasiadek and Srinivasan (1986, 1987, 1989), Spector and Flashner (1989), Shamsa and Flashner (1990), Singh and Schy (1986), Wang and Vidyasagar (1986).

The finite element approach assumes that manipulator links are divided into finite elements, whose elastic behaviour is described by arbitrarily established shape functions. This method of approximation of the deformed shape of link, is combined with various procedures of generating equations of motion. The main techniques used are: Lagrangian methods, Newton-Euler methods, virtual work methods, application of Gibbs- Appell equations and application of Kane's equations, as stated by Huston (1991).

A lot of studies were carried out, where the response of the robotic mechanism was calculated as the superposition of the large rigid body motion and small perturbation components. In this type of contribution, the large and small motions of the robot are separated. The finite element method was used by Jonker (1989) for mechanisms with deformable links and Liou and Erdman (1989) for a flexible planar four-bar linkage. Sunada and Dubowsky (1981; 1983) described dynamic equations formulation and their solution, using the finite element method, where a small elastic motion of links is imposed on the given rigid body motion of the entire mechanism. The finite element method was used for a flexible manipulator by Naganathan and Soni (1987; 1988). They presented an application of the Newton-Euler formulation, for beam finite elements with linear shape function. Usoro and Nadira (1984) and Usoro et al. (1986) discussed the formulation of dynamic equations and simulation results

for a planar finite element manipulator model with beam-type elements. Asada *et al.* (1990) expanded the deformation of links by using directly the Rayleigh-Ritz method.

Turcic and Midha (1984) as well as Nagarajan and Turcic (1990) derived a set of equations of motion for the elastic mechanism systems. The finite element method, with three-dimensional beam elements, was used to descretize the deformation of links. Unfortunately no numeric implementation of the models presented was discussed.

Li *et al.* (1990) presented a computational algorithm for the linearized dynamic model of a robot manipulator. The method deals with the rigid link manipulator, but the algorithm mentioned above makes fast computation possible. Golla *et al.* (1989) described an algorithm for the dynamic calculation of the flexible manipulator, on the basis of the Newton-Euler method applied to chain multibody dynamics problems. Ku and Chen (1990) applied the Timoshenko beam model to take into account the shear deformation of a link. Gordaninejad *et al.* (1991) studied the motion of a planar robot arm consisting of one revolute and one prismatic joint. Chang and Hamilton (1991a; 1991b) introduced the concept of the Equivalent Rigid Link System (ERLS) for modelling the kinematics and the dynamics of the robotic manipulators with flexible links. The improved calculation methods of derivation of a manipulator inertia matrix were discussed by Lilly and Orin (1991). Kalra and Sharan (1991) formulated dynamic equations of a flexible manipulator using the Galerkin method. Bouttaghou and Erdman (1991) described a unified approach to systematic derivation of the dynamic equations for flexible bodies. The proposed approach is independent of the field kinematic representation of deformations. The Euler-Bernouli and Timoshenko beam models were considered in the paper. Castelain and Bernier (1990) presented a program for the computation of motion equations for robot manipulators using hypercomplex algebra methods.

Shabana (1986) discussed the inertia coupling between the large motion and elastic deformations in flexible multibody systems and a proper choice of the generalized coordinates, so that the numerical procedure could be simplified. Shabana (1990) described the general dynamic model of flexible multibody systems, using the generalized Newton-Euler equations. Deformable bodies in this model can undergo large translational and rotational displacements. Tzou and Wan (1990) focused on the damping of the vibration of flexible manipulators modelled by the finite elements.

During the last few years a considerable amount of new contributions to the problem of simulation of the flexible mechanism and robot structures with the finite element method have been published. Among others, the papers by Bricout *et al.* (1990), Gao *et al.* (1989), Hac (1991), Yang and Sadler (1990) are mentioned.

In this paper, the finite element approach is used to formulate the kinematic and dynamic model of a flexible *n*-link manipulator. The Lagrange recursive technique was used for formulation of the system equations of motion. As an example of the application of this method, one- and two-link planar robotic systems were chosen for analysis. The equations presented were developed independently from the formulae published by Nagarajan and Turcic (1990). This paper demonstrates an attempt to apply the direct approach to the problem of dynamics of a flexible manipulator.

The links of the manipulator analysed in this paper may undergo large relative rotational motions. These links are considered as slim beams connected in one degree-of-freedom, frictionless, revolute joints. The mass of a link is uniformly distributed within each finite element. Different distributions of mass along the link could also be incorporated in this method. Control torques are applied at manipulator joints only. Linearly elastic robot structures having small relative displacements due to dynamic loads have been assumed. It has been also assumed that there is no interaction between axial, torsional and flexural deformations. The deflections of links and relative rotations of nodal points are small, so that the direction cosine matrix could be simplified, Beres *et al.* (1993).

The global motion of the flexible manipulator links was described as the superposition of a large rigid body motion of links due to the rotations in joints and small perturbation components due to elasticity of links. Therefore, the set of generalized coordinates was divided into two subsets. The subset of joint coordinates describes the large relative rotational motion of links. The subset of perturbation coordinates describe the deformation of links, with respect to the undeformed shape of that link, treated as a rigid body.

The consistent mass matrix for a flexural link deformation is generally composed of two parts, namely the translational and rotational inertia terms. The translational inertia terms are much bigger than the rotational ones, therefore only the former are considered and all the rotational deformation terms are neglected. For simplicity, the torsional kinetic energy of links is neglected. These neglected terms can be easily taken into account by analogy to translational inertia terms.

The beam-type finite elements, with interpolating polynomials of degree three and six generalized perturbation coordinates per finite element nodal point, were used for description of the link small displacement fields. All the matrices included in the kinetic energy equation were explicitly calculated. The matrices in the potential energy equation are calculated with the application of a finite element software package. Joint and link transformation matrices, describing the kinematic transformation of coordinate systems between the ends of the deformed manipulator link, have been used. Two sets of differential equations of motion were obtained. These are: large motion equations and small perturbation equations. The dynamic model of the manipulator motion obtained in this paper takes into account couplings between the large rigid body motion and flexible elastic deformations of links. The equations are derived consistently, which allows for creating software for the manipulator dynamics and control computation.

On the basis of the algorithm presented in this paper, a computer program was built. As a programming environment, the software package MATLAB was used. To generate the stiffness matrices for each link, another software package — MSC/pal 2 — was applied. The calculation results were presented.

2. Finite Element Model of Manipulator Links

It is assumed that each flexible link of the robot is composed of beam type finite elements. The first and last nodal points of the link coincide with the joints of the manipulator. A set of six scalar perturbation coordinates, i.e. three translations and

three rotations, is assumed for each node. The deformed shape of the link could be described by using a set of perturbation coordinates for the entire link and a set of shape functions that characterize the displacements of the link points located between the finite element nodes.

The decomposition of a link into finite elements is presented in Fig. 1. Two coordinate systems are used for each link. A global link coordinate system $y0\eta$ is attached to the link, whereas a local finite element coordinate system $y0x$ is attached to each finite element of the link.
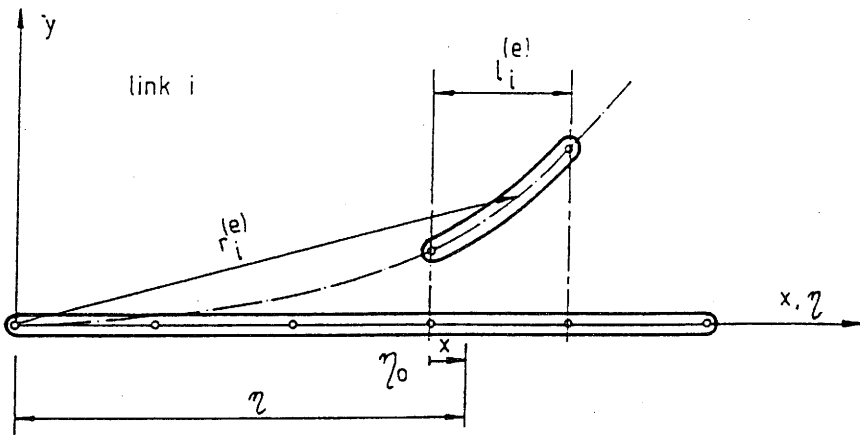

Fig. 1. Deformed shape of the manipulator link, decomposed into finite elements.

The analytical method discussed in this paper uses the Hartenberg-Denavit 4×4 matrix method to describe the kinematics of the flexible manipulator. Position vectors are 4×1 vectors of Cartesian coordinates augmented by an additional element equal to 1.

A point P located at η on the neutral axis of the link i is, after deformation of the link, located at $r_i^{i(e)}(\eta)$

$$r_i^{i(e)}(\eta) = \begin{bmatrix} 1 \\ \eta \\ 0 \\ 0 \end{bmatrix} + Np^{(e)} \tag{1}$$

where (e) is the ordinal number of the element in the link i, N is the matrix of shape functions in the element coordinate system, $p^{(e)}$ is the vector of generalized nodal perturbation coordinates for the element (e), $r_i^{i(e)}(\eta)$ is the position vector of the point P in the element (e) of the link i expressed in the coordinate system of the link i.

The shape function matrix N that relates nodal perturbation coordinates to the deformed position of the point P has a general form

$$
N = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
N_1 & 0 & 0 & 0 & 0 & 0 & N_4 & 0 & 0 & 0 & 0 & 0 \\
0 & N_2 & 0 & 0 & 0 & N_3 & 0 & N_5 & 0 & 0 & 0 & N_6 \\
0 & 0 & N_2 & 0 & -N_3 & 0 & 0 & 0 & N_5 & 0 & -N_6 & 0
\end{bmatrix}
\tag{2}
$$

The flexural deformation shape functions has been chosen as polynomials of degree three, whereas the longitudinal deformation shape functions are linear

$$
N_1 = 1 - \xi
\tag{3}
$$

$$
N_2 = 2\xi^3 - 3\xi^2 + 1
\tag{4}
$$

$$
N_3 = (\xi^3 - 2\xi^2 + \xi)L
\tag{5}
$$

$$
N_4 = \xi
\tag{6}
$$

$$
N_5 = -2\xi^3 + 3\xi^2
\tag{7}
$$

$$
N_6 = (\xi^3 - \xi^2)L
\tag{8}
$$

In the above equations ξ denotes the normalized coordinate, which is defined by

$$
\xi = \frac{x}{L}
\tag{9}
$$

where L is the length of the finite element.

The fourth and tenth columns of the shape function matrix are zero vectors, because the torsional inertia and torsional potential energy of the system were initially neglected for simplicity. These neglected terms can be easily taken into account by inserting a linear torsional shape functions into the fourth and the tenth column.

3. Set of Generalized Coordinates of the Manipulator Link

3.1. Generalized Coordinates

The generalized coordinate set describes the position of any point of a link with respect to the undeformed configuration of that link, with respect to the other link, or the base of the manipulator. It is divided into two subsets: the subset of joint variables q_i $(i = 1, 2)$ that describes the revolute displacement in joints and the subset of generalized perturbation coordinates, $p_{i\beta}$, reflecting the link deformation. The subscript i relates to the link number, while the subscript β relates to an ordinal number of the perturbation coordinate in the perturbation coordinate vector.

Figure 2 presents a finite element in the local coordinate system. The set of local perturbation coordinates, numbered consecutively from 1 to 12 within each finite element, forms a vector of local generalized coordinates for the element (e) that describes the relative motion of nodal points of this element

$$
p^{(e)}(t) = \left[p_1^{(e)}(t), \; p_2^{(e)}(t), \ldots, p_{11}^{(e)}(t), \; p_{12}^{(e)}(t) \right]^T
\tag{10}
$$

Now, a set of global generalized perturbation coordinates for the link i is built as follows

$$p_{i\beta}(t)\bigg|_{\beta=6(e-1)+j} = p_j^{(e)}(t) \qquad \text{for} \quad \begin{aligned} i,j &= 1,2,...,12 \\ \beta &= 1,2,...,NP(i) \end{aligned} \qquad (11)$$

where $NP(i)$ is the number of generalized coordinates and $NG(i)$ is the number of nodal points in the link i. The procedure of assembling the local finite element coordinates to create the global vector of coordinates is typical for the finite element method. Because six generalized coordinates per nodal point were assumed, the following equation holds
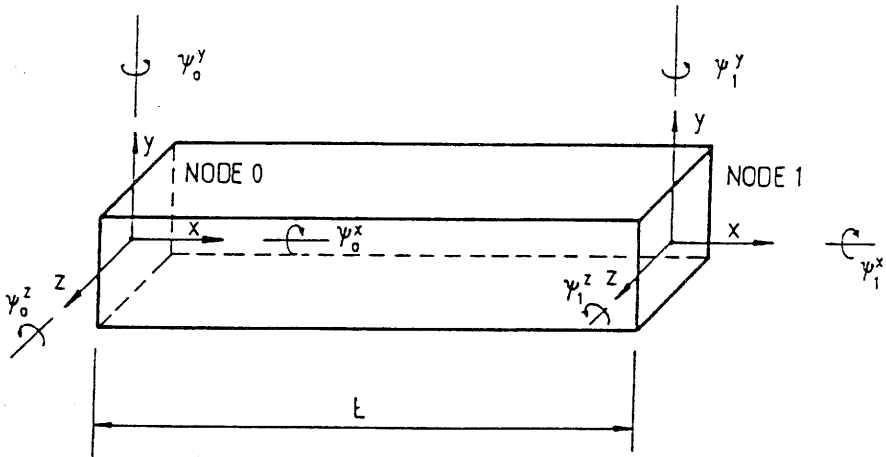
$$NP(i) = 6 \times NG(i) \qquad (12)$$

Fig. 2. The beam finite element in the local coordinate system.

After comparing the local perturbation coordinates for two adjacent finite elements it can be seen that the following formula can be written down

$$p_j^{(e)}(t) = p_{j+6}^{(e-1)}(t), \qquad j = 1,2,...,6 \qquad (13)$$

For future reference, the vector of global generalized perturbation coordinates for the entire manipulator is built, using the joint coordinates, q_i and the global generalized perturbation coordinates for the link i, $p_{i\beta}$, as follows

$$z(t) = \Big[q_1(t), q_2(t), ..., q_n(t), p_{1,1}(t), ..., p_{1,NP(1)},$$
$$p_{2,1}(t), ..., p_{2,NP(2)}, ..., p_{n,1}(t), ..., p_{n,NP(n)}\Big]^T \qquad (14)$$

This set of the generalized perturbation coordinates is general and does not take into account necessary boundary conditions, that must be imposed on coordinates to maintain the compatibility criteria between coordinate values.

3.2. Boundary Conditions

In order to relate the deformed and undeformed shapes of a manipulator link, a set of boundary conditions must be imposed on the set of global generalized perturbation coordinates. These boundary conditions must satisfy criteria of compatibility between the positions, velocities and accelerations of the manipulator points that are common in two links. The fixed, cantilever beam boundary condition model was adopted. In this model, it is assumed that all six generalized perturbation coordinates attached to the first finite element node of the link are set to zero. This can be written in terms of the system global perturbation coordinate vector as

$$p_{i\beta}\Big|_{\beta=1,2,...,6} = 0, \qquad i = 1,2 \tag{15}$$

By imposing these boundary conditions, kinematic realations were used to eliminate the dependent coordinates, so that the resulting dynamic model contains the minimum number of independent variables.

3.3. Generalized Velocities and Accelerations

Analogically, sets of global link generalized velocities $\dot{z}(t)$ and accelerations $\ddot{z}(t)$ are obtained by differentiating eqn. (14) with respect to time. These generalized coordinates, velocities and accelerations are the basis for the calculation of the potential and kinetic energies of the system.

4. Transformation Matrices

4.1. Joint Transformation Matrices

Joint transformation matrices $\boldsymbol{A}_{2i-1}$ are defined for the manipulator system analysed in this paper. These matrices are the standard homogeneous rotation transformation matrices. A matrix $\boldsymbol{A}_{2i-1}$ relates coordinate systems in the joint that connects links $i-1$ and i.

In the case of the plane motion of the manipulator system, the joint transformation matrices reduce to simple matrix expressions

$$\boldsymbol{A}_{2i-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos q_i & -\sin q_i & 0 \\ 0 & \sin q_i & \cos q_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{16}$$

where q_i $(i = 1, 2, ...n)$ is the joint variable in the joint i.

The time derivatives of the joint transformation matrices have the general form

$$\dot{\boldsymbol{A}}_{2i-1} = \boldsymbol{U}_i^{(1)} \dot{q}_i \tag{17}$$

$$\ddot{\boldsymbol{A}}_{2i-1} = \boldsymbol{U}_i^{(2)} \dot{q}_i^2 + \boldsymbol{U}_i^{(1)} \ddot{q}_i \tag{18}$$

where the matrices $U_i^{(1)}$ and $U_i^{(2)}$ stand for partial derivatives

$$U_i^{(1)} = \frac{\partial A_{2i-1}}{\partial q_i} \tag{19}$$

$$U_i^{(2)} = \frac{\partial^2 A_{2i-1}}{\partial q_i^2} \tag{20}$$

Using the derivative matrix Q, the time derivatives of joint transformation matrices can be expressed in the following way

$$\dot{A}_1 = Q A_1 \dot{q}_1 \tag{21}$$

$$\ddot{A}_1 = Q^2 A_1 \dot{q}_1^2 + Q A_1 \ddot{q}_1 \tag{22}$$

$$\dot{A}_3 = Q A_3 \dot{q}_2 \tag{23}$$

$$\ddot{A}_3 = Q^2 A_3 \dot{q}_2^2 + Q A_3 \ddot{q}_2 \tag{24}$$

These derivatives together with the time derivatives of link transformation matrices presented in the following subsection are used to calculate the kinetic and potential energies of the system.

4.2. Link Transformation Matrices

In addition to the joint transformation matrices described above, matrices that describe the transformation of coordinate systems between two ends of the deformed manipulator link were introduced as in (Book, 1984) and (Judd and Falkenburg, 1985).

This transformation is expressed as the transformation between two local coordinate systems, connected to the first and the last finite element node of the link

$$A_{2i} = H_i + Z_i \tag{25}$$

where

$$H_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_i & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{26}$$

and l_i is the length of the undeformed link i.

Taking into account the boundary conditions presented in the previous section, the general expression for the matrix Z_i takes the form

$$Z_i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ p_{i,j_1} & 0 & -p_{i,j_6} & p_{i,j_5} \\ p_{i,j_2} & p_{i,j_6} & 0 & -p_{i,j_4} \\ p_{i,j_3} & -p_{i,j_5} & p_{i,j_4} & 0 \end{bmatrix} \tag{27}$$

The indexes attached to the last finite element of the link are expressed as

$$j_k = 6\Big(NG(i) - 1\Big) + k, \qquad k = 1, 2, \ldots, 6 \tag{28}$$

These link transformation matrix could be presented in a consistent form as

$$A_{2i} = H_i + \sum_{\beta=1}^{NP(i)} p_{i\beta} \Lambda_{i\beta} \tag{29}$$

where $\Lambda_{i\beta}$ are the 4×4 transformation matrices. If the boundary conditions specified above are taken into account, only six of these transformation matrices are not equal to zero for each link.

The general form of these matrices is given in (Beres and Sasiadek, 1990; 1991). The first and second derivatives of the link transformation matrix are easily obtained from eqn. (29).

$$\dot{A}_{2i} = \sum_{\beta=1}^{NP(i)} \dot{p}_{i\beta} \Lambda_{i\beta} \tag{30}$$

$$\ddot{A}_{2i} = \sum_{\beta=1}^{NP(i)} \ddot{p}_{i\beta} \Lambda_{i\beta} \tag{31}$$

The derived formulae are components for the formulation of the general kinematic expressions relating the position of any point in any link to the manipulator base.

5. Kinematics of Flexible Manipulator Link

The position of the point P in the link i, expressed in terms of the fixed inertial coordinate system of the base, is given by

$$r_i^{(e)} = W_{2i-1} r_i^{i(e)} \tag{32}$$

where W_{2i-1} is the homogeneous transformation matrix between the link i and the base.

In general, the transformation matrix W_j is expressed as a product of transformations due to the displacement in the joint i, A_{2i-1}, and transformations due to the deformation of the link i, A_{2i}, as proposed by Book (1984).

From the notations shown in Fig. 3 it could be seen that the following formula holds

$$W_j = \prod_{h=1}^{j} A_h = A_1 A_2 A_3 \ldots A_j \tag{33}$$

Also the following recursive formulae can be written down

$$W_j = W_{j-1} A_j \tag{34}$$

$$\dot{W}_j = \dot{W}_{j-1}A_j + W_{j-1}\dot{A}_j \tag{35}$$

$$\ddot{W}_j = \ddot{W}_{j-1}A_j + 2\dot{W}_{j-1}\dot{A}_j + W_{j-1}\ddot{A}_j \tag{36}$$
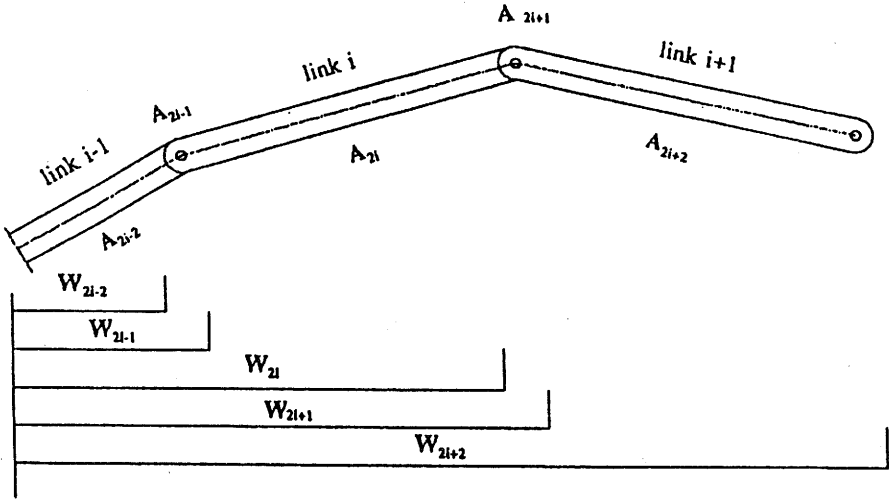

Fig. 3. Link and joint transformation matrices.

If j is an odd number, $(j = 2i - 1)$, the transformation A_{2i-1} represents the joint transformation; if j is an even number, $(j = 2i)$, then the transformation A_{2i} represents the transformations due to link deformation. Both of these transformations have been described in the previous section.

For the case of a two-link manipulator, the transformation matrices W_j and their time derivatives have the following form

$$W_1 = A_1 \tag{37}$$

$$W_2 = A_1 A_2 \tag{38}$$

$$W_3 = A_1 A_2 A_3 \tag{39}$$

$$\dot{W}_1 = \dot{A}_1 \tag{40}$$

$$\dot{W}_2 = \dot{A}_1 A_2 + A_1 \dot{A}_2 \tag{41}$$

$$\dot{W}_3 = \dot{A}_1 A_2 A_3 + A_1 \dot{A}_2 A_3 + A_1 A_2 \dot{A}_3 \tag{42}$$

$$\ddot{W}_1 = \ddot{A}_1 \tag{43}$$

$$\ddot{W}_2 = \ddot{A}_1 A_2 + 2\dot{A}_1 \dot{A}_2 + A_1 \ddot{A}_2 \tag{44}$$

$$\ddot{W}_3 = \ddot{A}_1 A_2 A_3 + A_1 \ddot{A}_2 A_3 + A_1 A_2 \ddot{A}_3$$
$$+ 2(\dot{A}_1 \dot{A}_2 A_3 + \dot{A}_1 A_2 \dot{A}_3 + A_1 \dot{A}_2 \dot{A}_3) \tag{45}$$

The state space form requires calculating the second derivatives of joint and perturbation variables from the general expressions (37)–(45). This can be done by applying decomposition used by Book (1984).

$$\dot{W}_j = \sum_{h=1}^{j} W_{h-1} \dot{A}_h W_j^{h+1} \tag{46}$$

where W_j^{h+1} denotes the product of elementary transformation matrices and

$$W_j^{h+1} = A_{h+1} A_{h+2} \ldots A_{j-1} A_j \tag{47}$$

The second derivatives of the transformation matrices can be written in the form

$$\ddot{W}_j = \sum_{h=1}^{j} W_{h-1} \ddot{A}_h W_j^{h+1} + \ddot{W}_{\nu,j} \tag{48}$$

where $W_{\nu,j}$ does not contain any second derivatives of joint and perturbation variables. It can be proven that

$$\ddot{W}_{\nu,2i-1} = \ddot{W}_{\nu,2i-2} A_{2i-1} + 2\dot{W}_{2i-2} \dot{A}_{2i-1} + W_{2i-2} U_i^{(2)} \dot{q}_i^2 \tag{49}$$

$$\ddot{W}_{\nu,2i} = \ddot{W}_{\nu,2i-1} A_{2i} + 2\dot{W}_{2i-1} \dot{A}_{2i} \tag{50}$$

For the case of a two-link manipulator it was obtained that

$$\ddot{W}_{\nu,1} = Q^2 A_1 \dot{q}_1^2 \tag{51}$$

$$\ddot{W}_{\nu,2} = Q^2 A_1 A_2 \dot{q}_1^2 + 2\dot{A}_1 \dot{A}_2 \tag{52}$$

$$\ddot{W}_{\nu,3} = Q^2 A_1 A_2 A_3 \dot{q}_1^2 + A_1 A_2 Q^2 A_3 \dot{q}_2^2 \\ + 2(\dot{A}_1 \dot{A}_2 A_3 + \dot{A}_1 A_2 \dot{A}_3 + A_1 \dot{A}_2 \dot{A}_3) \tag{53}$$

Having computed the transformation matrices W_j and their derivatives, the inertial velocity of the point P in the link i, that must be used to calculate the kinetic energy of the system, can be derived from the equation

$$\dot{r}_i^{(e)} = \dot{W}_{2i-1} r_i^{i(e)} + W_{2i-1} \dot{r}_i^{i(e)} \tag{54}$$

This equation expresses the velocity of any point of the manipulator in terms of the link and joint transformation matrices, as well as the velocities expressed in the local link coordinate system.

6. Kinetic Energy of the System

To obtain the total kinetic energy of the manipulator system, the kinetic energy of each elemental mass dm is first determined and then the individual kinetic energies are summed up to form the kinetic energy of the entire manipulator.

An elementary kinetic energy of a mass element $\mathrm{d}m$ in the finite element (e) of link i can be expressed as

$$\mathrm{d}K_i^{(e)} = \frac{1}{2}\mathrm{d}m\,\mathrm{tr}\left\{\dot{\boldsymbol{r}}_i^{(e)}\dot{\boldsymbol{r}}_i^{(e)T}\right\} \tag{55}$$

which, using eqn. (54), can be written in the form

$$\begin{aligned}
\mathrm{d}K_i^{(e)} = \frac{1}{2}\mathrm{d}m\,\mathrm{tr}\Big\{ &\dot{\boldsymbol{W}}_{2i-1}\boldsymbol{r}_i^{i(e)}\boldsymbol{r}_i^{i(e)T}\dot{\boldsymbol{W}}_{2i-1}^T \\
&+2\dot{\boldsymbol{W}}_{2i-1}\boldsymbol{r}_i^{i(e)}\dot{\boldsymbol{r}}_i^{i(e)T}\boldsymbol{W}_{2i-1}^T + \boldsymbol{W}_{2i-1}\dot{\boldsymbol{r}}_i^{i(e)}\dot{\boldsymbol{r}}_i^{i(e)T}\boldsymbol{W}_{2i-1}^T\Big\}
\end{aligned} \tag{56}$$

where $\boldsymbol{W}_{2i-1}$ and $\dot{\boldsymbol{W}}_{2i-1}$ are the transformation matrices described in the previous section.

The kinetic energy of the finite element (e) in the link i is determined by integration over the element length, which yields

$$K_i^{(fe)} = \int_{\eta_0^{(e)}}^{\eta_1^{(e)}} \mathrm{d}K_i^{(e)} \tag{57}$$

The kinetic energy of the entire manipulator is obtained by adding contributions from all the finite elements in all the links

$$K = \sum_{i=1}^{n}\sum_{e=1}^{m_i} \mathrm{d}K_i^{(e)} \tag{58}$$

In this equation the following notation was used: n is the number of links, m_i is the number of elements on the link i, whereas $\eta_0^{(e)}$ and $\eta_1^{(e)}$ are the coordinates of the finite element end points of the element (e) in the link i.

The integrations and summations may be performed within the brackets. Thus, eqn. (58) may be rewritten in the following way

$$K = \sum_{i=1}^{n}\mathrm{tr}\left\{\dot{\boldsymbol{W}}_{2i-1}\boldsymbol{B}_i^{(0)}\dot{\boldsymbol{W}}_{2i-1}^T + 2\dot{\boldsymbol{W}}_{2i-1}\boldsymbol{B}_i^{(1)}\boldsymbol{W}_{2i-1}^T + \boldsymbol{W}_{2i-1}\boldsymbol{B}_i^{(2)}\boldsymbol{W}_{2i-1}^T\right\} \tag{59}$$

where the expressions for $\boldsymbol{B}_i^{(k)}$ $(k = 0, 1, 2)$ are calculated as the sums of contributions from each finite element

$$\boldsymbol{B}_i^{(k)} = \sum_{e=1}^{m_i}\boldsymbol{B}_i^{(k)(e)}, \qquad k = 0, 1, 2 \tag{60}$$

6.1. Kinetic Energy of the System Expressed in Terms of the Local Finite Element Generalized Coordinates

To calculate the kinetic energy of the entire manipulator in terms of the local finite element generalized coordinates, each term in eqn. (59) is expressed as a function of

the finite element generalized coordinates $p_j(e)$ and their derivatives. Each term of the sum in eqn. (59) was calculated as

$$B_i^{(0)(e)} = \frac{1}{2} \int_{\eta_0^{(e)}}^{\eta_1^{(e)}} \mu r_i^{i(e)} r_i^{i(e)T} \, d\eta \tag{61}$$

$$B_i^{(1)(e)} = \frac{1}{2} \int_{\eta_0^{(e)}}^{\eta_1^{(e)}} \mu r_i^{i(e)} \dot{r}_i^{i(e)T} \, d\eta \tag{62}$$

$$B_i^{(2)(e)} = \frac{1}{2} \int_{\eta_0^{(e)}}^{\eta_1^{(e)}} \mu \dot{r}_i^{i(e)} \dot{r}_i^{i(e)T} \, d\eta \tag{63}$$

where μ denotes the mass of a unit length of the link.

Now, the three terms that constitute eqn. (59) are expanded. For the third of these terms, the product under the integral sign in eqn. (63) can be written in the form

$$\dot{r}_i^{i(e)} \dot{r}_i^{i(e)T} = N \dot{p}^{(e)} \dot{p}^{(e)T} N^T \tag{64}$$

where $p^{(e)}$ is the 12×1 vector of generalized velocities of the element (e), numbered in the local element numbering system.

Renumbering the terms and calculating the integrals, the following formula is obtained

$$B_i^{(2)(e)} = \sum_{j=1}^{12} \sum_{k=1}^{12} \dot{p}_j^{(e)} \dot{p}_k^{(e)} S_{jk} \tag{65}$$

where S_{jk} are the 4×4 matrices that include integrals of the products of shape functions, N_i, multiplied by the mass of the unit length of the element of the link i.

The matrices S_{jk} are symmetric and the formula $S_{jk} = S_{kj}$ holds. These matrices were calculated for $j, k = 1, 2, ..., 12$. They are specified in (Beres and Sasiadek, 1991). As can be seen, the formula for $B_i^{(2)(e)}$ has the shape of a quadratic form with respect to generalized velocities $p_j^{(e)}$ $(j = 1, 2, ..., 12)$. The matrices S_{jk} are the generalized coefficients of this quadratic form.

The product under the integral sign in eqn. (62) can be written in the form

$$r_i^{i(e)} \dot{r}_i^{i(e)T} = \left[\begin{bmatrix} 1 \\ \eta \\ 0 \\ 0 \end{bmatrix} + N p^{(e)} \right] \dot{p}^{(e)T} N^T \tag{66}$$

In order to include finite elements in the calculations and after taking into account a simple geometrical relation illustrated in Fig. 2, the position vector is decomposed into two terms

$$\begin{bmatrix} 1 & \eta & 0 & 0 \end{bmatrix}^T = d^{(e)} + b^{(e)} \tag{67}$$

where

$$d^{(e)} = \begin{bmatrix} 1 & \eta_0 & 0 & 0 \end{bmatrix}^{(e)T} \tag{68}$$

$$b^{(e)} = \begin{bmatrix} 0 & x & 0 & 0 \end{bmatrix}^{(e)T} \tag{69}$$

This decomposition expresses the geometrical coordinate of the point P as a sum of two distances: the position η_0 of the finite element, expressed in terms of the link coordinate system, and the distance x measured within the finite element. Thus, the product in eqn. (62) can be rewritten as follows

$$r_i^{i(e)} \dot{r}_i^{i(e)T} = d^{(e)} \dot{p}^{(e)T} N^T + b^{(e)} \dot{p}^{(e)T} N^T + N p^{(e)} \dot{p}^{(e)T} N^T \tag{70}$$

After rearranging these terms and performing calculation of the integral, the first term in eqn. (70) can be written as

$$\frac{1}{2} \int_{\eta_0^{(e)}}^{\eta_1^{(e)}} \mu d^{(e)} \dot{p}^{(e)T} N^T \, d\eta = \sum_{j=1}^{12} \dot{p}_j^{(e)} S_j^{(e)} \tag{71}$$

where S_j, $j = 1, 2, ..., 12$ are the 4×4 matrices that include integrals of the shape functions. These twelve matrices differ for each finite element. They are specified in (Beres and Sasiadek, 1991).

Similarly, the second term in eqn. (70) can be written as

$$\frac{1}{2} \int_{\eta_0^{(e)}}^{\eta_1^{(e)}} \mu b^{(e)} \dot{p}^{(e)T} N^T \, d\eta = \sum_{j=1}^{12} \dot{p}_j^{(e)} R_j \tag{72}$$

where R_j, $j = 1, 2, ..., 12$ are the 4×4 matrices, which include integral of the shape functions multiplied by the variable x. These twelve matrices are specified in (Beres and Sasiadek, 1991).

Summarizing the last results, the integral in eqn. (62) may be expressed as

$$B_i^{(1)(e)} = \sum_{j=1}^{12} \dot{p}_j^{(e)} \left(S_j^{(e)} + R_j \right) + \sum_{j=1}^{12} \sum_{k=1}^{12} p_j^{(e)} \dot{p}_k^{(e)} S_{jk} \tag{73}$$

where the matrices S_j, R_j and S_{jk}, $j, k = 1, 2, ... 12$ have already been specified.

The integral in eqn. (61) is calculated similarly. The product under the integral sign can be written as

$$r_i^{i(e)} r_i^{i(e)T} = \left[d^{(e)} + b^{(e)} + N p^{(e)} \right] \left[d^{(e)} + b^{(e)} + N p^{(e)} \right]^T \tag{74}$$

The integrals of five of nine terms have already been calculated, thus only four remaining integrals have to be computed. These integrals, after performing calculations have the following forms:

$$C^{(0)(e)} = \frac{1}{6} \mu L^3 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{75}$$

$$C^{(1)(e)} = \frac{1}{4} \mu L^2 \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \eta_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{76}$$

$$C^{(2)(e)} = C^{(1)(e)T} \tag{77}$$

$$C^{(3)(e)} = \frac{1}{2} \mu L \begin{bmatrix} 0 & \eta_0 & 0 & 0 \\ \eta_0 & \eta_0^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{78}$$

Using previously obtained results, the integral in eqn. (61) can be expressed as follows

$$B_i^{(0)(e)} = \sum_{k=0}^{3} C^{(k)(e)} + \sum_{j=1}^{12} p_j^{(e)} \left[S_j^{(e)} + S_j^{(e)T} + R_j + R_j^T \right] + \sum_{j=1}^{12} \sum_{k=1}^{12} p_j^{(e)} p_k^{(e)} S_{jk} \tag{79}$$

Summarizing the obtained results, the kinetic energy of the entire manipulator system can be calculated from eqn. (59) using the coefficients $B_i^{(0)}$, $B_i^{(1)}$ and $B_i^{(2)}$, as well as the transformation matrices W_j and their derivatives. The coefficients $B_i^{(0)}$, $B_i^{(1)}$ and $B_i^{(2)}$ are expressed in terms of local, finite element coordinate systems.

To prepare an effective algorithm for calculation of the kinetic energy of the manipulator systems, the expressions for the coefficients $B_i^{(0)}$, $B_i^{(1)}$ and $B_i^{(2)}$ must be transformed to the global coordinate systems.

6.2. Kinetic Energy of the System Expressed in Terms of the Global Link Generalized Coordinates

Transforming the above equations to the global coordinate system the expressions in eqns. (65), (73) and (82) can be written as follows

$$B_i^{(2)(e)} = \sum_{\beta=\beta_s}^{\beta_t} \sum_{\gamma=\gamma_s}^{\gamma_t} \dot{p}_{i\beta} \dot{p}_{i\gamma} S_{\beta\gamma}^{(e)} \tag{80}$$

$$B_i^{(1)(e)} = \sum_{\beta=\beta_s}^{\beta_t} \dot{p}_{i\beta} \left(S_\beta^{(e)} + R_\beta^{(e)} \right) + \sum_{\beta=\beta_s}^{\beta_t} \sum_{\gamma=\gamma_s}^{\gamma_t} p_{i\beta} \dot{p}_{i\gamma} S_{\beta\gamma}^{(e)} \tag{81}$$

$$B_i^{(0)(e)} = C^{(e)} + \sum_{\beta=\beta_s}^{\beta_t} p_{i\beta} \left(S_\beta^{(e)} + S_\beta^{(e)T} + R_\beta^{(e)} + R_\beta^{(e)T} \right) + \sum_{\beta=\beta_s}^{\beta_t} \sum_{\gamma=\gamma_s}^{\gamma_t} p_{i\beta} p_{i\gamma} S_{\beta\gamma}^{(e)} \tag{82}$$

where

$$\beta_s = 6(e-1)+1, \qquad \beta_t = 6(e-1)+12 \tag{83}$$

$$S_{\beta\gamma}^{(e)} = S_{jk}, \qquad R_\beta^{(e)} = R_j \tag{84}$$

$$\beta = 6(e-1)+j, \qquad \gamma = 6(e-1)+k \tag{85}$$

The second-order terms, i.e. the terms which contain the products of generalized coordinates, although given in eqn. (82), were omitted in the final expression for kinetic energy, since their values are small in comparison with other terms.

To get the final formulae for the coefficients $B_i^{(0)}$, $B_i^{(1)}$ and $B_i^{(2)}$ in eqn. (59), the assemblage process, typical for the finite element method, is used.

7. Potential Energy of the System

Two terms contribute to the potential energy of the flexible manipulator system: the elastic deformation of links and the gravity.

7.1. Potential Energy of Elasticity

The elastic potential energy is independent of joint variables q_i and can be expressed as a sum of terms for each link

$$V_e = \frac{1}{2} \sum_{i=1}^{n} \sum_{\beta=1}^{NP(i)} \sum_{\gamma=1}^{NP(i)} p_{i\beta} p_{i\gamma} K_{i\beta\gamma} \tag{86}$$

where the matrices $K_{i\beta\gamma}$ are link stiffness matrices. They are symmetric, banded and have dimension of $(NP(i) \times NP(i))$.

The calculation of the element stiffness matrices and their assemblage to the link stiffness matrix can be done using any standard finite element computer package that incorporates beam elements.

In the paper, the finite element software package MSC/pal2 was used for generation of the link stiffness matrix. An application of this package allowed taking into account the boundary conditions formulated in Section 3.2.

7.2. Potential Energy of Gravity

The potential energy of one finite element due to gravity takes the form

$$V_g^{(e)} = -\mu \int_{\eta_0^{(e)}}^{\eta_1^{(e)}} g^T r_i^{(e)} \, \mathrm{d}\eta \tag{87}$$

where $r_i^{(e)}$ is the position vector in the base coordinate systems and

$$g = \begin{bmatrix} 0 & g_x & g_y & g_z \end{bmatrix}^T \tag{88}$$

is the gravity vector.

The position vector can be written as a function of the transformation and the shape function matrix as

$$r_i^{(e)} = W_{2i-1} r_i^{(e)} = W_{2i-1} \left[d^{(e)} + b^{(e)} + N p^{(e)} \right] \tag{89}$$

Calculating integrals and rearranging the terms, the potential energy of gravity, for a finite element is

$$V_g^{(e)} = -\mu g^T W_{2i-1} L_i^{(e)} \times \left[\begin{bmatrix} 1 \\ \eta_0 \\ 0 \\ 0 \end{bmatrix}^{(e)} + \begin{bmatrix} 0 \\ 1/2 L_i^{(e)} \\ 0 \\ 0 \end{bmatrix}^{(e)} + N_I p^{(e)} \right] \tag{90}$$

The potential energy of gravity for one link is expressed as

$$V_{gi} = -\mu g^T W_{2i-1} \sum_{e=1}^{m_i} L_i^{(e)} N_I p^{(e)} - g^T W_{2i-1} M_i r_{ic} \tag{91}$$

where $L_i^{(e)}$ is the length of the finite element (e) in the link i, M_i is the mass of the link,

$$r_{ic} = \begin{bmatrix} 1 & x_{ic} & 0 & 0 \end{bmatrix}^T \tag{92}$$

x_{ic} is the distance, along the neutral axis, from the joint i to the centre of gravity of the undeformed link i, N_I is the matrix of integrated shape functions.

$$N_I = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2}L & 0 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2}L \\ 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2}L & 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2}L & 0 \end{bmatrix} \tag{93}$$

Finally, the total potential energy of gravity for the entire manipulator can be given as

$$V_g = g^T \sum_{i=1}^{n} W_{2i-1} s_i \tag{94}$$

where

$$s_i = M_i r_{ic} + \sum_{\beta=1}^{NP(i)} p_{i\beta} \alpha_{i\beta} \tag{95}$$

and the vectors $\alpha_{i\beta}$ are calculated using the matrix N_I and the values $L_i^{(e)}$ by assembling element coefficients simultaneously to those in the kinetic energy calculation process.

In the last equation, the sum on its right-hand side describes the influence of the link deformation on the change of the potential energy of gravity. This effect could be neglected as being small. Therefore, the total potential energy of gravity for the manipulator can be expressed as

$$V_g = -g^T \sum_{i=1}^{n} M_i W_{2i-1} r_{ic} \tag{96}$$

8. Partial Derivatives of the Kinetic and Potential Energies

8.1. Partial Derivatives of the Potential Energy of Elasticity with Respect to Joint Variables

Since the potential energy of elasticity is independent of joint variables, the partial derivatives of this energy with respect to joint variables are equal to zero.

$$\frac{\partial V_e}{\partial q_j} = 0, \qquad j = 1, 2, \ldots, n \tag{97}$$

8.2. Partial Derivatives of the Potential Energy of Elasticity with Respect to Deflection Variables

Derivatives of the potential energy of elasticity with respect to the deflection variables are found from eqn. (86) using the formula of quadratic form differentiation

$$\frac{\partial V_e}{\partial p_{j\beta}} = \sum_{\gamma=1}^{NP(j)} p_{j\beta} K_{j\beta\gamma} \tag{98}$$

8.3. Partial Derivatives of the Potential Energy of Gravity with Respect to Joint Variables

Derivatives of the potential energy of gravity with respect to the joint variables can be obtained from eqn. (94)

$$\frac{\partial V_g}{\partial q_j} = -g^T \sum_{i=j}^{n} \frac{\partial W_{2i-1}}{\partial q_j} s_i \tag{99}$$

As was mentioned, for simplicity, it was assumed that the quantities s_i do not depend on perturbation variables.

8.4. Partial Derivatives of the Potential Energy of Gravity with Respect to Deflection Variables

Derivatives of the potential energy of gravity with respect to the deflection variables can be written as

$$\frac{\partial V_g}{\partial p_{j\beta}} = -g^T \left[\sum_{i=j+1}^{n} \frac{\partial W_{2i-1}}{\partial p_{j\beta}} + W_{2j-1}\alpha_{j\beta} \right], \quad 1 < j < n-1 \tag{100}$$

$$\frac{\partial V_g}{\partial p_{n\beta}} = -g^T W_{2n-1}\alpha_{n\beta} \tag{101}$$

8.5. Partial Derivatives of the Transformation Matrices with Respect to Joint and Deflection Variables

Partial derivatives of the joint transformation matrices W_{2i-1} with respect to joint and deflection variables are described in Section 4.

8.6. Partial Derivatives of the Kinetic Energy with Respect to Joint Variables and Their Derivatives

To prepare for the application of Lagrange's equations, the partial derivatives of the system kinetic energy with respect to joint variables and their derivatives were

calculated. Also, the time derivative of the second expression was found and both terms were subtracted. The results are as follows.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial \dot{q}_j}\right) - \frac{\partial K}{\partial q_j} = 2\sum_{i=j}^{n} \mathrm{tr}\left\{\frac{\partial \boldsymbol{W}_{2i-1}}{\partial q_j}\left[\boldsymbol{G}^{(i)}\ddot{\boldsymbol{W}}_{2i-1}^{T}+\right.\right.$$

$$\left.\left. + \sum_{\beta=1}^{NP(i)} \ddot{p}_{i\beta}\boldsymbol{D}_{\beta}^{(i)}\boldsymbol{W}_{2i-1}^{T} + \sum_{\beta=1}^{NP(i)} \dot{p}_{i\beta}\boldsymbol{D}_{\beta}^{(i)}\dot{\boldsymbol{W}}_{2i-1}^{T}\right]\right\} \qquad (102)$$

where

$$\boldsymbol{D}_{\beta}^{(i)} = \boldsymbol{S}_{\beta}^{(i)} + \boldsymbol{R}_{\beta}^{(i)} + \sum_{\gamma=1}^{NP(i)} p_{i\gamma}\boldsymbol{S}_{\beta\gamma}^{(i)} \qquad (103)$$

$$\boldsymbol{G}^{(i)} = \boldsymbol{C}^{(i)} + \sum_{\beta=1}^{NP(i)} p_{i\beta}\left[\boldsymbol{S}_{\beta}^{(i)} + \boldsymbol{S}_{\beta}^{(i)T} + \boldsymbol{R}_{\beta}^{(i)} + \boldsymbol{R}_{\beta}^{(i)T}\right] \qquad (104)$$

In the above equations the products $p_{i\beta}\,p_{i\gamma}$ were neglected as the terms of second order. In these equations the upper index denotes the link number.

8.7. Partial Derivatives of the Kinetic Energy with Respect to the Deflection Variables and Their Derivatives

Analogically, the partial derivatives of the system kinetic energy with respect to perturbation variables and their time derivatives were calculated. Also, the derivative of the second expression was found, and both terms were subtracted. The following result was obtained.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial \dot{p}_{j\beta}}\right) - \frac{\partial K}{\partial p_{j\beta}} = 2\sum_{i=j+1}^{n} \mathrm{tr}\left\{\frac{\partial \boldsymbol{W}_{2i-1}}{\partial p_{j\beta}}\left[\sum_{\beta=1}^{NP(i)} \ddot{p}_{i\beta}\boldsymbol{D}_{\beta}^{(i)}\boldsymbol{W}_{2i-1}^{T}\right.\right.$$

$$\left.\left. + 2\sum_{\beta=1}^{NP(i)} \dot{p}_{i\beta}\boldsymbol{D}_{\beta}^{(i)}\dot{\boldsymbol{W}}_{2i-1}^{T} + \boldsymbol{G}^{(i)}\ddot{\boldsymbol{W}}_{2i-1}^{T}\right]\right\} + 2\,\mathrm{tr}\left\{\left[\boldsymbol{W}_{2j-1}\sum_{\gamma=1}^{NP(j)} \ddot{p}_{j\gamma}\boldsymbol{S}_{\beta\gamma}^{(j)}\right.\right. \qquad (105)$$

$$\left.\left. + 2\dot{\boldsymbol{W}}_{2j-1}\sum_{\gamma=1}^{NP(j)} \dot{p}_{j\gamma}\boldsymbol{S}_{\beta\gamma}^{(j)} + \ddot{\boldsymbol{W}}_{2j-1}\boldsymbol{D}_{\beta}^{(j)}\right]\boldsymbol{W}_{2j-1}^{T}\right\}, \qquad j = 1, 2, ..., n$$

The coefficients $\boldsymbol{D}_{\beta}^{(i)}$ and $\boldsymbol{G}_{\beta}^{(i)}$ in these equations have already been calculated. The partial derivatives described by eqns. (102)–(104), (106)–(107) and (110) contribute to the Lagrange equation of motion of the manipulator system.

9. Lagrange's Equations of Motion

The partial derivatives calculated in previous section are now assembled to form the
Lagrange equations of motion of the entire flexible manipulator system.

For joint variables

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial \dot{q}_j}\right) - \frac{\partial K}{\partial q_j} + \frac{\partial V_e}{\partial q_j} + \frac{\partial V_g}{\partial q_j} = \tau_j, \qquad j = 1, 2, ..., n \qquad (106)$$

For perturbation variables

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial \dot{p}_{j\beta}}\right) - \frac{\partial K}{\partial p_{j\beta}} + \frac{\partial V_e}{\partial p_{j\beta}} + \frac{\partial V_g}{\partial p_{j\beta}} = 0, \qquad \begin{matrix} j = 1, 2, ..., n \\ \beta = 1, 2, ..., m_i \end{matrix} \qquad (107)$$

The generalized forces, τ_j, that correspond to the joint variables q_j are joint torques.

Substituting eqns. (97)–(105) into general expressions (106) and (107), the La-
grange equations of motion of the entire robot have the following forms:

• for joint variables

$$2\sum_{i=j}^{n} \mathrm{tr}\left\{\frac{\partial \mathbf{W}_{2i-1}}{\partial q_j}\left[\mathbf{G}^{(i)}\sum_{h=1}^{2i-1}\left(\mathbf{W}_{h-1}\ddot{\mathbf{A}}_h \mathbf{W}_{2i-1}^{h+1}\right)^T + \sum_{\beta=1}^{NP(i)}\mathbf{D}_{\beta}^{(i)}\mathbf{W}_{2i-1}^T\ddot{p}_{i\beta}\right]\right\}$$

$$= 2\sum_{i=j}^{n} \mathrm{tr}\left\{\frac{\partial \mathbf{W}_{2i-1}}{\partial q_j}\left[\mathbf{G}^{(i)}\ddot{\mathbf{W}}_{\nu,2i-1}^T + 2\sum_{\beta=1}^{NP(i)}\mathbf{D}_{\beta}^{(i)}\dot{\mathbf{W}}_{2i-1}^T\dot{p}_{i\beta}\right]\right\} \qquad (108)$$

$$+ g^T\sum_{i=j}^{n}\frac{\partial \mathbf{W}_{2i-1}}{\partial q_j}s_i + \tau_j, \qquad j = 1, 2, ..., n$$

where q_j are embedded in the matrices $\mathbf{A}_h$.

• for perturbation variables

$$2\sum_{i=j+1}^{n}\mathrm{tr}\left\{\frac{\partial \mathbf{W}_{2i-1}}{\partial q_{j\beta}}\left[\sum_{\beta=1}^{NP(i)}\mathbf{D}_{\beta}^{(i)}\mathbf{W}_{2i-1}^T\ddot{p}_{i\beta} + \mathbf{G}^{(i)}\sum_{h=1}^{2i-1}\left(\mathbf{W}_{h-1}\ddot{\mathbf{A}}_h \mathbf{W}_{2i-1}^{h+1}\right)^T\right]\right\}$$

$$+ 2\,\mathrm{tr}\left\{\left[\mathbf{W}_{2j-1}\sum_{\gamma=1}^{NP(j)}\mathbf{S}_{\beta\gamma}^{(j)}\ddot{p}_{j\gamma} + \sum_{h=1}^{2j-1}\mathbf{W}_{h-1}\ddot{\mathbf{A}}_h \mathbf{W}_{2j-1}^{h+1}\mathbf{D}_{\beta}^{(j)}\right]\mathbf{W}_{2j-1}^T\right\}$$

$$= -2\sum_{i=j+1}^{n}\mathrm{tr}\left\{\frac{\partial \mathbf{W}_{2i-1}}{\partial q_{j\beta}}\left[2\sum_{\beta=1}^{NP(i)}\mathbf{D}_{\beta}^{(i)}\dot{\mathbf{W}}_{2i-1}^T\dot{p}_{i\beta} + \mathbf{G}^{(i)}\ddot{\mathbf{W}}_{\nu,2i-1}^T\right]\right\} \qquad (109)$$

$$-2\,\mathrm{tr}\left\{\left[2\dot{\boldsymbol{W}}_{2j-1}\sum_{\gamma=1}^{NP(j)}\boldsymbol{S}^{(j)}_{\beta\gamma}\dot{p}_{j\gamma}+\ddot{\boldsymbol{W}}_{\nu,2i-1}\boldsymbol{D}^{(j)}_{\beta}\right]\boldsymbol{W}^{T}_{2j-1}\right\}$$

$$-\sum_{\gamma=1}^{NP(i)}p_{j\gamma}\boldsymbol{K}_{j\beta\gamma}+\boldsymbol{g}^{T}\left[\sum_{i=j+1}^{n}\frac{\partial\boldsymbol{W}_{2i-1}}{\partial p_{j\beta}}+\boldsymbol{W}_{2j-1}\boldsymbol{\alpha}_{j\beta}\right],\qquad\begin{array}{l}j=1,2,...,n\\\beta=1,2,...,m_i\end{array}$$

These Lagrange equations form a set of non-linear differential equations of second order with respect to the joint and the perturbation variables. In order to transform this set of equations to a standard state space representation, a global flexible manipulator set of generalized coordinates defined in Section 3 is used.

$$\boldsymbol{z}(t)=\Big[q_1(t),q_2(t),...,q_n(t),p_{11}(t),...,p_{1,NP(1)},$$
$$p_{21}(t),...,p_{2,NP(2)},...,p_{n,1}(t),...,p_{n,NP(n)}\Big]^{T} \qquad (110)$$

Also, a state space vector, that comprises the set of global manipulator generalized coordinates and their derivatives, is arranged in the following manner

$$\boldsymbol{Q}(t)=\left[\begin{array}{c}\boldsymbol{z}(t)\\\dot{\boldsymbol{z}}(t)\end{array}\right] \qquad (111)$$

Using this notation, the set of Lagrange's equations of motion (108) and (109) can be written as follows

$$\boldsymbol{M}\ddot{\boldsymbol{z}}=\boldsymbol{f}(\boldsymbol{z},\dot{\boldsymbol{z}})+\boldsymbol{\tau} \qquad (112)$$

where

$$\boldsymbol{\tau}=[\tau_1,\tau_2,\ldots,\tau_n]^{T} \qquad (113)$$

is the vector of external torques applied in joints, augmented by additional zeros that correspond to generalized coordinates $p_{j\beta}$.

The $\boldsymbol{M}$ matrix is the generalized inertia matrix whose elements are calculated from the left-hand sides of eqns. (108) and (109). For example,

$$\boldsymbol{M}_{1,1}=2\,\mathrm{tr}\Big\{\boldsymbol{Q}\boldsymbol{A}_1\boldsymbol{G}^{(1)}\boldsymbol{A}_1^{T}\boldsymbol{Q}^{T}\Big\}+2\,\mathrm{tr}\Big\{\boldsymbol{Q}\boldsymbol{A}_1\boldsymbol{A}_2\boldsymbol{A}_3\boldsymbol{G}^{(2)}\boldsymbol{A}_3^{T}\boldsymbol{A}_2^{T}\boldsymbol{A}_1^{T}\boldsymbol{Q}^{T}\Big\} \qquad (114)$$

The generalized inertia matrix $\boldsymbol{M}$ is symmetric. Its coefficients are extracted from the expressions describing the partial derivatives of the kinetic energy of the system. The details regarding the matrix coefficients $\boldsymbol{M}$, for the specific case of a two-link, two-finite-element-per-link flexible manipulator, are presented in (Beres and Sasiadek, 1991).

To reduce the size of the problem, the static condensation, as described by Guyan (1964), or the generalized dynamic reduction techniques can be applied. A similiar technique could be used to eliminate the perturbation variables that were zero to

satisfy the boundary conditions. The reduced generalized inertia matrix, for a real manipulator structure is positive definite, thus is non-singular. Therefore, eqn. (112) can be expressed as

$$\ddot{z} = M^{-1}[f(z, \dot{z}) + \tau] \tag{115}$$

where M^{-1} denotes the inversion of the generalized inertia matrix. This inversion can be calculated numerically only. Here f denotes the vector of the right-hand sides of the system of differential equations. It is a function of the vectors of generalized coordinates and generalized velocities. It does not contain second derivatives of the generalized coordinates, i.e. the generalized accelerations. The details regarding the f vector element for the specific case of a two-link, two-finite-element-per-link flexible manipulator, are presented in the report by Beres and Sasiadek (1991).

Finally, eqn. (115) may be formulated as follows

$$\dot{Q} = f_1(Q) + f_2(\tau) \tag{116}$$

This equation form is ready for numerical integration.

To get numerical results, the presented algorithm must be programmed for numerical calculations.

10. Computer Program

To implement the computation algorithm presented in previous sections a computer program was designed. As the programming environment the commercial software MATLAB was used. The MSC/pal2 software was applied to generate the stiffness matrix K.

The program integrates the set of differential equations that describe the dynamics of a two-link manipulator presented in this paper. This set consists of fourteen highly non-linear differential equations of second order that were transformed, using the state space method, into a set of twenty eight differential equations of the first order. Given geometrical dimensions of links, material data, cross sections of link shapes and the input torques, the program calculates the responses of generalized variables in time.

The Runge-Kutta formulae of the fourth and fifth order were used for numerical integration of these differential equations. A particular implementation of these formulae — the Fehlberg method — allowed application of an automatic step size of integration.

The preliminary results of computations are presented in the next section.

11. Calculation Results

The results of calculations are presented in Figs. 4–7. The following input data were used:

length of each link	$l = 100\,\text{cm}$
link material	aluminum
link cross section	circular
diameter	1 cm
linear density of link material	$0.2132\,\text{kg/m}$
time range	0–0.5 s
torque in the first joint	10 Nm $\quad t < 0.25\,\text{s}$
	-10 Nm $\quad t > 0.25\,\text{s}$
torque in the second joint	4 Nm $\quad t < 0.25\,\text{s}$
	-4 Nm $\quad t > 0.25\,\text{s}$

The calculation time was set in the range of 0–0.5 s. To all the generalized coordinates, except for q_1, zero initial conditions were applied. As an initial condition for the first joint variable q_1 the value of $-\pi/2$ rad was used.

Figure 4 presents the time response of the first generalized coordinate. It shows the angular position of the first joint, after a step function disturbance induced at time equal to zero. The input signal — torque — rises at zero, remains steady up to time equal to 0.25 s and then falls down to a negative value, remains steady and returns to zero at time equal to 0.5 s. The curve is not symmetric with respect to the point $t = 0.25\,\text{s}$, because the total moment of inertia of the system changes with time. This is due to a change of the geometrical configuration of the manipulator.

Figure 5 illustrates the time response of the second generalized coordinate, that is the angular position of the second joint, after it was disturbed by the input torque applied at this joint. Analogically, this input signal raises at zero and changes its sign at time equal to 0.25 s. It should be noted that the shape of the curve is influenced by the inertia of both links. These two variables represent the large motion of the manipulator links.

Figure 6 shows the time responses of the generalized coordinates $p_{1,8}$ and $p_{1,14}$ that represent the deflection of the middle and the last finite element node in the first link. These coordinates represent the small perturbation motion of the link with respect to the imaginary rigid link. The two main vibration frequencies that were excited by input torques are visible in the form of different curve shapes.

Figure 7 presents the time responses of the generalized coordinates $p_{1,12}$ and $p_{1,18}$ that represent the rotation of the middle and the last finite element node in the first link. The smaller values represent the middle node rotation, while the larger values correspond to the last node rotation. Again, the two main frequencies are visible on both curves.
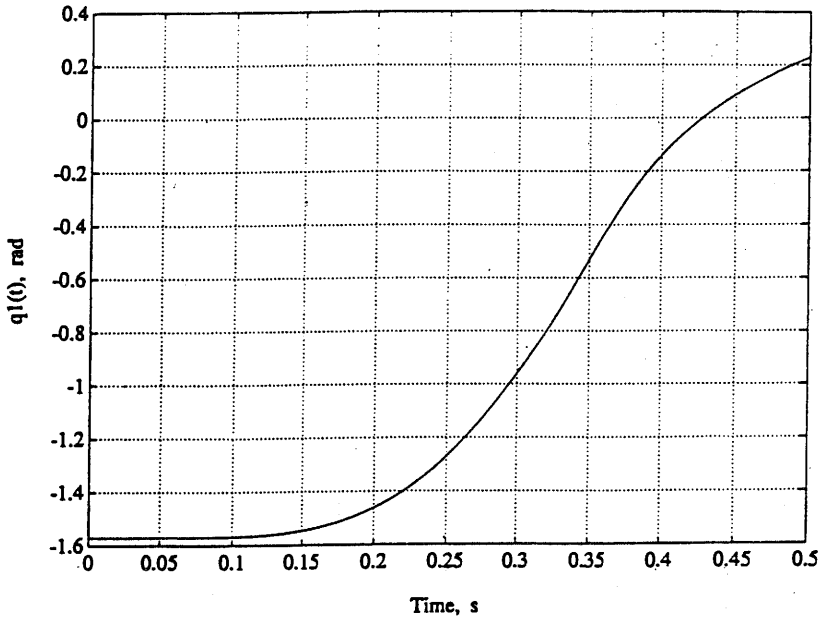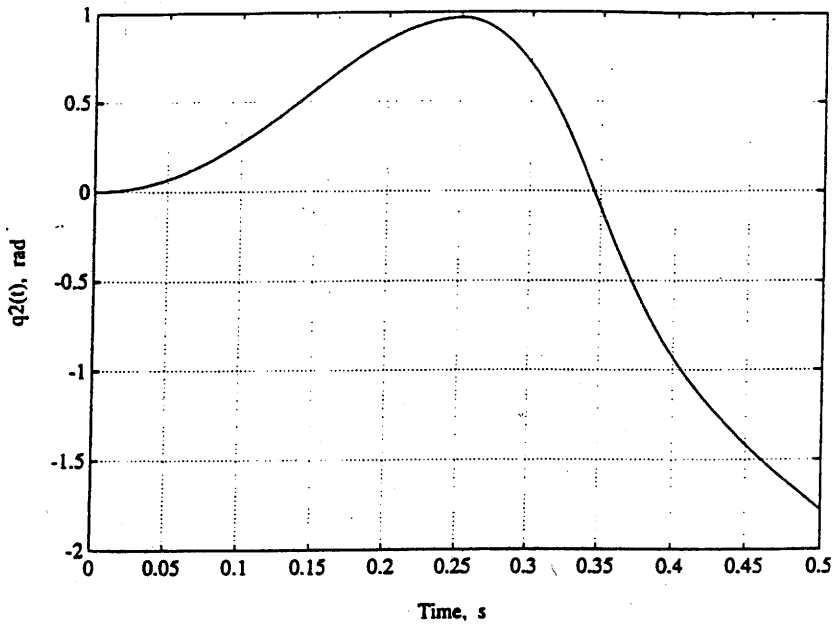
Fig. 4. Time response of the first joint coordinate, q_1.

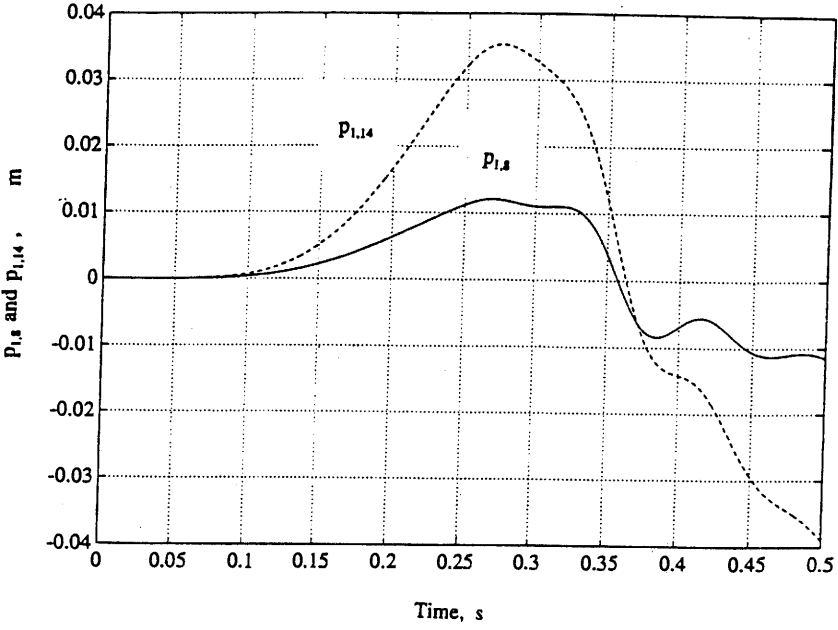Fig. 5. Time response of the second joint coordinate, q_2.

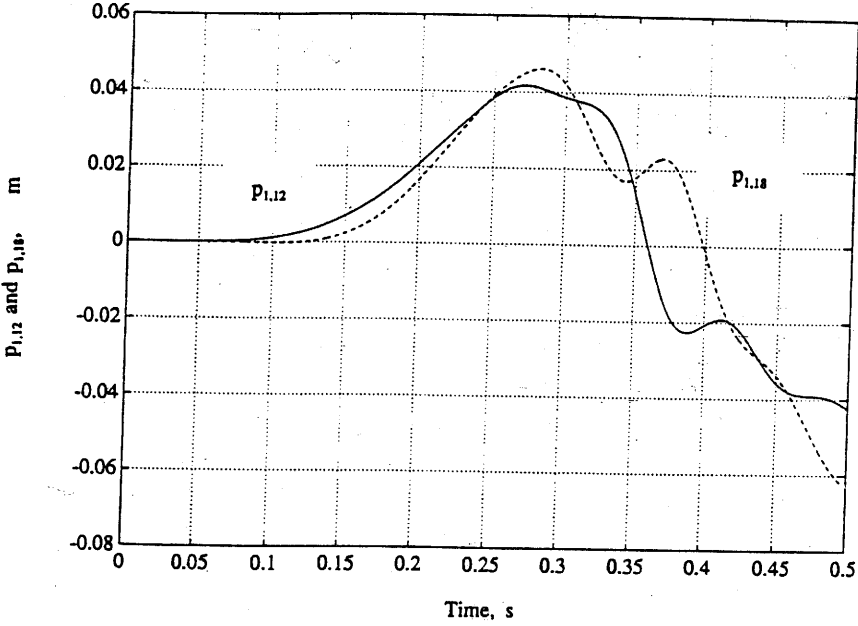Fig. 6. Time response of the generalized coordinates $p_{1,8}$ and $p_{1,14}$.

Fig. 7. Time response of the generalized coordinates $p_{1,12}$ and $p_{1,18}$.

12. Conclusions

In this paper the Lagrange finite element approach was used to formulate the dynamic model of an n-link flexible manipulator. The derived differential equations of motion take into account a coupling between the rigid body motion and elastic deformations of links. Specific formulae were presented for the two-link manipulator. Two finite elements in each link were used. The presented differential equations were derived in a consistent manner, which allows a further generalization of the model. In particular, this model can be used for an arbitrary number of links, arbitrary number of finite elements in the link, arbitrary cross section of each link, as well as arbitrary mass and stiffness distributions along the link. On the basis of the presented algorithm, a computer program was created and simulation results were obtained.

The main advantage of the method presented in this paper is the model generality. The application of the finite element methodology to calculate the manipulator dynamics gives an opportunity to take into account various complicated shapes of the manipulator link. Also, in addition to bending and shear deformations, various modes of link deflection can be considered, including small bending rotation inertia terms, axial deformation terms and a torsion about a longitudinal axis of a link.

The main drawback of this model is its computational burden that affects the computing time. This may be improved by considering all possible simplifications, as e.g. recursive schemes, special multiplication subroutines, appropriate numerical matrix inversion and integration methods. Some of recursive relations were already used in the algorithm presented in this paper. Also, the rapid development of the computer technology allows using more complex models of the flexible manipulators, that could be run fast enough for real time control applications.

An appropriate computer program has been developed. As the programming environment the commercial software MATLAB was used. To generate the stiffness matrix K, another commercial program — MSC/pal2 — was applied.

The results obtained during the initial simulation confirm that the model could be considered for investigations of n-link flexible manipulator dynamics. Further work in this area should concentrate on the improvement of the model efficiency. New algorithms that would reduce computation time will be of special consideration.

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