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# ON MULTIPURPOSE CONTROL SYSTEMS SYNTHESIS FOR NON-SQUARE DISCRETE-TIME MULTIVARIABLE PLANTS

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The paper is devoted to the synthesis of multipurpose control systems for linear $m$-input $l$-output $(m \geq l)$ discrete-time models of plants defined by proper right-invertible rational transfer matrices in $z \in \mathbb{C}$. The under consideration control systems simultaneously ensure: (i) complete dynamic decoupling, (ii) arbitrary closed-loop pole placement and (iii) steady-state output rejection of deterministic part of disturbances and zero steady-state regulations (or tracking) errors. Two cases of control systems are considered assuming that the state vector of the plant is either accessible or inaccessible for direct measurement. In the case of inaccessible plant's state vector, a stationary Kalman filter or a Luenberger observer is applied to estimate (to reconstruct) the state vector of the plant, which is then used in a linear state variable feedback. In a stochastic case with the inaccessible state vector, two control system structures can be designed with the filtered output $\hat{y}(k)$ or the original output $y(k)$ which are then used to define the error signal vectors $e(k) = y_0 - \hat{y}(k)$ or $e(k) = y_0 - y(k)$, respectively. The idea of the method and the algorithm for the synthesis of the proposed control systems as well as a short analysis of control system properties are presented.

## 1. Introduction

Applications of computer techniques along with recent synthesis methods developed for designing multivariable control systems give new possibilities for solving complex control problems. They are especially useful for multipurpose systems that ensure simultaneously the design goals as input-output dynamical decoupling, zero steady-state control errors with rejection of disturbances, and *a priori* assumed dynamical properties of the system defined by location of its closed-loop poles.

The first results, in which the above purposes were really achieved, were obtained by Wolovich (1981). His paper was concerned with deterministic, continuous multivariable plants described by strictly proper and invertible transfer matrices. Next steps have been made by Bańka (1991a; 1991b; 1994a) to expand the results of Wolovich to more general proper continuous plants with both stochastic and deterministic disturbances. In these papers, however, it has been proved that such multipurpose systems can be unstable if the square plants have non-minimum phase zeros (Bańka, 1991b; Bańka and Moskwa, 1994). Thus in the paper (Bańka, 1994b) a new

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algorithm is presented for designing multipurpose control systems which ensure all of the above-mentioned properties for non- square (right-invertible) continuous plants which can be both unstable and non-minimum phase.

In this paper, the recent results obtained for such type of discrete-time control systems with plants described by proper rational rectangle (right-invertible) transfer matrices $\boldsymbol{T}(z)$ with full rank $\boldsymbol{T}(z) = l$ have been presented. The multipurpose control systems under consideration achieve simultaneously:

i) complete dynamic decoupling of the closed-loop control system,
ii) arbitrary closed-loop pole displacement into the unit disc of $|z| < 1$ (including *deadbeat* poles located at the centre of that disc) and
iii) complete steady-state output rejection of the deterministic part of disturbances and zero steady-state regulation (or tracking) errors.

In the stochastic case, three control system structures can be designed as follows: (1) a control system structure with an access to the original output $\boldsymbol{y}(k)$ and an accessible state vector $\boldsymbol{x}(k)$ of the plant, (2) a system with the filtered output $\hat{\boldsymbol{y}}(k)$ and the filtered state vector $\hat{\boldsymbol{x}}(k)$ and (3) a structure with the filtered state vector $\hat{\boldsymbol{x}}(k)$ and the original output $\boldsymbol{y}(k)$. In these structures the filtered output $\hat{\boldsymbol{y}}(k)$ and the filtered state vector $\hat{\boldsymbol{x}}(k)$ are obtained from a stationary Kalman filter. For a deterministic case only two structures with (1) an accessible and (2) inaccessible state vector of the plant can be applied. In this case, a Luenberger observer (either reduced or full-order) may be used to reconstruct the unmeasured state vector of the plant.

In order to solve the problem considered we apply the direct polynomial matrix approach using simultaneously the following principles: diagonal dynamic decoupling and pole placement by linear state variable feedback (l.s.v.f.) with "input dynamics" (Wolovich, 1974), the "internal model" principle in the form of sufficient conditions given in (Callier and Desoer, 1982) and the well-known "separation" principle for solving the state estimation (reconstruction) problem irrespectively of solving the control tasks (i)–(iii).

## 2. The Plant Model and Structures of the Control Systems

We consider fully controllable and observable linear, multi-input, multi-output (MIMO) discrete-time models of plants defined by the state and output equations

$$\begin{aligned} \boldsymbol{x}(k+1) &= \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{u}(k) + \boldsymbol{E}\boldsymbol{r}(k) + \boldsymbol{G}\boldsymbol{w}(k) \\ \boldsymbol{y}(k) &= \boldsymbol{C}\boldsymbol{x}(k) + \boldsymbol{D}\boldsymbol{u}(k) + \boldsymbol{\nu}(k) \end{aligned} \tag{1}$$

where $\boldsymbol{y}(k) \in \mathbb{R}^n$, $\boldsymbol{u}(k) \in \mathbb{R}^m$ and $\boldsymbol{y}(k) \in \mathbb{R}^l$ $(m \geq l)$ are the state, the input and the output vectors, respectively. In the stochastic case, the vectors $\boldsymbol{w}(k) \in \mathbb{R}^p$ and $\boldsymbol{\nu}(k) \in \mathbb{R}^l$ represent zero-mean, discrete "white" random processes with the mutual covariance matrix

$$E\left\{\begin{bmatrix} \boldsymbol{w}(i) \\ \boldsymbol{\nu}(i) \end{bmatrix} \begin{bmatrix} \boldsymbol{w}^T(j) \, \boldsymbol{\nu}^T(j) \end{bmatrix}\right\} = \begin{bmatrix} \boldsymbol{W} & \boldsymbol{S} \\ \boldsymbol{S}^T & \boldsymbol{V} \end{bmatrix} \delta_{i-j} \tag{2}$$

This block matrix can be semi-positive definite for a positive definite density matrix $\boldsymbol{V} > 0$. The vector $\boldsymbol{r}(k) \in \mathbb{R}^r$ describes deterministic (non-diminishing) disturbances which in the stochastic case (with $\boldsymbol{E} = \boldsymbol{G}$) can also be treated as non-zero mean values of the disturbances $\boldsymbol{w}(k)$.

Adopting the polynomial matrix approach we transform the plant model (1) to the relatively prime matrix fraction description in $z \in \mathbb{C}$ as follows

$$\boldsymbol{y} = \boldsymbol{B}_1(z)\boldsymbol{A}_1^{-1}(z)\boldsymbol{u} + \boldsymbol{A}_2^{-1}(z)\boldsymbol{B}_2(z)\boldsymbol{w} + \boldsymbol{A}_3^{-1}(z)\boldsymbol{B}_3(z)\bar{\boldsymbol{r}} + \boldsymbol{\nu} \tag{3}$$

where

$$\boldsymbol{B}_1(z)\boldsymbol{A}_1^{-1}(z) = \boldsymbol{C}(z\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}, \tag{4}$$

$$\boldsymbol{A}_1^{-1}(z)\boldsymbol{B}_2(z) = \boldsymbol{C}(z\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{G} \tag{5}$$

and

$$\boldsymbol{A}_3^{-1}(z)\boldsymbol{B}_3(z) = \boldsymbol{C}(z\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{E}\boldsymbol{r}(z) \tag{6}$$

Since the transformed disturbance vector $\boldsymbol{r}(z)$ is included into the transfer matrix (6) the symbol $\bar{\boldsymbol{r}}$ in eqn. (3) denotes a "fictitious" impulsive input signal to the deterministic disturbance model. Similarly, the reference signal vector is generated from the reference model defined by

$$\boldsymbol{y}_0(z) = \boldsymbol{A}_0^{-1}(z)\boldsymbol{B}_0(z)\bar{\boldsymbol{y}}_0 \tag{7}$$

This is a diagonal and strictly proper transfer function matrix with the impulsive signal input $\bar{\boldsymbol{y}}_0$.

Three possible structures of stochastic control systems in the time domain are presented in Figs. 1, 2 and 3.

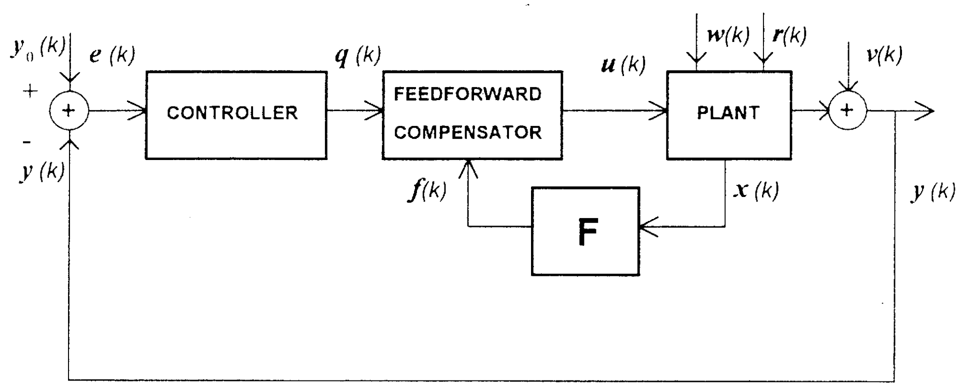


Fig. 1. Structure of the control system with an accessible state vector of the plant.

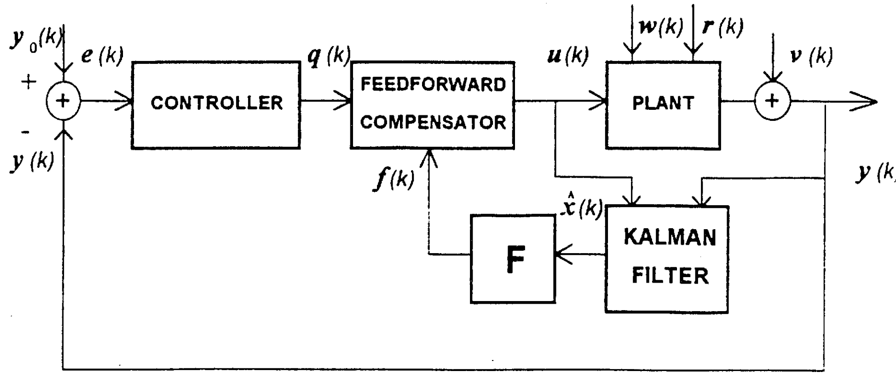


Fig. 2. Structure of the system with an inaccessible plant's state vector and direct feedback from output of the plant.

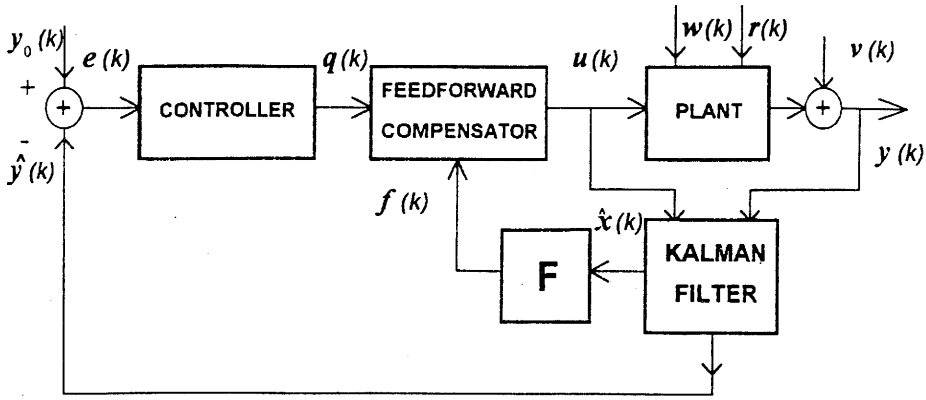


Fig. 3. Structure of the system with feedback from the filtered plant's output.

In Fig. 1 where the direct state feedback $\boldsymbol{f}(k) = \boldsymbol{F}\boldsymbol{x}(k)$ is employed, the proposed "two-part" compensator/controller includes a diagonal controller and a feedforward compensator ("input dynamics"). When the plant's state vector is not available for measurement, then the control system structures pictured in Figs. 2 and 3 contain additionally the third part which is either a stationary Kalman filter or a Luenberger observer for the deterministic case, respectively. In this structure, the error signal vector is defined simply as $\boldsymbol{e}(k) = \boldsymbol{y}_0(k) - \boldsymbol{y}(k)$. Moreover, as it is seen in Fig. 3, the additional filter's output $\hat{\boldsymbol{y}}(k)$ can also be used to define the error signal vector $\boldsymbol{e}(k) = \boldsymbol{y}_0(k) - \hat{\boldsymbol{y}}(k)$.

In the deterministic case only two distinct structures can be applied. These are generally consistent with the schemes given in Figs. 1 and 2 with the substitution of a Luenberger observer for a Kalman filter.

The structure of the multipurpose control system in the $z$-domain is presented in Fig. 4.

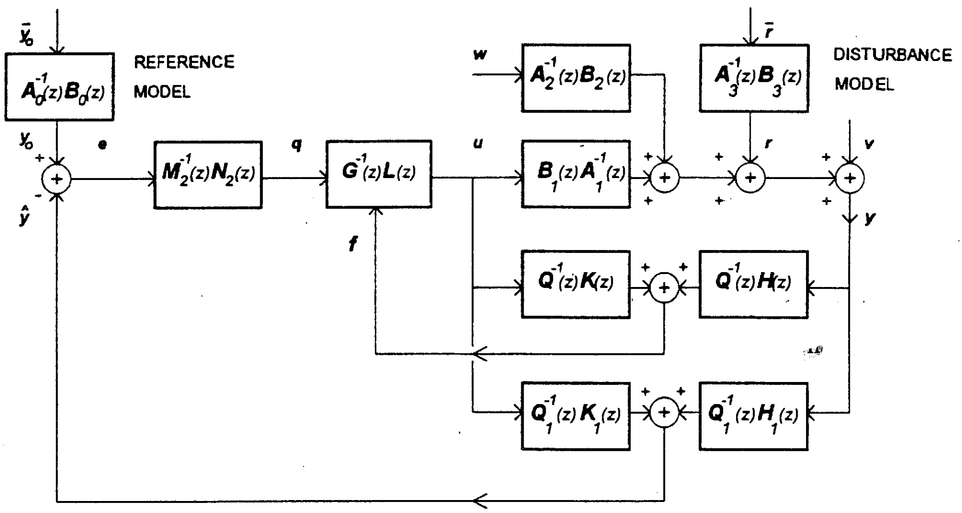


Fig. 4. Structure of the system in the $z$-domain for the stochastic case.

It is compatible with the most developed control system structure given in Fig. 3 for the stochastic case. Particular parts of the system are suitably defined in the $z$-domain by

1) the strictly proper (diagonal) transfer matrix $M_2^{-1}(z)N_2(z)$ for the controller,
2) the proper and possible "low-order" transfer matrix $G_I^{-1}(z)L(z)$ for the dynamic feedforward compensator,
3) the strictly proper transfer matrices

$$Q^{-1}(z)H(z) = F(zI_n - A + KC)^{-1}K \tag{8}$$

and

$$Q^{-1}(z)K(z) = F(zI_n - A + KC)^{-1}(B - KD) \tag{9}$$

for the Kalman filter along with the feedback matrix $F$,
4) the strictly proper transfer matrix

$$Q_1^{-1}(z)H_1(z) = C(zI_n - A + KC)^{-1}K \tag{10}$$

and proper transfer matrix

$$Q_1^{-1}(z)K_1(z) = C(zI_n - A + KC)^{-1}(B - KD) + D \tag{11}$$

for the Kalman filter along with the plant's output matrix $C$.

For a deterministic problem (or for a case of Fig. 2) the last transfer matrices are not present. Moreover, when a reduced Luenberger observer is applied, strict properness of the transfer matrices $Q^{-1}(z)\left[H(z)\vdots K(z)\right]$ defined by eqns. (8) and (9) is not valid. In this case both of them may be proper.

All of the above-mentioned polynomial matrix fractions are relatively left prime (RLP) with non-singular, row-proper, denominator matrices $M_2(z)$, $G(z)$, $Q(z)$ and $Q_1(z)$. The transfer matrices (8)–(11) for the Kalman filter (or (8)–(9) for a Luenberger observer) must be stable with the poles located within the unit disc. We also assume that the polynomial matrices $M_2(z)$ and $L(z)$ as well as $M_2(z)$ and $Q(z)$ are relatively right prime (RRP).

The problem is as follows. Given the transfer matrices of a plant defined by eqns. (4)–(6) and (7) determine the transfer matrices $M_2^{-1}(z)N_2(z), G^{-1}(z)$ $L(z), Q^{-1}(z)[H(z)\vdots K(z)]$ and $Q_1^{-1}(z)[H_1(z)\vdots K_1(z)]$ for the above-mentioned parts of the control system so as to realize the design objectives (i)–(iii).

## 3. Layout of the Method

The idea of the proposed method for solving the considered control tasks is as follows. Using the linear state variable feedback (l.s.v.f.) along with the dynamic feedforward $G^{-1}(z)L(z)$ we decouple the "inner" part of the system between the signals $q(k)$ and $y(k)$ to obtain the diagonal transfer matrix

$$T_{yq}(z) = B_1(z)[G(z)A_1(z) - F(z)]^{-1}L(z) = N(z)D^{-1}(z) \tag{12}$$

where

$$D^{-1}(z) = B(z)\left[G(z)A_1(z) - F(z)\right]^{-1} L(z) \tag{13}$$

Then we can apply the "internal model" principle for designing a feedback system with the "decoupled plant" defined by eqn. (12). According to the sufficient conditions of that principle given by (Callier and Desoer, 1982), see also (Bengtsson, 1977; Wolovich, 1974), the denominator matrix of the controller can be chosen as $M_2(z) = I_l m(z) = \mathrm{diag}[m(z)]$, where $m(z)$ is the least common multiplier of polynomials for all the unstable parts of the transfer matrices in eqns. (4)–(7). Hence it results in the Diophantine (diagonal) polynomial matrix equation

$$M_2(z)D(z) + N(z)N_2(z) = \Delta(z) \tag{14}$$

where $\Delta(z) = \mathrm{diag}[\delta_i(z)]$, $i = 1, 2, \ldots, l$, with $\delta_i(z)$ chosen as stable (monic) polynomials of degree $\deg[m(z)] + \deg[d_i(z)]$ matched to the assumed configurations of the closed-loop control system poles into the unit disc. The minimal degree solution (with respect to $N_2(z)$) yields both the numerator matrix $N_2(z)$ for the controller and the denominator matrix $D(z)$ for the decoupled "inner" part of the system.

The main problem is to find a method for diagonal decoupling of a rectangular (non-square) plant with $(m \geq l)$ so as to obtain the transfer matrix (12) free of cancellation of unstable "hidden" modes (uncontrollable and/or unobservable poles of $T_{yq}(z)$). They may occur if a plant has "interconnection transmission zeros" (Wiliams and Antsaklis, 1986). In order to do it we adopt the following lemma and the theorem given by Hikita (Hikita, 1987), slightly modified to the case considered.

**Lemma 1.** *The diagonal matrix $D(z) \in \mathbb{R}[z]^{l\times l}$ that satisfies relation (13) exists iff there are polynomial matrices $\bar{L}(z) \in \mathbb{R}[z]^{m\times(m-l)}$ and $\bar{B}(z) \in \mathbb{R}[z]^{m\times(m-l)}$ of full rank such that*

$$G(z)A_1(z) - F(z) - L(z)D(z)B(z) = \bar{L}(z)\bar{B}(z) \tag{15}$$

■

**Theorem 1.** *The closed-loop poles of the decoupled system $T_{yq}(z)$ realized by l.s.v.f. with dynamic feedforward consist of the zeros of $|[L(z), \bar{L}(z)]|$, which are uncontrollable, the zeros of $|[B^T(z), \bar{B}^T(z)]^T|$, which are unobservable and the zeros of $|D(z)|$ which are controllable and observable.* ■

To avoid unstable cancellations in the transfer matrix $T_{yq}(z)$ it is necessary for all the uncontrollable and unobservable poles to lie inside the unit disc. If the polynomial matrix $\widetilde{G}(z) \in \mathbb{R}[z]^{l\times l}$, which is a g.c.l.d. of $B(z)$ defined by the relation

$$B_1(z) = N(z)B(z) = N(z)\widetilde{G}(z)\widetilde{B}(z) \tag{16}$$

is not unimodular and if its zeros lie outside of the unit disc, the poles of the decoupled system corresponding to these unobservable zeros are fixed and unstable. To remove those unobservable poles we use the compensation scheme together with an additional dynamic feedforward compensator obtained by augmenting the plant model with a serial dynamic element $R_a(z)P_a^{-1}(z)$ connected to the input of the original plant. The transfer matrix of the "augmented plant" is given by

$$B_1(z)A_1^{-1}(z)R_a(z)P_a^{-1}(z) = N(z)\widetilde{J}(z)U_1(z)\hat{B}(z)\bar{P}^{-1}(z)P_a^{-1}(z) \tag{17}$$

where the matrices listed on the right-hand side of this equation should be calculated using the proposed design algorithm. Substituting $N(z)$ for $N(z)\widetilde{J}(z)$, $\widetilde{G}(z)$ for $I_l$, $B(z)$ for $U_1(z)\hat{B}(z)$ and $A_1(z)$ for $P_a(z)\bar{P}(z)$, a decoupled system $T_{yq}(z)$ without fixed poles caused by $\widetilde{G}(z)$ is obtained, see (Bańka, 1991b; 1994a) and (Hikita, 1987). Thus we obtain the following design algorithm for the control system under consideration.

**The algorithm**

**Step 1.** Define the matrix $M_2(z) = I_l m(z)$ with $m(z)$, a completely unstable (monic) polynomial generated from unstable poles of the transfer matrices (4)–(7). Let $\bar{m} = \deg[m(z)]$.

**Step 2.** Define $N(z) = \deg[n_i(z)]$, where $n_i(z)$, $i = 1, 2, \dots l$, is a g.c.l.d. of the $i$-th row of $B_1(z)$. Calculate the matrix $B(z) \in \mathbb{R}[z]^{l\times m}$ from the relationship $B_1(z) = N(z)B(z)$ and determine the matrix $\widetilde{G} \in \mathbb{R}[z]^{l\times l}$, a g.c.l.d. of columns of the matrix $B(z) = \widetilde{G}(z)\widetilde{B}(z)$. If $\widetilde{G}(z)$ is unimodular (or stable), then go to Step 3.

If a decoupled system without fixed poles is desired, the following additional steps should be taken.

**Step 2.1.** Given $\widetilde{G}(z)$ and vectors $e_i$, defined by the unit matrix $I_l = [e_1, e_2, ..., e_l]$, derive by the RRP fraction the vector transfer functions

$$\widetilde{G}^{-1}(z)e_i = \widetilde{R}_i(z)\widetilde{J}_{ii}^{-1}(z), \quad i = 1, 2, ..., l \tag{18}$$

Define

$$\widetilde{R}(z) = \left[\widetilde{R}_1(z), ..., \widetilde{R}_l(z)\right] \text{ and } \widetilde{J}(z) = \mathrm{diag}[J_{ii}(z)] \tag{19}$$

to obtain

$$\widetilde{G}(z)\widetilde{B} = \widetilde{J}(z)U_1(z) \text{ with } U_1(z) = I_l \tag{20}$$

**Step 2.2.** Given $\widetilde{B}(z) \in \mathbb{R}[z]^{l\times m}$ and $\widetilde{R}(z) \in \mathbb{R}[z]^{l\times l}$ calculate $\hat{B}(z) \in \mathbb{R}[z]^{l\times m}$ and $\hat{R}(z) \in \mathbb{R}[z]^{m\times m}$by the RRP fraction $\widetilde{R}^{-1}(z)\widetilde{B}(z) = \hat{B}(z)\hat{R}^{-1}(z)$.

**Step 2.3.** Derive $R_a(z) \in \mathbb{R}[z]^{m\times m}$ and $\bar{P} \in \mathbb{R}[z]^{m\times m}$ such that $A_1(z)\hat{R}(z) = R_a(z)\bar{P}(z)$ where $|R_a(z)| = \beta|\hat{R}(z)|$ and $|\bar{P}(z)| = 1/\beta|A_1(z)|$. In order to do it we set the matrix $A_1(z)\hat{R}(z)$ to the Smith form $H(z)$ defined by the relationship $A_1(z)\hat{R}(z) = U_2(z)H(z)U_3(z) = U_2(z)H_R(z)H_A(z)U_3(z)$.

Set $R_a(z) = U_2(z)H_R(z)$ and $\bar{P}(z) = H_a(z)U_3(z)$, where $H_R(z)$ and $H_A(z)$ are $m \times m$ diagonal polynomial matrices such that $|H_R(z)| = \alpha_1|\hat{R}(z)|$ and $|H_A(z)| = \alpha_2|A_1(z)|$. (The matrices $U_2(z)$ and $U_3(z)$ are unimodular, $\alpha_1$ and $\alpha_2$ are some scalars).

**Step 2.4.** By selecting an unimodular matrix $U_4(z) \in \mathbb{R}[z]^{m\times m}$ such that $R_a(z)U_4(z)$ is column-proper, derive the matrix $P_a(z) = U_4(z)\Lambda(z)$, where $\Lambda(z) = \mathrm{diag}[\lambda_i]$, $i = 1, 2, ..., m$. The polynomials $\lambda_i(z)$ are arbitrary so long as $\deg[\lambda_i(z)] = \deg_{ci}[R_a(z)U_4(z)]$.

Substitute $A_1(z) := P_a(z)\bar{P}(z)$, $B(z) := U_1(z)\hat{B}(z)$ and $N(z) := N(z)\widetilde{J}(z)$.

**Step 3.** Choose $\bar{B}(z) \in \mathbb{R}[z]^{(m-l)\times m}$ so that

$$\begin{bmatrix} B(z) \\ \bar{B}(z) \end{bmatrix} \in \mathbb{R}[z]^{m\times m} \tag{21}$$

is unimodular.

**Step 4.** Perform the RLP factorization of

$$A_1(z)\begin{bmatrix} B(z) \\ \bar{B}(z) \end{bmatrix}^{-1} = \widetilde{Q}^{-1}(z)\widetilde{P}(z) \tag{22}$$

to obtain $\widetilde{P}(z) \in \mathbb{R}[z]^{m\times m}$ and $\widetilde{Q}(z) \in \mathbb{R}[z]^{m\times m}$ which are row-reduced.

Determine $\nu_j = \deg_{rj}[\widetilde{Q}(z)], j = 1,2,...,m$ and $\nu = \max[\nu_j]$. Given the $\nu_j$ and $\nu$ derive

$$\hat{Q}(z) = \begin{bmatrix} z^{\nu-\nu_1} & & \\ & \ddots & \\ & & z^{\nu-\nu_m} \end{bmatrix} \widetilde{Q}(z) \tag{23}$$

$$\hat{P}(z) = \begin{bmatrix} z^{\nu-\nu_1} & & \\ & \ddots & \\ & & z^{\nu-\nu_m} \end{bmatrix} \widetilde{P}(z) \tag{24}$$

and define $\left[\hat{P}^F(z), \hat{P}^L(z)\right] = \hat{P}(z)$, where $\hat{P}^F(z) \in \mathbb{R}[z]^{m\times l}$ and $\hat{P}^L(z) \in \mathbb{R}[z]^{m\times(m-l)}$.

**Step 5.** Determine the degree $\bar{d}_i = \deg[d_i(z)]$ for each diagonal element of $D(z) = \deg[d_i(z)]$, $i = 1,2,...,l$, from the constraint

$$\bar{d}_i = \max\left\{\deg_{ci}\left[\hat{P}^F(z)\right] - \nu, 0\right\} \tag{25}$$

**Step 6.** Set $\Delta(z) = \mathrm{diag}[\delta_i(z)]$, $i = 1,2,...,l$ and solve the Diophantine eqn. (14) for $D(z)$ and $N_2(z)$. Each diagonal element of $\Delta(z)$ can be chosen as a stable polynomial of the degree $\bar{m} + \bar{d}_i$ suited to the assumed displacement of the closed-loop poles, inside the unit disc, for each (decoupled) loop of the control system. The zeros of $\Delta(z)$ and $N(z)$ should be disjoint.

**Step 7.** Perform the RLP factorization of

$$A_1(z)\begin{bmatrix} D(z)B(z) \\ \bar{B}(z) \end{bmatrix} = \Phi_D^{-1}(z)\Phi_N(z) \tag{26}$$

to obtain $\Phi_N(z) \in \mathbb{R}[z]^{m\times m}$ and $\Phi_D(z) \in \mathbb{R}[z]^{m\times m}$ which are row-reduced.

Determine $\mu_j = \deg_{rj}[\Phi_D(z)]$, $j = 1,2,...,m$ and $\mu = \max[\mu_j]$. Given $\mu_j$ and $\mu$ derive

$$\hat{\Phi}_D(z) = \begin{bmatrix} z^{\mu-\mu_1} & & \\ & \ddots & \\ & & z^{\mu-\mu_m} \end{bmatrix} \Phi_D(z) \tag{27}$$

$$\hat{\Phi}_N(z) = \begin{bmatrix} z^{\mu-\mu_1} & & \\ & \ddots & \\ & & z^{\mu-\mu_m} \end{bmatrix} \Phi_N(z) \tag{28}$$

Select a unimodular matrix $\hat{W}(z) \in \mathbb{R}[z]^{m\times m}$ such that $\hat{\Phi}_N(z)\hat{W}(z)$ is column-proper.

**Step 8.** Determine the degree $\bar{l}_j = \deg[\hat{l}_j(z)]$, $j = 1,2,...,m$, for each diagonal element of $\hat{L}(z) = \text{diag}[\hat{l}_j(z)]$, from the constraint

$$\bar{l}_j = \max\left\{\deg_{cj}\left[\hat{\Phi}_N(z)\hat{W}(z)\right] - \mu, 0\right\} \tag{29}$$

and set the matrix $\hat{L}(z)$. The elements of $\hat{L}(z)$ can be chosen freely as stable (monic) polynomials suited to the assumed uncontrollable poles of the transfer matrix $T_{yq}(z)$.

**Step 9.** Calculate $[L(z), \bar{L}(z)] = \hat{L}(z)\hat{W}(z)$. The matrices $L(z) \in \mathbb{R}[z]^{m\times l}$ and $\bar{L}(z) \in \mathbb{R}[z]^{m\times(m-l)}$ are given by the first $l$ columns and the last $m-l$ columns of $\hat{L}(z)\hat{W}(z)$, respectively.

**Step 10.** Perform the right division of

$$[L(z)D(z)B(z) + \bar{L}(z)\bar{B}(z)]A_1^{-1}(z) = G(z) - F(z)A_1^{-1}(z) \tag{30}$$

where $G(z) \in \mathbb{R}[z]^{m\times m}$ is the quotient and $-F(z) \in \mathbb{R}[z]^{m\times m}$ is a strictly proper part of the division that satisfies $\deg_{cj}[F(z)] < \deg_{cj}[A_1(z)]$, $j = 1,2,..,m$.

If the additional Steps 2.1–2.4 were taken, the following step would be performed to obtain the feedback matrix $F$ and the dynamic feedforward $G^{-1}(z)L(z)$.

**Step 10.1.** Derive by RLP factorization the matrices $\bar{G}(z) \in \mathbb{R}[z]^{m\times m}$ and $\bar{L}_0(z) \in \mathbb{R}[z]^{m\times m}$ such that

$$\bar{G}^{-1}(z)\bar{L}_0(z) = R_a(z)\,[G(z)P_a(z)]^{-1} \tag{31}$$

and calculate $\bar{L}(z) = \bar{L}_0(z)L(z)$. If $\bar{L}_0(z) \neq I_m$, then perform a right division of

$$\bar{L}_0(z)F(z)A_1^{-1}(z) = \bar{X}_P(z) + \bar{X}_R(z)A_1^{-1}(z) \tag{32}$$

by the original denominator matrix $A_1(z)$ and set the matrices $G(z) := \bar{G}(z) - \bar{X}_P(z)$, $F(z) := \bar{X}_R(z)$, $L(z) := \bar{L}(z)$ and $L_0(z) := I_m$.

**Step 11.** Given the column structure of the plant's denominator matrix $A_1(z)$ and the obtained matrix $F(z)$, determine the feedback matrix $F$ from the relationship

$$F(z) = F\,\hat{T}\,\hat{S}(z) \tag{33}$$

where

$$\hat{S}(z) = \text{ block diag}\left\{\begin{bmatrix} 1 \\ z \\ \vdots \\ z^{\mu_i - 1}\end{bmatrix} \quad \text{for } i = 1,2,...,m\right\} \tag{34}$$

and $\widetilde{T}$ is a similarity transformation matrix which results from the well-known Wolovich structure theorem (Wolovich, 1974). The polynomial matrix $\hat{S}(z)$ and the transformation matrix $\hat{T}$ are calculated during the RLP factorization in eqn. (4).

This step finishes solving the problem with an accessible state vector of the plant. If the plant's state vector is not accessible for a direct measurement, then the following steps should be made.

**Step 12.** In a stochastic case with the Kalman filter, perform the spectral factorization of

$$A_2(z)VA_2^*(z) + A_2(z)S^TB_2^*(z) + B_2(z)SA_2^*(z) + B_2(z)WB_2^*(z) \\ = \bar{C}_2(z)UU^T\bar{C}_2^*(z) \quad (35)$$

which yields a stable (left) factor $\bar{C}_2(z) \in \mathbb{R}[z]^{l\times l}$ ($U$ is an orthogonal matrix), (Kučera, 1981; Ježek and Kučera, 1985).

In the deterministic case with a full-order Luenberger observer set the diagonal matrix

$$\bar{C}_2(z) = \text{diag}[c_j(z)], \quad j = 1, 2, ..., l \quad (36)$$

where

$$c_j(z) = (z - z_1)(z - z_2)...(z - z_{d_j}) \quad (37)$$

are (stable) polynomials generated for the assumed values of poles for the observer. The subscripts $d_i$ are observability indices equal to the row degrees of the denominator matrix $A_2(z)$.

**Step 13.** Given the row structure of $A_2(z)$ transform the matrix $\bar{C}_2(z)$ to a matrix $C_2(z)$ with the same (row) structure as $A_2(z)$. Then determine the "gain" matrix $K$ for the Kalman filter (or $L$ for the full order Luenberger observer) from the equation

$$C_2(z) - A_2(z) = \widetilde{S}(z)\widetilde{T}K \quad (38)$$

where

$$\widetilde{S}(z) = \text{ block diag } \left\{ \left[1 \; z \; ... \; z^{\nu_i - 1}\right] \text{ for } i = 1, 2, ..., l \right\} \quad (39)$$

and $\widetilde{T}$ is a similarity transformation matrix. They are calculated during the RLP factorization in eqn. (5).

■

Using the matrices $F$ and $K$ (or $L$ for the full order observer) we can calculate the transfer matrices of the feedback part of the Kalman filter from eqns. (8) and (9). They should satisfy the matrix polynomial equation

$$K(z)A_1(z) + H(z)B_1(z) = Q(z)F(z) \quad (40)$$

Similarly, we can also calculate the additional transfer matrices (10) and (11) for the remaining part of the Kalman filter which should satisfy the other matrix polynomial equation

$$\boldsymbol{K}_1(z)\boldsymbol{A}_1(z)+\boldsymbol{H}_1(z)\boldsymbol{B}_1(z)=\boldsymbol{Q}_1(z)\boldsymbol{B}_1(z) \tag{41}$$

The above-mentioned transfer matrices are obtained without solving any polynomial equation. However, if a reduced order observer is employed to reconstruct the feedback signal $\boldsymbol{f}(k)=\boldsymbol{F}\boldsymbol{x}(k)$, the polynomial equation (40) must be solved explicitly. Then the reduced Wolovich compensator (Wolovich, 1974) or the minimal degree Rao-Chen compensator (Rao and Chen, 1987) can be used.

To solve the polynomial matrix equation (40) many different methods may be applied, see e.g. algebraic methods (Rao and Chen, 1987; Wolovich, 1974), polynomial methods (Kaczorek, 1984; 1986; Chang *et al.*, 1986), and interpolation methods (Antsaklis and Gao, 1993). Minimal-degree solutions of Diophantine equations are usually employed.

**Comments.** In order to elucidate the design algorithm presented above we provide the following additional comments.

C1. Step 6 plays the main role in the design algorithm. In this step an arbitrary closed-loop pole placement for the entire control system into the unit disc is achieved by a judicious choice of poles for the decoupled "inner" part of the system between signals $\boldsymbol{y}(k)$ and $\boldsymbol{q}(k)$ (determined by $\boldsymbol{D}(z)$) and the zeros of the controller (defined by $\boldsymbol{N}_2(z)$). An "optimal" displacement of the closed-loop poles within the appropriate regions of the unit disc is obtained in an interactive manner by solving the Diophantine equation (14) and performing simulations of the designed control system. Since the transfer matrix $\boldsymbol{T}_{yy_0}(z)$ of the decoupled closed-loop system is completely determined by the matrices $\boldsymbol{N}_2(z)$, $\Delta(z)$ and $\boldsymbol{N}(z)$ (see eqn. (65)), the dynamical behaviour of the control system can be evaluated just when a solution of eqn. (14) is found.

C2. The (minimal-degree) solutions of the Diophantine equation (14) exist if the appropriate polynomials in $\boldsymbol{M}_2(z)$ and $\boldsymbol{N}(z)$ are prime. But these solutions can be unsatisfactory from a practical point of view if non-minimum phase zeros of the designed control systems caused by $\boldsymbol{N}_2(z)$ occur. They can be changed only by suitable selecting closed-loop poles of the system during solving eqn. (14). However, some restrictions on these selections exist since the zeros of the diagonal matrix $\Delta(z)$ should be disjoint with the zeros of the matrix $\boldsymbol{N}(z)$. This condition is sufficient but it is not necessary.

C3. In Step 3 the unimodular matrix $\left[\boldsymbol{B}^T(z)\;\bar{\boldsymbol{B}}^T(z)\right]^T$ is usually chosen by adding $m-l$ rows to the original matrix $\boldsymbol{B}(z)$ calculated in Step 2. This choice, however, is not unique since this matrix can also be chosen as a square (stable) polynomial matrix.

C4. Zeros of the denominator matrices $\boldsymbol{Q}(z)\in\mathbb{R}[z]^{m\times m}$ (and $\boldsymbol{Q}_1(z)\in\mathbb{R}[z]^{l\times l}$ in the stochastic case) should lie inside the unit disc. If the zeros of the observer denominator matrix $\boldsymbol{Q}(z)$ assumed in Step 12 are to be located exactly in the centre of the unit disc, then the Luenberger observers have deadbeat properties.

In a similar manner, the deadbeat properties for the entire control system can also be achieved if the poles of the closed-loop system (i.e. the zeros of $\Delta(z)$) are located exactly at the centre of the unit disc.

C5. The full control system structure for the stochastic case depicted in Fig. 3 and 4 can be used only if the obtained stationary Kalman filter is "sufficiently fast", i.e. if the zeros of the denominator matrix $\boldsymbol{Q}(z)$ obtained from the spectral factorization of eqn. (35) are located closer to the centre of the unit disc than the assumed poles of a closed-loop control system. It depends on the intensity of the noises $\boldsymbol{w}(k)$ and $\boldsymbol{\nu}(k)$. If this condition is not satisfied, the structure of Fig. 2 (or of Fig. 1 if the plant's state vector $\boldsymbol{x}(k)$ is accessible) should be used. The benefits resulting from the structure of Fig. 3 where the Kalman filter is used twice consist mainly in additional reduction of modulations of the control signals $\boldsymbol{u}(k)$.

## 4. Short Anlysis of Control System Properties

The complete dynamic behaviour of the multipurpose control system in the $z$-domain is described by the (block) polynomial matrix form

$$\begin{aligned} \boldsymbol{P}(z)\boldsymbol{X}_p &= \boldsymbol{Q}(z)\boldsymbol{U} \\ \boldsymbol{Y} &= \boldsymbol{R}(z)\boldsymbol{X}_p + \boldsymbol{V}(z)\boldsymbol{U} \end{aligned} \tag{42}$$

where*

$$\boldsymbol{P}(z) = \begin{bmatrix} \boldsymbol{G}\boldsymbol{A}_1 & -\boldsymbol{L} & -\boldsymbol{I}_m & 0 & 0 & 0 & 0 \\ 0 & \boldsymbol{M}_2 & 0 & -\boldsymbol{N}_2 & 0 & 0 & 0 \\ -\boldsymbol{K}\boldsymbol{A}_1 - \boldsymbol{H}\boldsymbol{B}_1 & 0 & \boldsymbol{Q} & 0 & -\boldsymbol{H} & -\boldsymbol{H} & 0 \\ -\boldsymbol{K}_1\boldsymbol{A}_1 - \boldsymbol{H}_1\boldsymbol{B}_1 & 0 & 0 & \boldsymbol{Q}_1 & -\boldsymbol{H}_1 & -\boldsymbol{H}_1 & 0 \\ 0 & 0 & 0 & 0 & \boldsymbol{A}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boldsymbol{A}_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boldsymbol{A}_0 \end{bmatrix}$$

$$\boldsymbol{X}_p = \begin{bmatrix} \boldsymbol{x}_p \\ \boldsymbol{q} \\ \boldsymbol{f} \\ \hat{\boldsymbol{y}} \\ \boldsymbol{w}' \\ \boldsymbol{r}' \\ \boldsymbol{y}_0 \end{bmatrix}, \quad \boldsymbol{Q}(z) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \boldsymbol{N}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & H \\ 0 & 0 & 0 & \boldsymbol{H}_1 \\ 0 & 0 & \boldsymbol{B}_2 & 0 \\ 0 & \boldsymbol{B}_3 & 0 & 0 \\ \boldsymbol{B}_0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{U} = \begin{bmatrix} \bar{\boldsymbol{y}}_0 \\ \bar{\boldsymbol{r}} \\ \boldsymbol{w} \\ \boldsymbol{\nu} \end{bmatrix}, \quad \boldsymbol{Y} = \begin{bmatrix} \boldsymbol{e} \\ \boldsymbol{y} \\ \hat{\boldsymbol{y}} \end{bmatrix}$$

$$\boldsymbol{R}(z) = \begin{bmatrix} 0 & 0 & 0 & -\boldsymbol{I}_l & 0 & 0 & \boldsymbol{I}_l \\ \boldsymbol{B}_1 & 0 & 0 & 0 & \boldsymbol{I}_l & \boldsymbol{I}_l & 0 \\ 0 & 0 & 0 & \boldsymbol{I}_l & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{V}(z) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{I}_l \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

*The operator $z$ is omitted for the sake of simplicity.

Assuming that the polynomial $m(z)$ is prime with respect to each element of the matrices $\boldsymbol{Q}(z)$ and $\boldsymbol{L}(z)$, there exist two pairs of the polynomial matrices $\eta_1(z) \in \mathbb{R}[z]^{m\times m}$ and $\eta_2(z) \in \mathbb{R}[z]^{m\times m}$, as well as $\eta_3(z) \in \mathbb{R}[z]^{l\times m}$ and $\eta_4(z) \in \mathbb{R}[z]^{l\times l}$ such that

$$\begin{bmatrix} \boldsymbol{Q} & \boldsymbol{I}_m \\ \eta_3 & \eta_4 \end{bmatrix} \begin{bmatrix} -\boldsymbol{I}_m \\ \boldsymbol{Q} \end{bmatrix} = \begin{bmatrix} 0 \\ \boldsymbol{I}_m \end{bmatrix} \tag{43}$$

and

$$\begin{bmatrix} \boldsymbol{M} & \boldsymbol{QL} \\ \eta_3 & \eta_4 \end{bmatrix} \begin{bmatrix} -\boldsymbol{QL} \\ \boldsymbol{M}_2 \end{bmatrix} = \begin{bmatrix} -m(z)\boldsymbol{I}_m\boldsymbol{QL} + \boldsymbol{QLI}_l m(z) \\ -\eta_3\boldsymbol{QL} + \eta_4\boldsymbol{I}_l m(z) \end{bmatrix} = \begin{bmatrix} 0 \\ \boldsymbol{I}_l \end{bmatrix} \tag{44}$$

The last relationship uses the matrix $\boldsymbol{M}_2(z) = \boldsymbol{I}_l m(z)$ for which there exists another matrix $\boldsymbol{M}(z) = m(z)\boldsymbol{I}_m$.

If we premultiply eqn. (42) by the unimodular matrices

$$\boldsymbol{U}_2(z)\boldsymbol{U}_1(z) = \begin{bmatrix} \boldsymbol{M} & \boldsymbol{QL} & 0 & 0 & 0 & 0 & 0 \\ \eta_3 & \eta_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{I}_m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{I}_l & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boldsymbol{I}_l & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boldsymbol{I}_l & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boldsymbol{I}_l \end{bmatrix} \begin{bmatrix} \boldsymbol{Q} & 0 & \boldsymbol{I}_m & 0 & 0 & 0 & 0 \\ 0 & \boldsymbol{I}_l & 0 & 0 & 0 & 0 & 0 \\ \eta_1 & 0 & \eta_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{I}_l & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boldsymbol{I}_l & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boldsymbol{I}_l & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boldsymbol{I}_l \end{bmatrix} \tag{45}$$

we obtain an equivalent description of the control system under consideration given by

$$\widetilde{\boldsymbol{P}}(z)=\boldsymbol{U}_2(z)\boldsymbol{U}_1(z)\boldsymbol{P}(z)$$

$$= \begin{bmatrix} m(z)(\boldsymbol{QGA}_1 - \boldsymbol{KA}_1 - \boldsymbol{HB}_1) & 0 & 0 & -\boldsymbol{QLN}_2 & -\boldsymbol{HM}_2 & -\boldsymbol{HM}_2 & 0 \\ \eta_3(\boldsymbol{QGA}_1 - \boldsymbol{KA}_1 - \boldsymbol{HB}_1) & \boldsymbol{I}_l & 0 & -\eta_4\boldsymbol{N}_2 & -\eta_3\boldsymbol{H} & -\eta_3\boldsymbol{H} & 0 \\ \eta_1\boldsymbol{GA}_1 - \eta_2(\boldsymbol{KA}_1 + \boldsymbol{HB}_1) & -\eta_1\boldsymbol{L} & \boldsymbol{I}_m & 0 & -\eta_2\boldsymbol{H} & -\eta_2\boldsymbol{H} & 0 \\ -\boldsymbol{K}_1\boldsymbol{A}_1 - \boldsymbol{H}_1\boldsymbol{B}_1 & 0 & 0 & -\boldsymbol{Q}_1 & -\boldsymbol{H}_1 & -\boldsymbol{H}_1 & 0 \\ 0 & 0 & 0 & 0 & \boldsymbol{A}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boldsymbol{A}_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boldsymbol{A}_0 \end{bmatrix} \tag{46}$$

and

$$\widetilde{Q}(z) = U_2(z)U_1(z)Q(z) = \begin{bmatrix} QLN_2 & 0 & 0 & 0 \\ \eta_4 N_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta_2 H \\ 0 & 0 & 0 & H_1 \\ 0 & 0 & B_2 & 0 \\ 0 & B_3 & 0 & 0 \\ B_0 & 0 & 0 & 0 \end{bmatrix} \tag{47}$$

Inverting $\widetilde{P}(z)$ and premultiplying and postmultiplying the result by the matrices $R(z)$ and $\widetilde{Q}(z)$ we obtain the (block) rational transfer matrix between the block vectors $Y$ and $U$ defined by

$$T(z) = R(z)\widetilde{P}^{-1}(z)\widetilde{Q}(z) + V(z) = \begin{bmatrix} T_{e\bar{y}_0} & T_{e\bar{r}} & T_{ew} & T_{ev} \\ T_{y\bar{y}_0} & T_{y\bar{r}} & T_{yw} & T_{yv} \\ T_{\hat{y}\bar{y}_0} & T_{\hat{y}\bar{r}} & T_{\hat{y}w} & T_{\hat{y}v} \end{bmatrix} \tag{48}$$

where the most interesting transfer matrices between the error vector $e(k)$ and the vectors of the signals $\bar{y}_0, \bar{r}, w(k)$ and $\nu(k)$as well as the transfer matrices between the original plant's output $y(k)$ and the filtered output $\hat{y}(k)$ with respect to the input vector of the reference signals $\bar{y}_0$ have the following forms:

$$T_{e\bar{y}_0} = I_l - Q_1^{-1}(K_1A_1 + H_1B_1)\Delta_f^{-1}QLN_2A_0^{-1}B_0 \tag{49}$$

$$\begin{aligned} T_{e\bar{r}} &= [Q_1^{-1}(K_1A_1 + H_1B_1) \\ &\quad \cdot\Delta_f^{-1}(QLN_2Q_1^{-1}H_1 - HM_2) - Q_1^{-1}H_1]A_3^{-1}B_3 \end{aligned} \tag{50}$$

$$\begin{aligned} T_{ew} &= [Q_1^{-1}(K_1A_1 + H_1B_1) \\ &\quad \cdot\Delta_f^{-1}(QLN_2Q_1^{-1}H_1 - HM_2) - Q_1^{-1}H_1]A_2^{-1}B_2 \end{aligned} \tag{51}$$

$$\begin{aligned} T_{ev} &= [Q_1^{-1}(K_1A_1 + H_1B_1) \\ &\quad \cdot\Delta_f^{-1}(QLN_2Q_1^{-1}H_1 - HM_2) - Q_1^{-1}H_1] \end{aligned} \tag{52}$$

$$T_{y\bar{y}_0} = B_1\Delta_f^{-1}QLN_2A_0^{-1}B_0 \tag{53}$$

$$T_{\hat{y}\bar{y}_0} = Q_1^{-1}(K_1A_1 + H_1B_1)\Delta_f^{-1}QLN_2A_0^{-1}B_0 \tag{54}$$

The remaining six transfer matrices take similar forms as the transfer matrices for continuous systems obtained by Bańka (1991b; 1994a).

By analogy to the continuous systems it can be proved that all the poles of the control system are given by the zeros of the determinants

$$|\widetilde{P}(z)| = |\Delta_1(z)| \times |A_2(z)| \times |A_3(z)| \times |A_0(z)| \tag{55}$$

with the "internal" closed-loop poles of the system defined by

$$|\Delta_1(z)| = \left|\begin{bmatrix} m(z)(\boldsymbol{QGA}_1-\boldsymbol{KA}_1-\boldsymbol{HB}_1) & 0 & 0 & -\boldsymbol{QLN}_2 \\ \eta_3(\boldsymbol{QGA}_1-\boldsymbol{KA}_1-\boldsymbol{HB}_1) & \boldsymbol{I}_l & 0 & -\eta_4\boldsymbol{N}_2 \\ \eta_1\boldsymbol{GA}_1-n_2(\boldsymbol{KA}_1+\boldsymbol{HB}_1) & -\eta_1\boldsymbol{L} & \boldsymbol{I}_m & 0 \\ -\boldsymbol{K}_1\boldsymbol{A}_1-\boldsymbol{H}_1\boldsymbol{B}_1 & 0 & 0 & -\boldsymbol{Q}_1 \end{bmatrix}\right|$$

$$= |\boldsymbol{Q}_1(z)| \times |\Delta_f(z)| \tag{56}$$

where

$$\Delta_f(z) = m(z)(\boldsymbol{QGA}_1-\boldsymbol{KA}_1-\boldsymbol{HB}_1) + \boldsymbol{QLN}_2\boldsymbol{Q}_1^{-1}(\boldsymbol{K}_1\boldsymbol{A}_1+\boldsymbol{H}_1\boldsymbol{B}_1) \tag{57}$$

In view of (40) and (41) we have

$$\boldsymbol{T}_{y\bar{y}_0}(z) = \boldsymbol{T}_{\hat{y}\bar{y}_0}(z) = \boldsymbol{B}_1(z)\Delta_f^{-1}(z)\boldsymbol{Q}(z)\boldsymbol{L}(z)\boldsymbol{N}_2(z)\boldsymbol{A}_0^{-1}(z)\boldsymbol{B}_0(z) \tag{58}$$

where

$$\Delta_f(z) = \boldsymbol{Q}(z)\Big[m(z)(\boldsymbol{G}(z)\boldsymbol{A}_1(z)-\boldsymbol{F}(z)) + \boldsymbol{L}(z)\boldsymbol{N}_2(z)\boldsymbol{B}_1(z)\Big] \tag{59}$$

Since

$$\boldsymbol{B}_1(z) = \boldsymbol{N}(z)\boldsymbol{B}(z) = \begin{bmatrix}\boldsymbol{N}(z) & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{B} \\ \bar{\boldsymbol{B}}\end{bmatrix} \tag{60}$$

and

$$\boldsymbol{L}(z)\boldsymbol{N}_2(z) = \begin{bmatrix}\boldsymbol{L}(z) & \bar{\boldsymbol{L}}(z)\end{bmatrix}\begin{bmatrix}\boldsymbol{N}_2(z) \\ 0\end{bmatrix} \tag{61}$$

it follows that

$$\begin{aligned}\boldsymbol{L}(z)\boldsymbol{N}_2(z)\boldsymbol{B}_1(z) &= \boldsymbol{L}(z)\boldsymbol{N}_2(z)\boldsymbol{N}(z)\boldsymbol{B}(z) \\ &= [\boldsymbol{L}(z)\,\bar{\boldsymbol{L}}(z)]\begin{bmatrix}\boldsymbol{N}_2(z)\boldsymbol{N}(z) & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{B}(z) \\ \bar{\boldsymbol{B}}(z)\end{bmatrix}\end{aligned} \tag{62}$$

and, in consequence, it results from (58) that

$$\Delta_f(z) = \boldsymbol{Q}(z)\begin{bmatrix}\boldsymbol{L}(z) & \bar{\boldsymbol{L}}(z)\end{bmatrix}\begin{bmatrix}\Delta(z) & 0 \\ 0 & \boldsymbol{I}_{m-l}\end{bmatrix}\begin{bmatrix}\boldsymbol{B}(z) \\ \bar{\boldsymbol{B}}(z)\end{bmatrix} \tag{63}$$

Thus all the "internal" poles of the control system defined by the zeros of the determinant

$$|\Delta_1(z)| = |\boldsymbol{Q}_1(z)| \times |\boldsymbol{Q}(z)| \times |[\boldsymbol{L}(z)\ \bar{\boldsymbol{L}}(z)]| \times |\Delta(z)| \times \left| \begin{bmatrix} \boldsymbol{B}(z) \\ \bar{\boldsymbol{B}}(z) \end{bmatrix} \right| \tag{64}$$

are stable and freely located inside the unit disc. They are uncontrollable poles except for the poles defined by $|\Delta(z)|$, which are controllable and observable, and the poles defined by $|[\boldsymbol{B}^T\ \bar{\boldsymbol{B}}^T]|^T$, which are unobservable (if the latter exist). Also, since all of the obtained transfer matrices are proper (or strictly proper) the control system considered is internally proper. This implies that the design goal (ii) is satisfied.

To show that condition (i) is fulfilled let us consider the transfer matrices $\boldsymbol{T}_{yy_0}(z)$ and $\boldsymbol{T}_{\hat{y}y_0}(z)$ defined by relationships (53), (54) and (58). Substituting eqn. (63) into eqn. (58) by using formulae (60) and (61) we have obtained

$$\begin{aligned} \boldsymbol{T}_{yy_0}(z) &= \boldsymbol{T}_{\hat{y}y_0}(z) = \boldsymbol{B}_1(z)\Delta_f^{-1}(z)\boldsymbol{Q}(z)\boldsymbol{L}(z)\boldsymbol{N}_2(z) \\ &= [\boldsymbol{N}(z)\ \ 0] \begin{bmatrix} \Delta^{-1}(z) & 0 \\ 0 & \boldsymbol{I}_{m-l} \end{bmatrix} \begin{bmatrix} \boldsymbol{N}_2(z) \\ 0 \end{bmatrix} = \boldsymbol{N}(z)\Delta^{-1}(z)\boldsymbol{N}_2(z) \end{aligned} \tag{65}$$

as a diagonal and stable transfer matrix with the poles given by the zeros of $|\Delta(z)|$ assumed in Step 6 of the Algorithm. This implies a complete dynamic decoupling of the considered control system.

The zeros of the closed-loop control systems are determined by the zeros of $|\boldsymbol{N}(z)|$ which consists of the plant's transmission zeros. So if the plant is non-minimum-phase, the closed-loop system remains non-minimum-phase, too. Additional system's zeros are generated by the zeros of $|\boldsymbol{N}_2(z)|$, which are zeros of the controller. The latter can cause a non-minimum-phase control system obtained even when the plant is minimum-phase.

To prove that condition (iii) is satisfied we now consider the transfer matrices (49)–(52). Noting that

$$\begin{aligned} \boldsymbol{I}_l - \boldsymbol{B}_1\Delta_f^{-1}\boldsymbol{Q}\boldsymbol{L}\boldsymbol{N}_2 &= \boldsymbol{I}_l - \boldsymbol{N}\Delta^{-1}\boldsymbol{N}_2 = \boldsymbol{N}\Delta^{-1}[\Delta - \boldsymbol{N}_2\boldsymbol{N}]\boldsymbol{N}^{-1} \\ &= \boldsymbol{N}\Delta^{-1}\boldsymbol{M}_2\boldsymbol{D}\boldsymbol{N}^{-1} = \Delta^{-1}\boldsymbol{D}m(z) \end{aligned} \tag{66}$$

and

$$\begin{aligned} \boldsymbol{B}_1\Delta_f^{-1}\boldsymbol{H}\boldsymbol{M}_2 &= [\boldsymbol{N}\ \ 0] \begin{bmatrix} \boldsymbol{B} \\ \bar{\boldsymbol{B}} \end{bmatrix} \begin{bmatrix} \boldsymbol{B} \\ \bar{\boldsymbol{B}} \end{bmatrix}^{-1} \begin{bmatrix} \Delta^{-1} & 0 \\ 0 & \boldsymbol{I}_{m-l} \end{bmatrix} [\boldsymbol{L}\ \bar{\boldsymbol{L}}]\boldsymbol{Q}^{-1}\boldsymbol{H}\boldsymbol{M}_2 \\ &= [\boldsymbol{N}\ \ 0] \begin{bmatrix} \Delta^{-1} & 0 \\ 0 & \boldsymbol{I}_{m-l} \end{bmatrix} [\boldsymbol{L}\ \bar{\boldsymbol{L}}]^{-1}\boldsymbol{Q}^{-1}\boldsymbol{H}m(z) \end{aligned} \tag{67}$$

and substituting the right-hand side of eqn. (41) into eqns. (49)–(52) after manipulations we finally obtain

$$T_{e\bar{y}}(z) = \Delta^{-1}(z)D(z)m(z)A_0^{-1}(z)B_0(z) \tag{68}$$

$$\begin{aligned} T_{e\bar{r}}(z) &= -\Big[\Delta^{-1}(z)D(z)Q_1^{-1}(z)H_1(z) \\ &+ \big[N(z)\ 0\big]\big[L(z)\Delta(z)\ \bar{L}(z)\big]^{-1}Q^{-1}(z)H(z)\Big]m(z)A_3^{-1}(z)B_3(z) \end{aligned} \tag{69}$$

$$\begin{aligned} T_{ew}(z) &= -\Big[\Delta^{-1}(z)D(z)Q_1^{-1}(z)H_1(z) \\ &+ \big[N(z)\ 0\big]\big[L(z)\Delta(z)\ \bar{L}(z)\big]^{-1}Q^{-1}(z)H(z)\Big]m(z)A_2^{-1}(z)B_2(z) \end{aligned} \tag{70}$$

$$\begin{aligned} T_{ev}(z) &= -\Big[\Delta^{-1}(z)D(z)Q_1^{-1}(z)H_1(z) \\ &+ \big[N(z)\ 0\big]\big[L(z)\Delta(z)\ \bar{L}(z)\big]^{-1}Q^{-1}(z)H(z)\Big]m(z) \end{aligned} \tag{71}$$

Since all the above-presented transfer matrices are strictly proper and all the unstable modes of "external" models defined by eqns. (5), (6) and (7) are cancelled with the zeros of $m(z)$, we have

$$E\left\{\lim_{k\to\infty} e(k)\right\} = 0$$

Therefore all the design goals are achieved.

## 5. Conclusions and Final Remarks

In the paper, we have presented an algorithm for the synthesis of multipurpose, discrete-time, control systems for a general class of linear plants with the number of inputs greater than the number of their outputs. Moreover, unlike the earlier algorithms given for square (invertible) plants, the proposed algorithm ensures internal stability and an internal property for non-minimum-phase plants.

On the whole, the discrete-time systems have similar properties as continuous systems. However, for discrete-time systems the *deadbeat* properties can also be achieved if poles of the closed-loop system (and/or poles of the *deadbeat* observers) are located exactly in the centre of the unit disc. Unfortunately, the multipurpose *deadbeat* control systems usually do not work in practice because of big amplitudes of signals encountered. In general, the proposed design procedure is more difficult to apply since the relationships between the dynamical system behaviour and the displacement of the system poles (and their zeros) are "richer" and poorly known.

Although the proposed design procedure is considerably more complex (following the new conditions and a general class of the plants under consideration) than the algorithms which have been proposed so far, the realization of the obtained control systems in time domain is still straightforward since all the transfer matrices are proper or strictly proper. Unfortunately, many operations used in the algorithm can cause its numerical instability. Hence, in practice, additional attempts should be made to develop useful software packages.

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