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ON THE APPROXIMATE SOLVING OF CONTROL PROBLEMS FOR SYSTEMS DESCRIBED BY VARIATIONAL INEQUALITIES WITH NONLINEAR OPERATORS

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In this paper an approximate solution of some extremal problem is constructed. The system is described by a variational inequality (VI), where the operator is of uniformly lower semilimited variation. For this purpose, we carry out the change of the initial problem for the family of auxiliary problems: the inequality is converted into an equality by adding a penalty term to the utility function $L(u, y)$, and the control space is extended. At each step we find the optimal control, which is an approximate solution of the initial problem. If there are some assumptions on the differentiability of the system operator and the utility function, then we can deduce the optimality conditions for the approximate problem. As examples, we consider some free-boundary problems on $W_p^1(\Omega)$, $p \geq 2$.

1. Introduction

Recently the interest in the variational inequalities with partial derivatives has been increasing. This is due to internal problems as well as to application requirements. The theory of variational inequalities has emerged as a powerful and effective technique to study a wide class of free-boundary problems, equlibrium, nonlinear optimization problems, etc. The variational inequalities have a wide spectrum in various branches of both pure and applied sciences, and in particular, in economics and engineering (Dafermos, 1992; Duvaut and Lions, 1972).

There are many works where the system is described by variational inequalities. In many of them the authors consider variational inequalities a with strong monotone operator which have the unique solution. Nevertheless, many mathematical models use operators with weaker conditions which assume non-uniqueness. Moreover, many algorithms use auxiliary conditions regarding the operators, functions and sets, e.g. the system quasilinearity (Tseng, 1992), the system differentiability (Tseng, 1991), the set polyhedrality (Pang, 1990) or compactness (Taji, 1993). Therefore, future studies of this problem are still needed. In this work, we study the variational inequalities where the operator is of uniformly lower semilimited variation. The results are obtained under the assumptions utilized for the proof of solvability (Ivanenko and

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Melnik, 1988), and the system differentiability is used to find the optimality conditions only.

To construct an approximate solution, we change the initial extremal problem for an auxiliary problem, where the control space is extended. This auxiliary problem consists of the operator equation and of a new utility function with a penalty term. Analogous methods were used for solving linear variational inequalities (Glowinski *el al.*, 1976), where a quadratic penalty function was constructed and for solving an extremal problem in (Barbu, 1993), where the separation of the penalty made it possible to construct an extremal problem for a linear operator equation. We consider the problem in which the operator is essentially nonlinear, i.e. we cannot construct a quadratic penalty function or extract a linear operator equation, etc. But by some properties of the new utility function, however, at least one solution of each approximate problem exists. The sufficient properties of this function are proved on the compactly embedded space $V_2^* \subset V_1^*$. Such solutions are approximate for the initial problem.

Futhermore, the aim of Section 5 is to show that under additional conditions on the differentiability of the operator and the utility function (analogous conditions are used to obtain optimality conditions for known functions) we can obtain the optimality conditions for the auxiliary extremal problem, i.e. the new penalty function is not weaker than the one analysed earlier.

The theory is suitable, in particular, for free-boundary problems, as will be shown in Section 6.

2. Problem Formulation

We consider the optimal control problem for a system described by the following variational inequality:

$$\Big\langle A(u,y), \xi - y \Big\rangle \geq \Big\langle f, \xi - y \Big\rangle \quad \forall \xi \in K \tag{1}$$

$$L(u,y) \to \inf_{u \in U} \tag{2}$$

Let V_1 be a reflexive Banach space and V_0 be a Hilbert space ('state spaces') such that $V_1 \subset V_0 \subset V_1^*$, where V_1^* is the dual of V_1, V_0^* is the dual of V_0. We identify V_0^* with V_0. Let us assume that these injections are dense and continuous. The duality pairing on $V_1 \times V_1^*$ will be denoted by $\langle \cdot, \cdot \rangle$. We identify it with the inner product on V_0. Let $\mathbb{U}$ denote a space adjoint to some Banach space or to a separable and normed space. Let K be a weakly closed convex subset in V_1, U be a weakly closed limited subset in $\mathbb{U}$ and $f \in V_1^*$. Moreover, let $L : U \times V_1 \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be a weakly lower semicontinuous function, i.e. if $U \ni u_n \to u$ weakly-$(*)$ in $\mathbb{U}$, $V_1 \ni y_n \to y$ weakly in V_1, then $\underline{\lim}_{n\to\infty} L(u_n, y_n) \geq L(u,y)$. Here and subsequently, we use the following terminology:

Definition 1. An operator $A : V_1 \to V_1^*$ is called

a) *radially continuous* if $\forall y \in D(A)$ and $\forall \xi, h \in V_1$ there is an $\varepsilon > 0$ such that the function $[0,\varepsilon] \ni t \mapsto \langle A(y + t\xi), \xi \rangle$ is continuous;

b) *demicontinuous* if for every convergent sequence $\{y_n\}$ the sequence $\{A(y_n)\}$ converges weakly;

c) *strongly monotone* if $\forall y_i \in V_1 (\|y_i\|_{V_1} \leq R,\ i = 1, 2)$ there is a $d > 0$ such that

$$\Big\langle A(y_1) - A(y_2), y_1 - y_2 \Big\rangle \geq d\|y_1 - y_2\|^2_{V_1}$$

Definition 2. An operator $A : U \times V_1 \to V_1^*$ is called

a) *coercive* if there exists $y_0 \in K$ such that

$$\Big\langle A(u, y), y - y_0 \Big\rangle \geq \gamma\Big(\|y\|_{V_1}\Big)\|y\|_{V_1} \tag{3}$$

where $\gamma(s) \to \infty$ as $s \to \infty$;

b) the operator of *uniformly lower semilimited variation* if $\forall y_i \in V_1 (\|y_i\|_{V_1} \leq R,\ i = 1, 2)$ the inequality

$$\Big\langle A(u, y_1) - A(u, y_2), y_1 - y_2 \Big\rangle \geq \inf_{v \in U} C_v\Big(R, \|y_1 - y_2\|'_{V_1}\Big) \quad \forall u \in U \tag{4}$$

holds, where $\|\cdot\|'$ is a compact seminorm with respect to $\|\cdot\|$, $C_v : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ are continuous, and $\forall h, R > 0$ we have $\frac{1}{\tau}C(R, \tau h) \to 0$ as $\tau \to 0$;

c) *uniformly locally limited* on X if $\forall y \in D(A)$ and $\forall u \in U$ (such that $\|u\|_{\mathbb{U}} \leq l < \infty$) there are $k_1 > 0$ and $k_2 > 0$, such that $\|A(u, \xi)\|_X \leq k_2$ as $\|y - \xi\| \leq k_1$.

Let us orient $A : U \times V_1 \to V_1^*$ by the following requirements:

i) A is coercive;

ii) $\forall u \in U$ the operator $A(u, \cdot) : V_1 \to V_1^*$ is radially continuous, and $\forall y \in V_1$ the operator $A(\cdot, y) : U \to V_1^*$ is continuous; moreover, if $U \ni u_n \to u$ weakly-($*$) in $\mathbb{U}$ and $D(A) \ni y_n \to y$ weakly in V_1, then $\langle A(u_n, y), y_n - y\rangle \to 0$;

iii) A is of uniformly lower semilimited variation.

Consequently, the existence of solutions to (1)–(2) is proved and A has the property (M): if $U \ni u_n \to u$ weakly-($*$) in U, $D(A) \ni y_n \to y$ weakly in V_1, $A(u_n, y_n) \to d$ weakly-($*$) in V_1^* and $\underline{\lim}_{n\to\infty}\langle A(u_n, y_n), y_n\rangle \leq \langle d, y\rangle$, then $d = A(u, y)$ (see Ivanenko and Melnik, 1988).

We consider the space V_2 such that the injection $V_1 \subset V_2$ is dense and compact. The injection $V_2^* \subset V_1^*$ is then dense and continuous. Moreover, we set $\|\cdot\|'_{V_1} = \|\cdot\|_{V_2}$.

3. Preliminaries

We shall use the following results:

Proposition 1. *Let X be a reflexive Banach space and $A : U \times (D(A) \subset X) \to X^*$ such that $A(u,\cdot) : D(A) \to X^*$ is a radially continuous operator $\forall u \in U$, $A(\cdot, y) : U \to X^*$ is continuous and limited, and $\forall y_i \in D(A)$, $(i = 1, 2, \|y_i\|_X \leq R)$ the following estimate holds:*

$$\Big\langle A(u,y_1) - A(u,y_2), y_1 - y_2 \Big\rangle_X \geq - \inf_{v \in U} C_v\Big(R, \|y_1 - y_2\|_X\Big) \quad \forall u \in U \qquad (5)$$

Then A is uniformly locally limited on X^* and, if there exist $K \subset X$ and $M \subset U$ such that $\|y\|_X \leq k_1$, $\|u\|_{\mathbb{U}} \leq l$ and $\langle A(u,y), y\rangle_X \leq k_2 \forall (u,y) \in K \times M$, then $\exists C > 0$ such that $\forall (u,y) \in K \times M$ the estimate $\|A(u,y)\|_{X^*} \leq C$ holds.

Proof. Suppose, contrary to our claim, that for some $y \in D(A)$ there is a sequence y_n such that $y_n \to y$ strongly in X and

$$\sup_{\|u\|_{\mathbb{U}} \leq l} \|A(u, y_n)\|_{X^*} \to \infty$$

It follows from (5) that for every $\omega \in X$ such that $y + \omega \in D(A)$, and for some $R > 0$, we have

$$\Big\langle A(u,y_n) - A(u, y+\omega), y_n - y - \omega \Big\rangle_X \geq - \inf_{v \in U} C_v\Big(R, \|y_n - y - \omega\|_X\Big)$$

$$\Big\langle A(u,y_n), y_n - y \Big\rangle_X - \Big\langle A(u,y_n), \omega \Big\rangle_X - \Big\langle A(u, y+\omega), y_n - y - \omega \Big\rangle_X$$

$$\geq - \inf_{v \in U} C_v\Big(R, \|y_n - y - \omega\|_X\Big)$$

Thus

$$\Big\langle A(u,y_n), \omega \Big\rangle_X \leq \Big\langle A(u,y_n), y_n - y \Big\rangle_X$$

$$+ \Big\langle A(u, y+\omega), y_n - y - \omega \Big\rangle_X + \inf_{v \in U} C_v\Big(R, \|y_n - y - \omega\|_X\Big)$$

Let us suppose that $\alpha_n = 1 + \sup_{\|u\|_{\mathbb{U}} \leq l} \|A(u,y_n)\|_{X^*} \|y_n - y - \omega\|_X$, $\alpha_n \geq 1$. By multiplying the previous inequality by α_n^{-1}, we obtain

$$\frac{1}{\alpha_n}\Big\langle A(u,y_n), \omega \Big\rangle_X \leq \frac{1}{\alpha_n}\Big\{\Big\langle A(u,y_n), y_n - y \Big\rangle_X$$

$$+ \Big\langle A(u, y+\omega), y_n - y - \omega \Big\rangle_X + \inf_{v \in U} C_v\Big(R, \|y_n - y - \omega\|_X\Big)\Big\}$$

Since $y_n \to y$ strongly in X, we have $\|y_n - y - \omega\|_X \to \|\omega\|_X$. The sequence α_n^{-1}

is limited. Then the last inequality is followed by

$$\frac{1}{\alpha_n}\Big\langle A(u,y_n),\omega\Big\rangle_X \le \frac{1}{\alpha_n}\Big\{\inf_{v\in U} C_v\Big(R, \|y_n - y - \omega\|_X\Big)$$

$$+ \|A(u, y+\omega)\|_{X^*}\|y+\omega-y_n\|_X + 1\Big\} \le N_1$$

where N is independent of u. Hence

$$\frac{1}{\alpha_n}\Big\langle A(u,y_n),\omega\Big\rangle_X \le N_1$$

Similarly, for $(-\omega)$ we have

$$\frac{1}{\alpha_n}\Big\langle A(u,y_n),-\omega\Big\rangle_X \le N_2$$

Since $\omega \in X$ is an arbitrary element, we obtain

$$\varlimsup_{n\to\infty}\Big|\frac{1}{\alpha_n}\Big\langle A(u,y_n),\omega\Big\rangle_X\Big| < \infty \quad \forall\omega\in X$$

Then from the resonance theorem it may be concluded that

$$\|A(u,y_n)\|_{X^*} \le N\alpha_n$$

where h is independent of both $u\in U$ and n, i.e.

$$\|A(u,y_n)\|_{X^*} \le N\Big(1+\sup_{\|u\|_{\mathbb{U}}\le l}\|A(u,y_n)\|_{X^*}\Big)$$

Let n_0 be such that $N\|y_n - y\|_X \le \frac{1}{2}$ for $n > n_0$. We thus get

$$\|A(u,y_n)\|_{X^*} \le N + \frac{1}{2}\sup_{\|u\|_{\mathbb{U}}\le l}\|A(u,y_n)\|_{X^*}$$

$$\|A(u,y_n)\|_{X^*} \le 2N < \infty$$

This contradicts the fact that $\sup_{\|u\|_{\mathbb{U}}\le l}\|A(u,y_n)\|_{X^*} \to \infty$. Since A is locally limited in X, it follows that $\exists\varepsilon > 0$ and $\exists M_\varepsilon > 0$ such that $\|A(u,\xi)\|_{X^*} \le M\ \forall\xi \in X$, $\|\xi\|_X \le \varepsilon$, $\forall\|u\|_{\mathbb{U}} \le l$. By the definition of the norm and properties of A we have

$$\|A(u,y)\|_{X^*} = \sup_{\|\xi\|_X\le\varepsilon}\frac{1}{\varepsilon}\Big\langle A(u,y),\xi\Big\rangle_X \le \sup_{\|\xi\|_X\le\varepsilon}\frac{1}{\varepsilon}\Big\{\Big\langle A(u,y),y\Big\rangle_X$$

$$+\Big\langle A(u,\xi),\xi\Big\rangle_X - \Big\langle A(u,\xi),y\Big\rangle_X + \inf_{v\in U} C_v\Big(R,\|y-\xi\|_X\Big)\Big\}$$

$$\le \frac{1}{\varepsilon}(k_2 + M_\varepsilon\varepsilon + M_\varepsilon k_1 + k_2) = M < \infty$$

where

$$\begin{aligned}
&\text{i)} \quad \Big\langle A(u,y),y\Big\rangle_X \le k_2\\
&\text{ii)} \quad \Big|\Big\langle A(u,\xi),\xi\Big\rangle_X\Big| \le \|A(u,\xi)\|_{X^*}\|\xi\|_X \le M_\varepsilon \varepsilon\\
&\text{iii)} \quad \Big|\Big\langle A(u,\xi),y\Big\rangle_X\Big| \le \|A(u,\xi)\|_{X^*}\|y\|_X \le M_\varepsilon k_1\\
&\text{iv)} \quad \inf_{v\in U} C_v\Big(R,\|y-\xi\|_X\Big) \le k_3
\end{aligned}$$

since the functions $C_v(R,\tau)$ are continuous and $\|y\|_X \le k_1, \|\xi\|_X \le \varepsilon$, U is limited.

Thus $\|A(u,y)\|_{X^*} \le M\ \forall (u,y) \in K \times M$, which completes the proof. ■

Proposition 2. *Let us assume that conditions (i)-(iii) of Section 2 hold and (u,y) is a solution to (1). Then $A(u,y) \in V_2^*$ for $f \in V_2^*$.*

Proof. Let $J : V_2 \to V_2^*$ be dual. We can show that $\exists \varepsilon_1 > 0$ such that $\forall \varepsilon_1 \ge \varepsilon > 0$ the inequality

$$\Big\langle J(y_\varepsilon - y) + \varepsilon A(u,y_\varepsilon), \xi - y_\varepsilon\Big\rangle \ge 0 \quad \forall \xi \in K \tag{6}$$

is solvable in K. Since V_2 is a reflexive Banach space, an equivalent norm exists such that V_2 is a strongly convex space (Lions, 1969). Therefore, the dual function $J : V_2 \to V_2^*$ is strongly monotone and demicontinuous (Lions, 1969). $J(\cdot - y) + \varepsilon A(u,\cdot)$ is coercive, since $\forall u \in U$

$$\Big\langle J(y_\varepsilon - y) + \varepsilon A(u,y_\varepsilon), y_\varepsilon - y_0\Big\rangle \ge \gamma\Big(\|y_\varepsilon\|_{V_1}\Big)\|y_\varepsilon\|_{V_1}$$

$$+\,\Phi\Big(\|y_\varepsilon - y\|_{V_2}\Big)\|y_\varepsilon - y_0\|_{V_2} \to \infty \quad \text{as} \quad \|y_\varepsilon\|_{V_1} \to \infty$$

where $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function, Φ generates J, $\Phi(0) = 0$ and $\Phi(s) \to \infty$ as $s \to \infty$. Also $J(\cdot - y) + \varepsilon A(u,\cdot)$ is a radially continuous operator of a uniformly lower semilimited variation. Therefore $\forall \varepsilon > 0$ the inequality (6) is solvable. Since A is coercive, the solution of (1) is limited, $\|y\|_{V_1} \le k_1$. Since $J(\cdot - y) + \varepsilon A(u,\cdot)$ is coercive, there is a limited solution to (6), $\|y_\varepsilon\|_{V_1} \le k_2$. Hence

$$\varepsilon\Big\langle A(u,y_\varepsilon), y - y_\varepsilon\Big\rangle \ge \Big\langle J(y - y_\varepsilon), y - y_\varepsilon\Big\rangle = \|y - y_\varepsilon\|_{V_2}^2$$

Moreover,

$$\Big\langle A(u,y), y_\varepsilon - y\Big\rangle \le \Big\langle A(u,y_\varepsilon), y_\varepsilon - y\Big\rangle + \inf_{v\in U} C_v\Big(R, \|y_\varepsilon - y\|_{V_2}\Big)$$

$$\le -\varepsilon^{-1}\|y_\varepsilon - y\|_{V_2}^2 + \inf_{v\in U} C_v\Big(R, \|y_\varepsilon - y\|_{V_2}\Big)$$

i.e.

$$\varepsilon^{-1}\|y_\varepsilon - y\|_{V_2}^2 \le \Big\langle A(u,y), y - y_\varepsilon\Big\rangle - \inf_{v\in U} C_v\Big(R, \|y_\varepsilon - y\|_{V_2}\Big)$$

$$\le \Big\langle f, y - y_\varepsilon\Big\rangle - \inf_{v\in U} C_v\Big(R, \|y_\varepsilon - y\|_{V_2}\Big)$$

Thus

$$\varepsilon^{-1}\|y_\varepsilon - y\|_{V_2} \le \|f\|_{V_2} - \|y_\varepsilon - y\|_{V_2}^{-1} \inf_{v\in U} C_v\Big(R, \|y_\varepsilon - y\|_{V_2}\Big)$$

If $\|y_\varepsilon - y\|_{V_2} \to 0$ as $\varepsilon \to 0$, then

$$\|y_\varepsilon - y\|_{V_2}^{-1} \inf_{v\in U} C_v\Big(R, \|y_\varepsilon - y\|_{V_2}\Big) \to 0 \qquad \text{as } \varepsilon \to 0$$

and since $C_v(R;\tau)$ are continuous, we see that $\varepsilon^{-1}(y - y_\varepsilon)$ is limited in V_2. Otherwise, we can choose a subsequence $\{\varepsilon_n\}$ such that $\|y_{\varepsilon_n} - y\|_{V_2} \ge C_2$. Since for any $v \in U$ continuous functions C_v are limited on the bounded interval $[C_2, C_1]$, there is a maximum $C_v(R, \|y_\varepsilon - y\|_{V_2}) \le C_3$, i.e.

$$\varepsilon^{-1}\|y_\varepsilon - y\|_{V_2} \le \|f\|_{V_2^*} + \frac{C_3}{C_2} < \infty$$

The sequence $\{y_\varepsilon\}$ is limited. If $J(y - y_\varepsilon) = \varepsilon A(u, y_\varepsilon)$, then $A(u, y_\varepsilon)$ is limited in V_2^*. Otherwise, since A is locally limited, there are δ and $\widehat{\xi} \in \operatorname{int} K$ such that $\|A(u,\xi)\|_{V_2^*} \le M \quad \forall \xi \in B_\delta(\widehat{\xi})$, $B_\delta(\widehat{\xi}) = \{\xi \in K : \|\xi - \widehat{\xi}\|_{V_2} \le \delta\}$. If $\{y_\varepsilon\} \subset B_\delta$, then $\{A(u, y_\varepsilon)\}$ is limited. Otherwise, we can choose $\xi \in B_{\delta/2}(\widehat{\xi})$ and consider $\zeta \in B_{\delta/2}(0)$ such that $\zeta + \xi \in D(A)$. Then

$$\Big\langle A(u,y_\varepsilon), \zeta\Big\rangle \le \Big\langle A(u,y_\varepsilon), y_\varepsilon - \xi\Big\rangle + \Big\langle A(u,\zeta+\xi), \zeta+\xi-y_\varepsilon\Big\rangle$$

$$+ \inf_{v\in U} C_v\Big(R, \|\zeta+\xi-y_\varepsilon\|_{V_2}\Big) \le \varepsilon^{-1}\Big\langle J(y-y_\varepsilon), y_\varepsilon - \xi\Big\rangle + \Big\langle A(u,\zeta+\xi), \zeta+\xi-y_\varepsilon\Big\rangle$$

$$+ \inf_{v\in U} C_v\Big(R, \|\zeta+\xi-y_\varepsilon\|_{V_2}\Big) \le \varepsilon^{-1}\|y - y_\varepsilon\|_{V_2}\|y_\varepsilon - \xi\|_{V_2}$$

$$+ \|A(u,\zeta+\xi)\|_{V_2^*}\|\zeta+\xi-y_\varepsilon\|_{V_2} + \inf_{v\in U} C_v\Big(R, \|\zeta+\xi-y_\varepsilon\|_{V_2}\Big) \le N < \infty$$

Consequently, $\{A(u, y_\varepsilon)\}$ is limited.

We can choose subsequences satisfying

$$\begin{aligned} y_{\varepsilon_m} &\to y && \text{weakly in } V_1 \\ A(u, y_{\varepsilon_m}) &\to d && \text{weakly-}(*) \text{ in } V_2^* \end{aligned}$$

But $V_1 \subset V_2$ is compact, and so

$$y_{\varepsilon_m} \to y \text{ strongly in } V_2$$

It still remains to prove that $A(u,y) = d \in V_2^*$. By property (iii) the following estimate holds:

$$\Big\langle A(u,y_\varepsilon) - A(u,\xi), y_\varepsilon - \xi\Big\rangle \ge -\inf_{v\in U} C_v\Big(R, \|y_\varepsilon - \xi\|_{V_2}\Big)$$

Set $\xi = (1-\lambda)y + \lambda x$, where x is an arbitrary fixed element of V_1. Then

$$-\Big\langle A(u,y_\varepsilon), y_\varepsilon - y\Big\rangle + \Big\langle A(u,\xi), y_\varepsilon - y - \lambda(x-y)\Big\rangle$$
$$\le \lambda\Big\langle A(u,y_\varepsilon), y - x\Big\rangle + \inf_{v\in U} C_v\Big(R, \lambda\|y-x\|_{V_2}\Big)$$

Letting $\varepsilon \to 0$ yields

$$\lambda\Big\langle A(u,\xi), y - x\Big\rangle \le \lambda\Big\langle d, y - x\Big\rangle + \inf_{v\in U} C_v\Big(R, \lambda\|y-x\|_{V_2}\Big)$$

and consequently

$$\Big\langle A(u,\xi) - d, y - x\Big\rangle \le \lim_{\lambda\searrow 0} \frac{1}{\lambda} \inf_{v\in U} C_v\Big(R, \lambda\|y-x\|_{V_2}\Big) = 0 \quad \forall \xi \in V_1$$

Since x is arbitrary, $A(u,y) = d$ by property (M), and the proof is complete. ■

4. Main Construction

Let us isolate some equation from the variational inequality.

Lemma 1. *Suppose that the problem*

$$A(u,y) = f + v \tag{7}$$

$$F(v) = \sup_{\xi\in B_1} \Big\langle v, y(v) - \xi\Big\rangle + I_K\Big(y(v)\Big) = 0 \tag{8}$$

where $B_1 = \{\xi \in K : \|\xi - y\|_{V_1} \le 1\}$, and I_K is the indicator of the set K (without loss of generality we can set $F(v) = \infty$ if B_1 is empty), has a solution for any arbitrary fixed u. Then the pair (u,y) is a solution of (1) if there is a v such that (u,y,v) is a solution of (7)–(8).

Proof. Let (u,y) be a solution of (1). Then $\langle v, y-\xi\rangle = \langle A(u,y) - f, y - \xi\rangle \le 0$ for any arbitrary $\xi \in K$ and $I_K(y(v)) = 0$, i.e. $F(v) = 0$. Consequently, (u,y,v) is the solution of (7)–(8).

Let (u,y,v) be a solution of (7)–(8). Then $F(v) = 0$, $\sup_{\xi\in B_1}\langle v, y-\xi\rangle = 0$ and $I_K(y(v)) = 0$. Thus $y \in K$. Suppose that $\exists \xi \in K$ such that $\langle v, y - \xi\rangle > 0$. Let $\xi_\lambda = \lambda\xi + (1-\lambda)y, \lambda \in (0,1)$. Then $\langle v, y - \xi_\lambda\rangle = \langle v, y - \lambda\xi - (1-\lambda)y\rangle = \langle v, \lambda y - \lambda\xi\rangle = \lambda\langle v, y - \xi\rangle$. Hence the sign of the function $\langle v, y-\xi\rangle$ remains constant along each ray outgoing from y. Since K is convex, any arbitrary ray intersects the unit ball with the centre at $y \in K$. If $\langle v, y - \xi\rangle > 0$, then $\exists \xi_\lambda \in K$ such that $\langle v, y - \xi_\lambda\rangle > 0$. We obtain the contradiction with $\langle v, y - \xi\rangle \le 0$. Consequently, (u,y) is the solution of (1). ■

Remark 1. We selected such B_1 so as to obtain $F(v) < \infty$ if $\|y\|_{V_1} < \infty$ and $\|v\|_{V_2^*} < \infty$. Moreover, if K is bounded, we can consider in (8) $\sup_{\xi\in K}\langle v, y - \xi\rangle$.

We can show that the function F possesses sufficient properties for the solvability.

Proposition 3. *$F(v)$ is a coercive, lower limited, weakly lower semicontinuous function on V_2^*.*

Proof. Since A is coercive and $\lambda\langle v, y-\xi\rangle = \langle v, y-\xi_\lambda\rangle$, where $\xi_\lambda = \lambda\xi + (1-\lambda)y$, there is a ξ_λ such that

$$F(v) \geq \sup_{\xi\in B_1}\Big\langle v, y-\xi\Big\rangle = \Big\langle v, y-\xi_\lambda\Big\rangle \geq \Big\langle A(u,y)-f, y-\xi_0\Big\rangle \|y-\xi_0\|_{V_1}^{-1}$$

$$\geq c\|y\|_{V_1}^{-1}\Big\langle A(u,y)-f, y-\xi_0\Big\rangle \geq c\Big(\gamma(\|y\|_{V_1}) - \|f\|_{V_1^*}\Big) \to \infty$$

as $\|y\|_{V_1} \to \infty$. Hence $F(v) \to \infty$ as $\|y\|_{V_1} \to \infty$.

We consider the behaviour of v as $\|y\|_{V_1} \to \infty$:

$$\|v\|_{V_2^*}\|y\|_{V_2} \geq \Big\langle v, y\Big\rangle = \Big\langle A(u,y)-f, y\Big\rangle \geq c\Big(\gamma(\|y\|_{V_1})\|y\|_{V_1}\Big) - \|f\|_{V_1^*}\|y\|_{V_1}$$

Since $V_1 \subset V_2^*$ is continuous, there exists $C > 0$ such that $\|y\|_{V_2} \leq C\|y\|_{V_1}$. Therefore, $\|v\|_{V_2^*} \to \infty$ as $\|y\|_{V_1} \to \infty$.

Now let us assume that $\|y\|_{V_1} \leq k_1$, $F(v) \leq k_2$ and $\|v\|_{V_2^*} \to \infty$. Since $F(v)$ and $\|y\|_{V_1}$ are limited, we obtain that $\langle v, y\rangle$ and $\|y\|_{V_2}$ are limited. Moreover, $\|u\|_{\mathrm{U}} \leq l$ since u belongs to a bounded set. Hence $\|v\|_{V_2^*} < \infty$ by Proposition 1. This proves that $F(v)$ is coercive.

Furthermore, $F(v)$ is lower limited. Indeed, if $y \notin K$, we have $I_K(y(v)) = 1$ and $\sup_{\xi\in B_1}\langle v, y-\xi\rangle \geq \langle v, y(v)-y\rangle = 0$, $F(v) \geq 1$. If $y \in K$ satisfies inequality (1), then $\langle v, y(v)-\xi\rangle \leq 0\ \forall\xi \in K$ and $I_K(y(v)) = 0$, $F(v) = 0$. On the other hand, $\exists\xi \in K$ such that either $\langle v, y-\xi\rangle > 0$ or $I_K(y(v)) = 1$, i.e. $F(v) > 0$. We obtain $F(v) \geq 0$.

It still remains to prove that F is weakly lower semicontinuous. Let $v_n \to v$ weakly-($*$) in V_2^*. By the solvability theorem (Ivanenko and Melnik, 1988) $\forall v_n\ \exists y_n$ such that $\{y_n\}$ is limited. Then we can choose a subsequence such that $y_m \to y$ weakly in V_1. But $V_1 \subset V_2^*$ is compact. Thus $y_m \to y$ in V_2. Then

$$\varliminf_{m\to\infty}\sup_{\xi\in B_1}\Big\langle v_m, y_m-\xi\Big\rangle = \lim_{m\to\infty}\Big\langle v_m, y_m\Big\rangle + \lim_{m\to\infty}\sup_{\xi\in B_1}\Big\langle v_m, -\xi\Big\rangle = \Big\langle v, y\Big\rangle$$

$$+\lim_{m\to\infty}\sup_{\xi\in B_1}\Big\langle v_m, -\xi\Big\rangle \geq \Big\langle v, y\Big\rangle + \sup_{\xi\in B_1}\Big\langle v, -\xi\Big\rangle = \sup_{\xi\in B_1}\Big\langle v, y-\xi\Big\rangle$$

Let us assume that $y(v) \notin K$. Because K is closed, there is a neighbourhood $U_y \subset V_1$ of y, where $U_y \cap K = \emptyset$, and $\exists N$ such that $\forall m \geq N$, $y_m \in U_y$, i.e. $I_K(y_m) = 1$, $\lim_{m\to\infty} I_K(y_m) = I_K(y) = 1$. In the other case $y \in K$, i.e. $I_K(y) = 0 \leq I_K(y_m)\ \forall m$. Thus $\varliminf_{m\to\infty} F(v_m) \geq F(v)$ and F is weakly lower semicontinuous. This completes the proof. ■

Remark 2. If K is a cone, the variational inequality is equivalent to the system

$$\left\langle A(u,y),\xi\right\rangle \geq \left\langle f,\xi\right\rangle \quad \forall \xi \in K$$

$$\left\langle A(u,y),y\right\rangle = \left\langle f,y\right\rangle$$

and we can use a more suitable penalty function, e.g.

$$F(v) = \left|\left\langle v, y(v)\right\rangle\right| + I_K(y(v))$$

The properties of this function can be proved analogously to Proposition 3.

Remark 3. The statement remains true, if we carry out the change of the indicator I_K for some weakly lower semicontinuous function $\widehat{\beta}$ such that $\widehat{\beta}(v) = 0$ if $y(v) \in K$ and $\widehat{\beta}(v) > 0$ if $y(v) \notin K$.

We introduce the new function

$$L_\varepsilon(u,y,v) = L(u,y) + \frac{1}{\varepsilon}F(v)$$

where $L(u,y)$ is the initial utility function. Let us consider the new problem:

$$A(u,y) = f + v \tag{9}$$

$$L_\varepsilon(u,y,v) \to \inf_{v\in V_2^*, u\in \mathbb{U}} \tag{10}$$

Proposition 4. *Under the above conditions,* $\forall \varepsilon > 0$ *the solution of (9)–(10) exists for every* $f \in V_1^*$.

Proof. By the properties of the operator $A(u,\cdot)$ $\forall u \in U$, $\forall v \in V_1^*$ $\exists y \in V_1$ such that $A(u,y) = f + v$.

To prove that $L_\varepsilon(u,y,v)$ is lower limited, it suffices to show that $L(u,y)$ is lower limited. Let us suppose the contrary, i.e. that there is a sequence $\{u_n, y_n, v_n\}$ such that

$$L\Big(u_n, y_n(v_n)\Big) < -n \quad \forall n$$

By assumption $\{u_n\}$ is limited; $\{y_n\}$ is limited, because the function $L_\varepsilon(u,y,v)$ is coercive. Then we can choose weakly convergent subsequences

$$y_m \to \widehat{y} \quad \text{weakly in } V_1$$

$$u_m \to \widehat{u} \quad \text{weakly-}(*) \text{ in } \mathbb{U}$$

Since L is weakly lower semicontinuous, we obtain

$$L(\widehat{u},\widehat{y}) \leq \underline{\lim}_{m\to\infty} L(u_m, y_m) < -\infty$$

But this is impossible because we have $L : U \times V_1 \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. Hence L is lower limited. Since $F(v) \geq 0$, $L(u,y) + \frac{1}{\varepsilon}F(v) \geq L(u,y)$. Thus, L_ε is a lower limited function.

Since $F(v)$ is coercive, the function $L_\varepsilon(u,y,v)$ is coercive. Furthermore, L_ε is weakly lower semicontinuous because both L and F are weakly lower semicontinuous.

Hence, for the function L_ε a minimizing sequence $\{u_n, y_n, v_n\}$ exists, and $\{y_n\}$, $\{v_n\}$ are limited by coercivity of L_ε, $\{u_n\}$ is limited by assumption. Also $\{A(u_n, y_n)\}$ is limited by Proposition 1. By reflexity of V_1, V_2^* and $\mathbb{U}$ we can choose weakly convergent subsequences

$$y_m \to \widehat{y} \quad \text{weakly in } V_1$$

$$v_m \to \widehat{v} \quad \text{weakly-}(*) \text{ in } V_2^*$$

$$u_m \to \widehat{u} \quad \text{weakly-}(*) \text{ in } \mathbb{U}$$

$$A(u_m, y_m) \to \kappa \quad \text{weakly-}(*) \text{ in } V_2^*$$

But $V_1 \subset V_2$ is compact. Thus

$$y_m \to \widehat{y} \ \text{ strongly in } V_2$$

Hence

$$\varliminf_{m\to\infty} \Big\langle A(u_m, y_m), y_m \Big\rangle = \varliminf_{m\to\infty} \Big\langle v_m + f, y_m \Big\rangle = \Big\langle \widehat{v} + f, \widehat{y} \Big\rangle$$

Since the operator A has property (M), we obtain $A(\widehat{u}, \widehat{y}) = f + \widehat{v}$.

Suppose that this solution is not optimal, i.e. $\exists (u,y,v) \in U \times K \times V_2^*$ such that $L_\varepsilon(u,y,v) < L_\varepsilon(\widehat{u}, \widehat{y}, \widehat{v})$. Let d be the least lower bound of function L_ε. Then

$$L_\varepsilon(u,y,v) < L_\varepsilon(\widehat{u}, \widehat{y}, \widehat{v}) \leq \varliminf_{m\to\infty} L_\varepsilon(u_m, y_m, v_m) = \lim_{m\to\infty} L_\varepsilon(u_m, y_m, v_m) = d$$

Thus, we obtain the contradiction, and the proof is complete. ∎

Since the problem (9)–(10) has a solution, we can prove the approximate theorem and thus obtain the method of approximate solving (1)–(2).

Theorem 1. *Let* $\forall \varepsilon > 0$ *the triple* $(u_\varepsilon, y_\varepsilon, v_\varepsilon)$ *be a solution of*

$$A(u,y) = f_\varepsilon + v$$

$$L_\varepsilon(u,y,v) = L(u,y) + \frac{1}{\varepsilon}\left(\sup_{\xi \in B_1} \Big\langle v, y(v) - \xi \Big\rangle + I_K\Big(y(v)\Big) \right) \to \inf_{v \in V_2^*, u \in \mathbb{U}}$$

where $f_\varepsilon \in V_2^*$ *and* $\|f - f_\varepsilon\|_{V_1^*} < \varepsilon$. *Then we can choose subsequences* $\{u_{\varepsilon_n}\}, \{y_{\varepsilon_n}\}, \{v_{\varepsilon_n}\}$ *such that* $u_{\varepsilon_n} \to \widehat{u}$ *weakly-*$(*)$ *in* $\mathbb{U}$, $y_{\varepsilon_n} \to \widehat{y}$ *weakly in* V_1, $v_{\varepsilon_n} \to \widehat{v}$ *weakly-*$(*)$ *in* V_2^*, *and* $(\widehat{u}, \widehat{y})$ *is a solution of (1)–(2).*

Proof. By Proposition 2, $\forall f_\varepsilon$ there is a $v \in V_2^*$, such that for any arbitrary fixed $u \in U$ there is a $y \in V_1$ such that $L_\varepsilon(u,y,v) = L(u,y)$. Then there is a minimizing sequence $\{u_\varepsilon, y_\varepsilon, v_\varepsilon\}$. Since L_ε is coercive and the set U is bounded, this sequence is limited. Hence we can choose weakly convergent subsequences

$$u_{\varepsilon_n} \to \widehat{u} \quad \text{weakly-}(*) \text{ in } \mathbb{U}$$

$$y_{\varepsilon_n} \to \widehat{y} \quad \text{weakly-}(*) \text{ in } V_1$$

$$v_{\varepsilon_n} \to \widehat{v} \quad \text{weakly-}(*) \text{ in } V_2^*$$

Moreover, $f_{\varepsilon_n} \to f$ strongly in V_1^*. Since $V_1 \subset V_2$ is compact, we have

$$y_{\varepsilon_n} \to \widehat{y} \text{ strongly in } V_2$$

Hence

$$\varliminf_{\varepsilon_n \to 0} \left\langle A(u_{\varepsilon_n}, y_{\varepsilon_n}), y_{\varepsilon_n} \right\rangle = \varliminf_{\varepsilon_n \to 0} \left\langle v_{\varepsilon_n} + f_{\varepsilon_n}, y_{\varepsilon_n} \right\rangle = \left\langle \widehat{v} + f, \widehat{y} \right\rangle$$

By property (M) $A(\widehat{u}, \widehat{y}) = f + \widehat{v}$.

Since U is weakly closed, we have $u \in U$. It still remains to show that $\langle v, \xi - y \rangle \geq 0$ and $y \in K$, i.e. $F(v) = 0$. Let $(u_1, y_1, v_1) \in U \times K \times V_2^*$ be an arbitrary triple which satisfies inequality (1). Then (without loss of generality $\varepsilon_n = \varepsilon$) we have

$$L_\varepsilon(u_\varepsilon, y_\varepsilon, v_\varepsilon) \leq L_\varepsilon(u_1, y_1, v_1) = L\Big(u_1, y_1(v_1)\Big)$$

and

$$F(v_\varepsilon) \leq \varepsilon\Big(L(u_1, y_1(v_1))\Big) - L(u_\varepsilon, y_\varepsilon, v_\varepsilon) \leq \varepsilon C$$

where C is independent of ε. Since L is weakly lower semicontinuous,

$$\varliminf_{\varepsilon \to 0} L_\varepsilon(u_\varepsilon, y_\varepsilon, v_\varepsilon) \geq \varliminf_{\varepsilon \to 0} L(u_\varepsilon, y_\varepsilon, v_\varepsilon) \geq L(\widehat{u}, \widehat{y})$$

and

$$F(\widehat{v}) \leq \varliminf_{\varepsilon \to 0} F(v_\varepsilon) \leq 0$$

Since $F(v) \geq 0$, we have $F(\widehat{v}) = 0$. Hence $\widehat{y} \in K$ and satisfies the variational inequality (1).

Suppose that this solution is not optimal with respect to the utility function L, i.e. $\exists (u, y, v) \in U \times K \times V_2^*$ such that $L(u, y(v)) < L(\widehat{u}, \widehat{y})$. But $\{(u_\varepsilon, y_\varepsilon, v_\varepsilon)\}$ is a minimizing sequence. We write $\varliminf_{\varepsilon \to 0} L_\varepsilon(u_\varepsilon, y_\varepsilon, v_\varepsilon) = d$. Then

$$L\Big(u, y(v)\Big) < L(\widehat{u}, \widehat{y}) \leq \varliminf_{\varepsilon \to 0} L_\varepsilon(u_\varepsilon, y_\varepsilon, v_\varepsilon) = d$$

We obtain the contradiction. The theorem is proved. ■

Remark 4. It follows from the theorem that $\forall f \in V_1^*$, if (u, y) is a solution of (1), then there is a $v \in V_2^*$ such that $A(u, y) = v + f$.

Remark 5. Let the initial problem make it possible to localize the approximate problem on the closed set $K' \subset V_2^*$, i.e. we can deduce that $A(u, y) - f \in K'$, when (u, y) is a solution of (1). Then we can consider the approximate problem on the control set $U \times K'$ and all the statements remain true.

5. Optimality Conditions

Set $F_1(v, y) = \sup_{\xi \in B_1} \langle v, y(v) - \xi \rangle$, and let us show that we can infer the optimality conditions for the function $L_\varepsilon(u, y, v) = L(u, y) + \varepsilon^{-1} F(v) = L(u, y) + \varepsilon^{-1}(F_1(v, y) + I_K(y(v)))$, if the system operator A and the utility function L, possess some relevant differentiability properties.

Let (u, v) be an optimal control of (1)–(2). Then

$$L_\varepsilon(u, y, v) \le L_\varepsilon\Big(w_u, y(w_u, w_v), w_v\Big) \quad \forall (w_u, w_v) \in U \times V_2^*$$

By construction this inequality may be written as follows:

$$L(u, y) \le L\Big(w_u, y(w_u, w_v)\Big) \tag{11}$$

$$F_1(v, y) \le F_1\Big(w_v, y(w_u, w_v)\Big) \tag{12}$$

$$I_K\Big(y(v)\Big) = 0 \tag{13}$$

Since F_1 is linear with respect to y and v, $F_1(v, y)$ has Gâteaux partial derivatives

$$D_y F_1(y, v) h_y = \Big\langle v, h_y \Big\rangle, \quad D_v F_1(y, v) h_v = \sup_\xi \Big\langle h_v, y - \xi \Big\rangle$$

Moreover, if L is subdifferentiable, then (11)–(13) become

$$\Big\langle \partial L_u(u, y), w_u - u \Big\rangle + \Big\langle \partial L_y(u, y), w_y - y \Big\rangle \ge 0$$

$$\Big\langle v, w_y - y \Big\rangle + \sup_\xi \Big\langle w_v - v, y - \xi \Big\rangle \ge 0$$

$$I_K\Big(y(v)\Big) = 0$$

If A and L satisfy stronger conditions, we can modify the last formulas. Let in a neighbourhood $\mathbf{V} \subset U \times V_1$ the operator A and the function L satisfy the conditions:

j) $A : U \times V_1 \to V_2^*$ has continuous Gâteaux partial derivatives

$$D_u A : \mathbf{V} \to L(U, V_2^*), \quad D_y A : \mathbf{V} \to L(V_1, V_2^*)$$

jj) $L : U \times V_1 \to \overline{\mathbb{R}}$ has continuous Gâteaux partial derivatives

$$D_y L : \mathbf{V} \to V_2^*, \quad D_u L : \mathbf{V} \to \mathbb{U}^*$$

jjj) $\forall (u, y) \in \mathbf{V}$ the operator $[D_y A(u, y)]^{-1}$ exists and is limited with respect to u.

For convenience we introduce the operator $\mathbf{A}(u, y, v) = A(u, y) - v$. It is easy to see that $D_u \mathbf{A}(u, y, v) = D_u A(u, y)$, $D_y \mathbf{A}(u, y, v) = D_y A(u, y)$. Moreover, we have $D_v \mathbf{A}(u, y, v) = -1$. Consequently, the conditions of the implicit function theorem are satisfied. Hence $\forall h = (h_u, h_v) \in U \times V_2^*$ the formulae for y' are

$$y'(u, v)h_u = -\Big[D_y A(u, y)\Big]^{-1} D_u A(u, y) h_u$$

$$y'(u, v)h_v = \Big[D_y A(u, y)\Big]^{-1} h_v$$

Hence it follows from (11)–(13) that the optimal control (u, v) is estimated as follows:

$$\Big\langle D_u L(u, y), w_u - u\Big\rangle + \Big\langle D_y L(u, y), y_u'(w_u - u)\Big\rangle + \Big\langle D_y L(u, y), y_v'(w_v - v)\Big\rangle \geq 0$$

$$\Big\langle D_v F_1(y, v), w_v - v\Big\rangle + \Big\langle D_y F_1(y, v), y_u'(w_u - u)\Big\rangle + \Big\langle D_y F_1(y, v), y_v'(w_v - v)\Big\rangle \geq 0$$

$$I_K\Big(y(v)\Big) = 0$$

Substitute y_u' and y_v' into the previous inequalities. Then

$$\Big\langle D_u L(u, y), w_u - u\Big\rangle - \Big\langle D_y L(u, y), [D_y A(u, y)]^{-1} D_u A(u, y)(w_u - u)\Big\rangle$$

$$+ \Big\langle D_y L(u, y), [D_y A(u, y)]^{-1}(w_v - v)\Big\rangle \geq 0$$

$$\Big\langle D_v F_1(y, v), w_v - v\Big\rangle - \Big\langle D_y F_1(y, v), [D_y A(u, y)]^{-1} D_u A(u, y)(w_u - u)\Big\rangle$$

$$+ \Big\langle D_y F_1(y, v), [D_y A(u, y)]^{-1}(w_v - v)\Big\rangle \geq 0$$

$$I_K\Big(y(v)\Big) = 0$$

Moreover, let us introduce the state $P = (P_u, P_v) \in V_1 \times V_1$ such that

$$D_y A(u, y) P_u = D_y L(u, y), \quad D_y A(u, y) P_v \doteq D_y F_1(v, y)$$

Then the inequality is written as follows:

$$\Big\langle D_u L(u, y) - D_u A(u, y)^* P_u, w_u - u\Big\rangle + \Big\langle P_u, w_v - v\Big\rangle \geq 0$$

$$\Big\langle D_v F_1(y, v) P_v, w_v - v\Big\rangle - \Big\langle D_u A(u, y)^* P_v, w_v - v\Big\rangle \geq 0$$

$$I_K\Big(y(v)\Big) = 0$$

6. Examples

The examples below are model steady-state problems of the thermal conductivity theory (with nonlinear Fourier law), of the elasticity theory (with inhomogeneous medium) and of the nonlinear viscoelasticity theory. In the linear case these models were described partially by (Duvaut and Lions, 1972).

Let us consider the Sobolev spaces $W_p^1(\Omega)$, where $p \geq 2$, Ω is a bounded domain in $\mathbb{R}^n$, Γ is the boundary of Ω, $\forall x \in \Gamma$ the vector of the external normal ν is defined, $A : U \times W_p^1(\Omega) \to W_q^{-1}(\Omega)$, $p^{-1} + q^{-1} = 1$, $Dy = (\frac{\partial y}{\partial x_1}, \dots, \frac{\partial y}{\partial x_n})$, $U = \{u \in L_q(\Omega) : \|u\|_{L_q(\Omega)} \leq M\}$, $V_1 = W_p^1(\Omega) \times W_p^{1/q}(\Gamma)$, $V_1^* = W_q^{-1}(\Omega) \times W_q^{-1/q}(\Gamma)$, $V_2^* = L_q(\Gamma)$, $K = \{y \in W_p^1(\Omega) : y_{|\Gamma} \geq 0\}$.

We can show that the function $\widehat{\beta}(v) = \beta \operatorname{dist}(y(v), K)$ $(\beta > 0)$ is weakly lower semicontinuous. Let $v_n \to v$ weakly in $L_q(\Gamma)$. Then $y_n = y(v_n)$ is a limited sequence. Without loss generality $y_n \to y$ weakly in $W_p^1(\Omega)$ (otherwise we can choose a weakly convergent subsequence). Moreover, since the limit is unique and A has property (M), $y = y(v)$. Hence $y_n \to y$ strongly in $L_p(\Gamma)$. Consequently, since K is closed, the function $\widehat{\beta}$ is weakly lower semicontinuous and we can carry out the change of I_K for $\widehat{\beta}$ (Remark 4).

Example 1. Let us consider the operator $A(u, y) = \mathbf{A}(y) + B(u)$, where $B(u) = -u$, $u \in L_q(\Omega)$, $\mathbf{A}(y) = -\sum_{i=1}^n \frac{\partial}{\partial x_i}\big(|\frac{\partial y}{\partial x_i}|^{p-2}\frac{\partial y}{\partial x_i}\big) + c(|y|)y$, $c : L_p(\Omega) \to L_{q'}(\Omega)$ is nonnegative with $q' = p/(p-2)$; $\frac{\partial y}{\partial \nu_A} = -\sum_{i=1}^n |\frac{\partial y}{\partial x_i}|^{p-2}\frac{\partial y}{\partial x_i}$. We consider the problem:

$$A(u, y) = f(x)$$

$$y_{|\Gamma} \geq 0, \qquad \frac{\partial y}{\partial \nu_A}\Big|_\Gamma \geq 0, \qquad y\frac{\partial y}{\partial \nu_A}\Big|_\Gamma = 0$$

$$L(u, y) = \|y - z_d\|_{L_p(\Omega)}^p + \|u\|_{L_q(\Omega)}^q \to \inf_{u \in U}$$

In variational inequality form, this problem is as follows:

$$a(y, y - \xi) \geq \int_\Omega \Big(f(x) + u(x)\Big)(y - \xi)\,\mathrm{d}x, \quad \forall \xi \in K \tag{14}$$

$$L(u, y) \to \inf_{u \in U} \tag{15}$$

where $a(y, y - \xi) = \int_\Omega \sum_{i=1}^n |\frac{\partial y}{\partial x_i}|^{p-2}\frac{\partial y}{\partial x_i}\frac{\partial (y-\xi)}{\partial x_i}\,\mathrm{d}x + \int_\Omega c(|y|)y(y - \xi)\,\mathrm{d}x$, $K = \{y \in W_p^1(\Omega) : y_{|\Gamma} \geq 0\}$. Let us show that the method is applicable in this case.

Evidently, $\mathbf{A}(y)$ is radially continuous, $B(u)$ continuous. For any arbitrary $y_j \in W_p^1(\Omega)$, $\|y_j\|_{W_p^1(\Omega)} \le R$, $j = 1, 2$, we have

$$-\int_\Omega \sum_{i=1}^n \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial y_1}{\partial x_i}\Big|^{p-2}\frac{\partial y_1}{\partial x_i}\Big)(y_1-y_2)\,\mathrm{d}x + \int_\Omega \sum_{i=1}^n \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial y_1}{\partial x_i}\Big|^{p-2}\frac{\partial y_1}{\partial x_i}\Big)(y_1-y_2)\,\mathrm{d}x$$

$$+\int_\Omega c(|y_1|)y_1(y_1-y_2)\,\mathrm{d}x - \int_\Omega c(|y_2|)y_2(y_1-y_2)\,\mathrm{d}x$$

$$\ge \int_\Omega \min_{j=1,2} c(|y_j|)(y_1-y_2)^2\,\mathrm{d}x + \int_\Omega \sum_{i=1}^n \Big|\frac{\partial y_1}{\partial x_i}\Big|^{p-2}\frac{\partial y_1}{\partial x_i}\frac{\partial(y_1-y_2)}{\partial x_i}\,\mathrm{d}x$$

$$-\int_\Omega \sum_{i=1}^n \Big|\frac{\partial y_2}{\partial x_i}\Big|^{p-2}\frac{\partial y_2}{\partial x_i}\frac{\partial(y_1-y_2)}{\partial x_i}\,\mathrm{d}x - \int_\Gamma \sum_{i=1}^n \Big|\frac{\partial y_1}{\partial x_i}\Big|^{p-2}\frac{\partial y_1}{\partial x_i}(y_1-y_2)\,\mathrm{d}\Gamma$$

$$+\int_\Gamma \sum_{i=1}^n \Big|\frac{\partial y_2}{\partial x_i}\Big|^{p-2}\frac{\partial y_2}{\partial x_i}(y_1-y_2)\,\mathrm{d}\Gamma \ge -\max_{j=1,2}\|Dy_j\|^{p-1}_{L_q(\Gamma)}\|y_1-y_2\|_{L_p(\Gamma)}$$

$$\ge -(cR)^{p-1}\,\|y_1-y_2\|_{L_p(\Omega)}$$

By the trace theorem, $\|Dy_j\|_{L_q(\Gamma)} \le c\|Dy_j\|_{L_q(\Omega)} \le cR$ (see Lions and Magenes, 1969). Moreover, $\|\cdot\|_{L_p(\Gamma)}$ is compact with respect to $\|\cdot\|_{W_p^{-1/p}(\Gamma)}$ (see Ladyzhenskaya and Uraltseva, 1973). Thus, A is a radially continuous operator of uniformly lower semilimited variation.

If $c(|y|)y \ge c(x)|y|^{p-2}y$, where $c(x) > 0$, then we can show that A is coercive (Congbao, 1994a; 1994b). Otherwise, $a(y,y) \ge \int_\Omega \sum_{i=1}^n |\frac{\partial y}{\partial x_i}|^p\,\mathrm{d}x = \|Dy\|^p_{L_p(\Omega)} \to \infty$ as $\|Dy\|^p_{L_p(\Omega)} \to \infty$, and $L(u,y) \ge \|y - z_d\|^p_{L_p(\Omega)} \to \infty$ as $\|y\|^p_{L_p(\Omega)} \to \infty$. This means that the problem is coercive. Consequently, the problem is solvable and we can construct the new extremal problem

$$A(u,y) = f(x), \qquad \frac{\partial y}{\partial \nu_A}\Big|_\Gamma = v \tag{16}$$

$$L_\varepsilon(u,y,v) = L(u,y) + \frac{1}{\varepsilon}\Big(\int_\Gamma v|y|\,\mathrm{d}\Gamma + \beta\|y^-\|^p_{L_p(\Gamma)}\Big) \to \inf_{u\in U, v\in K'} \tag{17}$$

where $K' = \{v \in L_q(\Gamma) | v \ge 0\}$, $y^- = \min(y,0)$. Since the system satisfies the assumptions of Proposition 4, the solutions of (16)–(17) exist for all $\varepsilon > 0$ and by Theorem 1 we can find subsequences such that $y_\varepsilon \to \widehat{y}$ weakly in $W_p^1(\Omega)$, $u_\varepsilon \to \widehat{u}$ weakly-($*$) in $L_q(\Gamma)$, i.e. $y_\varepsilon \to \widehat{y}$ strongly in $L_p(\Omega)$ as $\varepsilon \to 0$, $(\widehat{u},\widehat{y})$ is a solution of (14)–(15).

Example 2. Let the operator $\mathbf{A}(y)$ be the same as in Example 1 and assume that the control is $U \ni u = (\varphi,\psi)$, where $\varphi \in W_p^{1/p}(\Gamma)$, $\psi \in L_q(\Gamma)$, $\varphi \ge 0$, $\psi \ge 0$. We consider the problem

$$\mathbf{A}(y) = f(x)$$

$$y|_\Gamma \geq -\varphi, \frac{\partial y}{\partial \nu_A}\Big|_\Gamma \geq -\psi, \qquad (y+\varphi)\Big(\frac{\partial y}{\partial \nu_A} + \psi\Big)\Big|_\Gamma = 0$$

$$L(u,y) = \|y - z_d\|^p_{W^1_p(\Omega)} + \|u\|^q_{W^{1/p}_p(\Gamma)\times L_q(\Gamma)} \to \inf_u$$

In variational inequality form this problem is as follows:

$$a(y, y-\xi) \geq \int_\Omega \Big(f(x) + u(x)\Big)(y-\xi)\,\mathrm{d}x, \quad \forall \xi \in K \tag{18}$$

$$L(u,y) \to \inf_{u\in U} \tag{19}$$

where $K = \{y \in W^1_p(\Omega) | y_{|\Gamma} \geq -\varphi\}$. Then the system satisfies Proposition 4 and Theorem 1. Consequently, we can construct the extremal problem

$$\mathbf{A}(y) = f(x), \qquad \frac{\partial y}{\partial \nu_A}\Big|_\Gamma = v \tag{20}$$

$$L_\varepsilon(u,y,v) = L(u,y) + \frac{1}{\varepsilon}F(v) \to \inf_{u\in U, v\in K'} \tag{21}$$

where $F(v) = \sup_{\xi\in K} \int_\Gamma v(y-\xi)\,\mathrm{d}\Gamma + \beta\|(y+\varphi)^-\|^p_{L_p(\Gamma)}$ and $K' = \{v \in L_q(\Gamma) | v \geq -\psi\}$. A solution to (20)–(21) exists for all $\varepsilon > 0$, and by Theorem 1 we can find subsequences such that $y_\varepsilon \to \widehat{y}$ weakly in $W^1_p(\Omega)$, $u_\varepsilon \to \widehat{u}$ weakly-(*) in $W^{1/p}_p(\Gamma) \times L_q(\Gamma)$, $\mathbb{U}$, i.e. $y_\varepsilon \to \widehat{y}$ strongly in $L_p(\Omega)$ as $\varepsilon \to 0$, $(\widehat{u}, \widehat{y})$ is a solution to (18)–(19).

Remark 6. Analogous results can be obtained for the operator

$$\mathbf{A}(y) = -\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(g(y)\left|\frac{\partial y}{\partial x_i}\right|^{p-2}\frac{\partial y}{\partial x_i}\right) + c(|y|)y$$

on $M(\Phi) = \{y(x), x \in \Omega : \Phi(y(x)) \in W^1_p(\Omega)\}$, see (Laptev, 1994).

Example 3. Let $u \in L_q(\Omega)$ and assume that the operator is of the form

$$A(u,y) = -\sum_{i=1}^n \frac{\partial}{\partial x_i} a_i(x,y,Dy) + a_0(x,u,y,Dy)$$

where $\frac{\partial y}{\partial \nu_A} = -\sum_{i=1}^n a_i(x,y,Dy)$. Let us consider the free-boundary problem:

$$A(u,y) = f(x) \tag{22}$$

$$y|_\Gamma \geq 0, \frac{\partial y}{\partial \nu_A}\Big|_\Gamma \geq 0, \qquad y\frac{\partial y}{\partial \nu_A}\Big|_\Gamma = 0 \tag{23}$$

$$L(u,y) = \|y - z_d\|^p_{W^1_p(\Omega)} + \|u\|^q_{L_q(\Omega)} \to \inf_{u\in U} \tag{24}$$

Let the functions $a_i,\ i=\overline{0,n}$, satisfy Caratheodory conditions, i.e. they are measurable with respect to all arguments and continuous with respect to y, Dy and u for almost all x. Suppose for simplicity that $2 \le p < n$ and the functions $a_i,\ i=\overline{0,n}$, satisfy the following conditions:

a) $\exists \delta_0 > 0$ such that $\forall 0 < \delta \le \delta_0$ and for some positive functions $C, C_1 : (0,\delta_0] \to \mathbb{R}_+$ we have

$$|a_i(x,y,\xi)| \le c|\xi|^{p-1} + \delta|y|^r + C(\delta)h(x), \quad i=\overline{1,n}$$

$$|a_0(x,u,y,\xi)| \le \delta(|\xi|^{r_0} + |y|^{r_1}) + c_1|u|^{q-1} + C_1(\delta)h_0(x)$$

where $c, c_1 > 0$, $r = n(p-1)/(n-p) - 1$, $h \in L_q(\Omega)$, $r_0 = p - 1 + p/n$, $r_1 = np/(n-p) - 1$, $h_0 \in L_{s'}(\Omega)$, $s' = np/(np - n + p)$;

b) the ellipticity condition:

$$\int_\Omega \sum_{i=1}^{n} \Big(a_i(x,y,\xi) - a_i(x,y,\nu)\Big)(\xi_i - \nu_i)\,\mathrm{d}x \ge 0$$

c) the coercivity condition: for almost all $x \in \Omega$ and $\forall u \in U$ we have

$$\sum_{i=1}^{n} a_i(x,y,\xi)\xi_i + a_0(x,u,y,\xi)y \ge c_0|\xi|^p - \delta g(x)|y|^p - C_2(\delta)g_1(x)$$

where $c_0 > 0$, $g_1 \in L_1(\Omega)$, $g \in L_{n/p}(\Omega)$, $C_2(0,\delta_0] \to \mathbb{R}_+$ is limited.

Then the operator satisfies the conditions of Propositions 3 and 4 (an analogous proof can be found in (Laptev, 1994)), and we may construct the new extremal problem:

$$A(u,y) = f(x), \qquad \frac{\partial y}{\partial \nu_A}\Big|_\Gamma = v \tag{25}$$

$$L_\varepsilon(u,y,v) = L(u,y) + \frac{1}{\varepsilon}\Big(\int_\Gamma v|y|\,\mathrm{d}\Gamma + \beta\|y^-\|^p_{L_p(\Gamma)}\Big) \to \inf_{u\in U, v\in K'} \tag{26}$$

where $K' = \{v \in L_q(\Gamma) : v \ge 0\}$.

The solutions to (25)–(26) exist for all $\varepsilon > 0$ and by the approximate theorem we can find subsequences such that $y_\varepsilon \to \widehat{y}$ weakly in $W^1_p(\Omega)$, $u_\varepsilon \to \widehat{u}$ weakly-(*) in $L_q(\Gamma)$, i.e. $y_\varepsilon \to \widehat{y}$ strongly in $L_p(\Omega)$ as $\varepsilon \to 0$, $(\widehat{u},\widehat{y})$ is a solution to (22)–(24).

Remark 7. Since L is coercive, we can consider an unbounded set U and all the statements remain true.

7. Conclusions

In this paper a solving framework for the extremal problem with a variational inequality is constructed. For this purpose, the variational inequality is converted into the equation by adding an adequate penalty term to the initial utility function (some of these equations are well-known in others theories). This idea was used, in particular, in (Glowinski *et al.*, 1976) for a monotone operator and in (Barbu, 1993) for subdifferential part of a variational inequality. In this paper we propose other penalty functions. By the penalty method weakly convergent subsequences are constructed. The limit of these subsequences is a solution of the initial problem. The main adventage of this method consists in using assumptions of the problem solvability proof only (Ivanenko and Melnik, 1988). Of course, for a slender group of systems, there exist more suitable methods (Liu and Rubio, 1991; Noor, 1991; Tseng, 1991; 1992), which allow us to calculate the next approximation and its error. Using this method, we cannot calculate them. Such a possibility appears under the assumption that in some neighbourhood the system operator A and the utility function L possess some differentiability properties, as has been stated in Section 5. Hence these conditions are not stronger than in the majority of known problems, and we do not reduce the information concerning the initial problem with respect to other penalty functions. This means that the method is applicable to a wider class of problems. Moreover, if we have some sequence of solutions to the optimization problems, we can reconstruct it by minimization of L_ε and find the direction of weak convergence. For most practical problems there are models in Sobolev spaces $W_p^k(\Omega)$. Then we can consider e.g. problems in $L_p(\Omega)$, and the sequence of approximate solutions converges strongly in $L_p(\Omega)$ to the solution of the initial problem.

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