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VOLTERRA SERIES EXPANSION FOR STATE QUADRATIC SYSTEMS

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A review of two methods to compute the kernels of a Volterra series expansion via flow and by the Carleman bilinearization for non-linear analytic systems with linear control is presented. We give explicit formulae especially for state quadratic systems. Furthermore, illustrative examples are given, and the two methods are briefly compared.

1. Introduction

A large number of plants of interest in many engineering applications have non-linear dynamics and may be modelled by ordinary differential equations of the form (SISO-case):

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{a}\Big(\boldsymbol{x}(t)\Big) + \boldsymbol{b}\Big(\boldsymbol{x}(t)\Big)u(t) \tag{1}$$

$$y(t) = \boldsymbol{c}^T\boldsymbol{x}(t), \qquad \boldsymbol{x}_0 = \boldsymbol{x}(t_0 = 0) \tag{2}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ denotes the state vector, $u(t)$ the control, and $y(t)$ the output of the system. The quantities $\boldsymbol{a}$ and $\boldsymbol{b}$ are analytic vector fields of $\boldsymbol{x}$. The theory of such systems has been well established in recent years. Many important properties for the analysis and design of control systems like observability, controllability, realization, as well as the observer and controller design were also examined and solved in a very general form (Isidori, 1989; Nijmeijer and van der Schaft, 1990; Schwarz, 1991). In practice however, it is convenient and useful to consider more simple subclasses of the general non-linear systems (1)–(2), e.g. linear, bilinear and quadratic systems.

The contribution of this paper is to show how the Volterra kernels can be computed explicitly. To this aim, two effective methods are presented. The Volterra series expansion represents the input-output behaviour of a non-linear system. It has found applications in optimal control, stability theory, stochastic control, identification, etc. The Volterra series are well studied in the literature. However, most studies give representations only in a very general form and ignore the structural features of the systems found in practice. In this paper, state quadratic systems as a specific class of non-linear systems are considered.

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The motivation to consider quadratic systems is that they have a traditional place in the mathematical literature and are of increasing importance in the field of systems theory. Many natural processes, e.g. rigid bodies, adaptive control systems and prey-predator phenomena (Frayman, 1974; Kang and Krener, 1991) can be described by quadratic models. We propose three methods by which non-linear dynamic technical systems may be approximated by state quadratic models, see (Jelali and Schwarz, 1995). Furthermore, the application of linear state feedback to a bilinear system results in a quadratic system. Thus, it is necessary to study quadratic systems in order to properly understand the feedback theory of bilinear ones.

2. Non-Linear State-Space Description and Volterra Series

The Taylor series expansion of the vector fields $\boldsymbol{a}(\boldsymbol{x})$ and $\boldsymbol{b}(\boldsymbol{x})$ in (1) may be represented using multiple Kronecker products

$$\boldsymbol{x}^{(k)}(t) = \underbrace{\boldsymbol{x}(t) \otimes \boldsymbol{x}(t) \otimes \ldots \otimes \boldsymbol{x}(t)}_{k \text{ times}} \tag{3}$$

in the form

$$\boldsymbol{a}(\boldsymbol{x}) = \boldsymbol{A}_1\boldsymbol{x}(t) + \boldsymbol{A}_2\boldsymbol{x}^{(2)} + \ldots + \boldsymbol{A}_N\boldsymbol{x}^{(N)} + \ldots \tag{4}$$

$$\boldsymbol{b}(\boldsymbol{x}) = \boldsymbol{B}_0 + \boldsymbol{B}_1\boldsymbol{x}(t) + \boldsymbol{B}_2\boldsymbol{x}^{(2)} + \ldots + \boldsymbol{B}_{N-1}\boldsymbol{x}^{(N-1)} + \ldots \tag{5}$$

Retaining the first r terms of $\boldsymbol{a}(\boldsymbol{x})$ and s of $\boldsymbol{b}(\boldsymbol{x})$ yields the polynomial systems with linear control

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r} \boldsymbol{A}_i\boldsymbol{x}^{(i)}(t) + \sum_{j=0}^{s} \boldsymbol{B}_j\boldsymbol{x}^{(j)}(t)u(t) \tag{6}$$

$$y(t) = \boldsymbol{c}^T\boldsymbol{x}(t), \qquad \boldsymbol{x}_0 = \boldsymbol{x}(0) \tag{7}$$

For $r = s = 2$ the state quadratic systems

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}_1\boldsymbol{x}(t) + \boldsymbol{A}_2\boldsymbol{x}(t) \otimes \boldsymbol{x}(t) \\ &\quad + \Big[\boldsymbol{b}_0 + \boldsymbol{B}_1\boldsymbol{x}(t) + \boldsymbol{B}_2\boldsymbol{x}(t) \otimes \boldsymbol{x}(t)\Big]u(t)\end{aligned} \tag{8}$$

$$y(t) = \boldsymbol{c}^T\boldsymbol{x}(t), \qquad \boldsymbol{x}_0 = \boldsymbol{x}(0) \tag{9}$$

are obtained. Those systems can also be described using conventional matrix notation as follows:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}_1\boldsymbol{x}(t) + \sum_{i=1}^{n} \boldsymbol{e}_i\boldsymbol{x}^T(t)\boldsymbol{Q}_i\boldsymbol{x}(t) \\ &\quad + \Big[\boldsymbol{b}_0 + \boldsymbol{B}_1\boldsymbol{x}(t) + \sum_{i=1}^{n} \boldsymbol{e}_i\boldsymbol{x}^T(t)\boldsymbol{R}_i\boldsymbol{x}(t)\Big]u(t)\end{aligned} \tag{10}$$

$$y(t) = \boldsymbol{c}^T \boldsymbol{x}(t), \qquad \boldsymbol{x}_0 = \boldsymbol{x}(0) \tag{11}$$

where $\boldsymbol{Q}_i$, $\boldsymbol{R}_i$ are $n \times n$-matrices and $\boldsymbol{e}_i$ is the standard basis vector in the i-th direction. For $\boldsymbol{A}_2 = \boldsymbol{0}$, $\boldsymbol{B}_2 = \boldsymbol{0}$ (8)–(9) is reduced to a bilinear system.

The input-output behaviour of (1) can be represented by a Volterra series expansion

$$\begin{aligned} y(t) &= w_0(t) + \sum_{\nu=1}^{\infty} \int_0^t \int_0^{\tau_1} \dots \int_0^{\tau_{\nu-1}} w_\nu(t, \tau_1, \dots, \tau_\nu) \\ &\quad \times u(\tau_1) \dots u(\tau_\nu) \, \mathrm{d}\tau_\nu \dots \mathrm{d}\tau_1 \end{aligned} \tag{12}$$

The kernels w_ν in (12) may be regarded as a generalization of weighting functions.

3. Computation of Kernels via Flow

The kernels are given, using Lie derivatives, by the following expressions (Isidori, 1989):

$$\begin{aligned} w_0(t) &= \boldsymbol{c}^T \boldsymbol{\Phi}_t^{\boldsymbol{a}}(\boldsymbol{x})\big|_{\boldsymbol{x}_0} \\ w_1(t, \tau_1) &= \mathcal{L}_{\boldsymbol{P}_{\tau_1}} \boldsymbol{c}^T \boldsymbol{\Phi}_t^{\boldsymbol{a}}(\boldsymbol{x})\big|_{\boldsymbol{x}_0} \\ w_2(t, \tau_1, \tau_2) &= \mathcal{L}_{\boldsymbol{P}_{\tau_2}} \mathcal{L}_{\boldsymbol{P}_{\tau_1}} \boldsymbol{c}^T \boldsymbol{\Phi}_t^{\boldsymbol{a}}(\boldsymbol{x})\big|_{\boldsymbol{x}_0} \\ &\vdots \\ w_i(t, \tau_1, \dots, \tau_i) &= \mathcal{L}_{\boldsymbol{P}_{\tau_i}} \dots \mathcal{L}_{\boldsymbol{P}_{\tau_1}} \boldsymbol{c}^T \boldsymbol{\Phi}_t^{\boldsymbol{a}}(\boldsymbol{x})\big|_{\boldsymbol{x}_0} \end{aligned} \tag{13}$$

where

$$\boldsymbol{P}_t(\boldsymbol{x}) = \frac{\partial \boldsymbol{\Phi}_{-t}^{\boldsymbol{a}}(\boldsymbol{x})}{\boldsymbol{x}} \boldsymbol{b} \circ \boldsymbol{\Phi}_t^{\boldsymbol{a}}(\boldsymbol{x}) \tag{14}$$

$$\mathcal{L}_{\boldsymbol{P}_{\tau_i}}(\cdot) = \frac{\partial(\cdot)}{\partial \boldsymbol{x}} \boldsymbol{P}_{\tau_i}(\boldsymbol{x}) \tag{15}$$

'$\circ$' denotes composition w.r.t. the argument $\boldsymbol{x}$, and $\boldsymbol{\Phi}_t^{\boldsymbol{a}}(\boldsymbol{x})$ is the flow of the drift vector field $\boldsymbol{a}(\boldsymbol{x})$, i.e. the solution of

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{a}\big(\boldsymbol{x}(t)\big), \qquad \boldsymbol{x}_0 = \boldsymbol{x}(0) \tag{16}$$

Of course, not every system (1)–(2) admits a convergent Volterra representation. For a detailed background regarding the problem of existence and convergence of Volterra series, the reader should consult (Czarniak, 1984; Isidori, 1989; Sandberg, 1985).

Accordingly, the computation of the Volterra series expansion requires the following steps.

1. Find the general solution $x(t) = f(t, C_i)$ of the differential equation $\dot{x}(t) = a(x(t))$ as a function of the time t and the integration constants C_i.
2. Eliminate the C_i by considering the initial condition $x(0) = x_0$. Then one has the solution $x(t) = f(t, x_0)$.
3. Set the flow as $\Phi_t^a(x_0) = x(t)$ or $\Phi_t^a(x) = f(t, x)$.
4. Compute the kernels from (13).
5. Compute the Volterra series expansion from (12).

In general, it is very difficult to compute the flow $\Phi_t^a(x)$. For quadratic systems the general solution of the quadratic vector-differential equation

$$\begin{aligned} \dot{x}(t) &= A_1 x(t) + A_2 x(t) \otimes x(t) \\ &= A_1 x(t) + \sum_{i=1}^{n} e_i x^T(t) Q_i x(t), \qquad x_0 = x(0) \end{aligned} \tag{17}$$

has to be found. In the literature, only special classes of linear and bilinear systems are considered. For these systems the flow is given by

$$\Phi_t^a(x) = e^{A_1 t} x, \qquad P_t(x) = e^{-A_1 t} b_0 \tag{18}$$

However, we have presented the following analytic solutions for special structures of (17) in (Jelali, 1994; Schwarz and Jelali, 1994).

Structure 1. For $A_1 = 0$ in (17) the quadratic differential equation

$$\begin{aligned} \dot{x}(t) &= A_2 x(t) \otimes x(t) \\ &= \left(\sum_{i=1}^{n} e_i x^T(t) Q_i \right) x(t), \qquad x_0 = x(0) \end{aligned} \tag{19}$$

has the general solution

$$\begin{aligned} x(t) &= \left[I_n - \left(\sum_{i=1}^{n} e_i x_0^T Q_i \right) t \right]^{-1} x_0 \\ &=: \left[I_n - S_Q t \right]^{-1} x_0 \end{aligned} \tag{20}$$

C1) for any x_0 for which S_Q is symmetric and commutes with each Q_i

or

C2) for all $\boldsymbol{x}_0$ iff the conditions

$$\boldsymbol{A}_2(\boldsymbol{A}_2\boldsymbol{x} \otimes \boldsymbol{x}) \otimes \boldsymbol{x} = \boldsymbol{A}_2\boldsymbol{x} \otimes (\boldsymbol{A}_2\boldsymbol{x} \otimes \boldsymbol{x}) \tag{21}$$

and

$$\boldsymbol{A}_2\Big[\boldsymbol{A}_2(\boldsymbol{A}_2\boldsymbol{x} \otimes \boldsymbol{x}) \otimes \boldsymbol{x}\Big] \otimes \boldsymbol{x} = \boldsymbol{A}_2(\boldsymbol{A}_2\boldsymbol{x} \otimes \boldsymbol{x}) \otimes (\boldsymbol{A}_2\boldsymbol{x} \otimes \boldsymbol{x}) \tag{22}$$

are satisfied. Of course, the property that matrices are commutable is a strong condition. However, if this is not satisfied, one can try to check C2) which complements C1), see Example 1.

Example 1. For a state quadratic system of the form (19) with

$$\boldsymbol{A}_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \boldsymbol{Q}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; \boldsymbol{Q}_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{23}$$

one can write

$$\boldsymbol{S}_{\boldsymbol{Q}} = \sum_{i=1}^{2} \boldsymbol{e}_i \boldsymbol{x}_0^T \boldsymbol{Q}_i = \begin{bmatrix} x_{02} & x_{01} \\ -x_{01} & x_{02} \end{bmatrix} \tag{24}$$

and therefore

$$\boldsymbol{x}(t) = \left[\boldsymbol{I}_2 - \begin{bmatrix} x_{02}t & x_{01}t \\ -x_{01}t & x_{02}t \end{bmatrix}\right]^{-1} \boldsymbol{x}_0$$

$$= \frac{1}{(1 - x_{02}t)^2 + x_{01}^2 t^2} \begin{bmatrix} x_{01} \\ x_{02} - (x_{01}^2 + x_{02}^2)t \end{bmatrix} \tag{25}$$

Thus, $\boldsymbol{S}_{\boldsymbol{Q}}$ is symmetric and commutes with $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ for $x_{01} = 0$. However, it is easy to check that the conditions (21)–(22) are satisfied such that the solution (25) holds for all $\boldsymbol{x}_0 \in \mathbb{R}^2$.

Structure 2. If the following relationship

$$\boldsymbol{A}_2\boldsymbol{x}(t) \otimes \boldsymbol{x}(t) = l(\boldsymbol{x})\boldsymbol{A}_1\boldsymbol{x}(t), \qquad l(\boldsymbol{x}) = l_1x_1 + l_2x_2 + \ldots + l_nx_n \tag{26}$$

between the linear and quadratic term in (17) exists, the solution of

$$\dot{\boldsymbol{x}}(t) = \Big[1 + l(\boldsymbol{x})\Big]\boldsymbol{A}_1\boldsymbol{x}(t) \tag{27}$$

is

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}_1\gamma(t)}\boldsymbol{x}_0 \tag{28}$$

where the function $\gamma(t)$ satisfies

$$\dot{\gamma}(t) = 1 + l\left(e^{\boldsymbol{A}_1\gamma(t)}\boldsymbol{x}_0\right) \tag{29}$$

Structure 3. The differential equation (17) with

$$\boldsymbol{A}_2 = \begin{bmatrix} \boldsymbol{q}^T & \boldsymbol{0} & \boldsymbol{0} & \dots & \dots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{q}^T & \boldsymbol{0} & \dots & \dots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & & & \ddots & \ddots & & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{0} & & \boldsymbol{q}^T \end{bmatrix} \tag{30}$$

or

$$\boldsymbol{Q}_i = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{q}^T \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{bmatrix} \begin{matrix} \\ \leftarrow i\text{-th row} \\ \\ \\ \\ \end{matrix} , \qquad \boldsymbol{q} \in \mathbb{R}^n \tag{31}$$

has the general solution

$$\boldsymbol{x}(t) = \frac{e^{\boldsymbol{A}_1 t}\boldsymbol{x}_0}{1 - \boldsymbol{q}^T \int_0^t e^{\boldsymbol{A}_1\tau}\,\mathrm{d}\tau\,\boldsymbol{x}_0} \tag{32}$$

If $\boldsymbol{A}_1$ is non-singular, then (32) can be rewritten as

$$\boldsymbol{x}(t) = \frac{e^{\boldsymbol{A}_1 t}\boldsymbol{x}_0}{1 + \boldsymbol{q}^T\boldsymbol{A}_1^{-1}(\boldsymbol{I}_n - e^{\boldsymbol{A}_1 t})\boldsymbol{x}_0} \tag{33}$$

Example 2. Consider the quadratic system with the following state and output equations:

$$\dot{x}_1(t) = -2x_1(t) + x_1(t)x_2(t) + u(t)$$

$$\dot{x}_2(t) = -x_2(t) + x_2^2(t) + u(t)$$

$$y(t) = x_1(t), \qquad \boldsymbol{x}_0 = \boldsymbol{0}$$

and calculate the first two Volterra kernels using the foregoing method. Since this system has the structure of (30), a short calculation with (33) shows that

$$\boldsymbol{\Phi}_t^a(\boldsymbol{x}) = \frac{1}{1-(1-e^{-t})x_2}\begin{bmatrix} e^{-2t}x_1 \\ e^{-2t}x_2 \end{bmatrix}, \quad \boldsymbol{P}_t(\boldsymbol{x}) = \begin{bmatrix} \dfrac{e^{2t}}{1-(1-e^{-t})x_2} \\ \dfrac{e^{t}}{[1-(1-e^{-t})x_2]^2} \end{bmatrix}$$

Applying (13) leads to

$$w_1(t,\tau_1) = e^{-2(t-\tau_1)}, \qquad w_2(t,\tau_1,\tau_2) = (1-e^{-t})\left[e^{-2(t-\tau_1-\tau_2)} + e^{-(t-2\tau_1)}\right]$$

4. Computation of Kernels by the Carleman Bilinearization

Another method to construct the Volterra kernels consists in using the truncated approximation (6)–(7) and defining the new state vector

$$\boldsymbol{z}(t) := \begin{bmatrix} \boldsymbol{x}^{(1)} \\ \boldsymbol{x}^{(2)} \end{bmatrix} \tag{34}$$

With the truncated time derivative of $\boldsymbol{x}^{(2)}$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}^{(2)} &= (\boldsymbol{A}_1 \otimes \boldsymbol{I}_n + \boldsymbol{I}_n \otimes \boldsymbol{A}_1)\boldsymbol{x}^{(2)} + \Big[(\boldsymbol{b}_0 \otimes \boldsymbol{I}_n + \boldsymbol{I}_n \otimes \boldsymbol{b}_0) \\ &\quad + (\boldsymbol{B}_1 \otimes \boldsymbol{I}_n + \boldsymbol{I}_n \otimes \boldsymbol{B}_1)\boldsymbol{x}^{(2)}(t)\Big]u(t)\end{aligned} \tag{35}$$

the quadratic system (8)–(9) can be transformed into a bilinear one of the type

$$\dot{\boldsymbol{z}}(t) = \boldsymbol{A}_c\boldsymbol{z}(t) + [\boldsymbol{N}_c\boldsymbol{z}(t) + \boldsymbol{b}_c]u(t) \tag{36}$$

$$y(t) = \boldsymbol{c}_c^T\boldsymbol{z}(t), \qquad \boldsymbol{z}_0 = \boldsymbol{z}(0) \tag{37}$$

where

$$\boldsymbol{A}_c = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 \\ \boldsymbol{0}_{n^2\times n} & \boldsymbol{A}_{2,2} \end{bmatrix}, \qquad \boldsymbol{N}_c = \begin{bmatrix} \boldsymbol{B}_1 & \boldsymbol{B}_2 \\ \boldsymbol{B}_{2,1} & \boldsymbol{B}_{2,2} \end{bmatrix} \tag{38}$$

$$\boldsymbol{b}_c = \begin{bmatrix} \boldsymbol{b}_0, & \boldsymbol{0}_{n^2} \end{bmatrix}, \qquad \boldsymbol{c}_c^T = \begin{bmatrix} \boldsymbol{c}^T, & \boldsymbol{0}_{n^2}^T \end{bmatrix} \tag{39}$$

The matrices $\boldsymbol{A}_{2,2}$, $\boldsymbol{B}_{2,1}$ and $\boldsymbol{B}_{2,2}$ are respectively given by

$$\boldsymbol{A}_{2,2} = \boldsymbol{A}_1 \otimes \boldsymbol{I}_n + \boldsymbol{I}_n \otimes \boldsymbol{A}_1 \tag{40}$$

$$\boldsymbol{B}_{2,1} = \boldsymbol{b}_0 \otimes \boldsymbol{I}_n + \boldsymbol{I}_n \otimes \boldsymbol{b}_0 \tag{41}$$

$$\boldsymbol{B}_{2,2} = \boldsymbol{B}_1 \otimes \boldsymbol{I}_n + \boldsymbol{I}_n \otimes \boldsymbol{B}_1 \tag{42}$$

where $\boldsymbol{I}_n$ denotes the identity matrix of order n. The type of approximation illustrated here is called the Carleman bilinearization and can also be applied to other classes of analytic systems with linear control (1). For more information see e.g. (Brockett, 1976; Rugh, 1981).

Assuming that the Volterra series for (8)–(9) exists, the first two terms and the truncated Carleman bilinearization must agree (Brockett, 1976). Thus, the Volterra kernels ($\boldsymbol{x}_0 = \boldsymbol{0}$) can be computed by

$$w_i(t, \tau_1, \ldots, \tau_i) = \boldsymbol{c}_c^T e^{\boldsymbol{A}_c(t-\tau_1)} \left(\prod_{k=1}^{i-1} \boldsymbol{N}_c e^{\boldsymbol{A}_c(\tau_k - \tau_{k+1})} \right) \boldsymbol{b}_c \qquad \forall t \geq \tau_1 \geq \ldots \geq \tau_i \geq 0 \tag{43}$$

One can make the change of variables $\sigma_i = t - \tau_i$ in order to yield the triangular kernels

$$w_{i,\mathrm{tri}}(\sigma_1, \ldots, \sigma_i) = \boldsymbol{c}_c^T e^{\boldsymbol{A}_c \sigma_1} \left(\prod_{k=2}^{i} \boldsymbol{N}_c e^{\boldsymbol{A}_c(\sigma_k - \sigma_{k-1})} \right) \boldsymbol{b}_c \qquad \forall \sigma_i \geq \sigma_{i-1} \geq \ldots \geq \sigma_1 \geq 0 \tag{44}$$

One possible difficulty is the computation of the matrix exponential which, as is well-known, is denoted by $e^{\boldsymbol{A}_c t} = \sum_{k=0}^{\infty} (k!)^{-1} (\boldsymbol{A}_c t)^k$. The theoretical and computational properties of the matrix exponential function have been treated extensively in (Moler and van Loan, 1978).

Example 3. Calculate the first two Volterra kernels for the quadratic system

$$\dot{x}_1(t) = -x_1(t) + u(t)$$

$$\dot{x}_2(t) = -x_2(t) + x_1(t)u(t) + x_2^2(t)u(t)$$

$$y(t) = x_2(t), \qquad \boldsymbol{x}_0 = \boldsymbol{0}$$

The bilinear approximation is of the type (36)–(37) with

$$A_c = -\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad N_c = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$b_c = [1,\ 1,\ 0,\ 0,\ 0,\ 0]^T, \qquad c_c^T = [1,\ 0,\ 0,\ 0,\ 0,\ 0]$$

Thus, some calculations lead to the Volterra kernels

$$w_1(t, \tau_1) = c_c^T e^{A_c(t-\tau_1)} b_c = 0$$

$$w_2(t, \tau_1, \tau_2) = c_c^T e^{A_c(t-\tau_1)} N_c e^{A_c(\tau_1-\tau_2)} b_c = e^{-(t-\tau_2)}$$

5. Conclusion

The computation of kernels via flow requires the solution of autonomous quadratic vector-differential equations which cannot be found for all quadratic systems. But if this solution exists, then the Volterra kernels can be quite easily computed. The computation could be done effectively using symbolic software packages like Maple, Macsyma, Mathematica, etc.

The computation of kernels by the Carleman bilinearization works well in most cases. It takes advantage of the fact that bilinear systems form a class of systems for which the Volterra kernels are relatively easy to compute. However, the disadvantage of this method is that the dimension of the obtained bilinear system increases more than proportionally to the dimension of the examined system such that the bilinear system, in general, does not represent a minimal realization. In order to reduce the dimension of the Carleman bilinearization and the complexity of the terms of the Volterra kernels, it is very efficient to use the reduced Kronecker product since it contains no redundant information.

The results presented here may be useful in a number of areas of systems theory such as correlation analysis and identification. The results presented for the SISO-case may be extended to the MIMO-case in future research. In addition to that, computation time and computation accuracy of both methods will be discussed.

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