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ON THE STABILITY OF NONLINEAR PD CONTROL

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A construction of nonlinear-PD control (NPD control) is considered which applies increased control effort when the system output is moving away from its desired value and reduced effort when the output is moving toward the goal point. Such NPD control has been described previously in the literature and experimentally demonstrated; but to date, no stability proof has been given.

For strictly proper, time invariant SISO systems with NPD control, stability is established by demonstrating a Lyapunov function. Design, and implementation issues are addressed and design examples are presented.

1. Introduction

Broadly speaking, Nonlinear-PD (NPD) control is any control structure of the form:

$$u(t) = k(\cdot)e(t) + b(\cdot)\dot{e}(t) \tag{1}$$

where $k(\cdot)$ and $b(\cdot)$ are time-varying stiffness and damping terms, which may depend on system state, input or other variables; and $u(t)$ and $e(t)$ are the system input and error, respectively. NPD control is illustrated in the block diagram of Fig. 1. The estimated state is shown as an input to the control block to support evaluation of the gains $k(\cdot)$ and $b(\cdot)$. NPD control has been proposed for a number of robotic applications, including the Utah/MIT hand (Jacobsen *et al.*, 1984) and the Sarcos Dextrous Arm (Xu *et al.*, 1993; 1994; 1995).

Xu *et al.* (1993; 1994; 1995) have presented a construction of NPD control which increases damping relative to what is achieved by linear PD control. This is done by increasing the error gain, $k(\cdot)$, during periods when the output is moving away from the goal point, and reducing the gain while the output is traveling toward the goal point, as illustrated in Fig. 2.

The proof of stability for the NPD control technique demonstrated by Xu *et al.* is addressed in this paper. In their construction, which they term NPD control, $k(\cdot)$ and $b(\cdot)$ are chosen according to

$$k(e,\dot{e}) = \frac{k_1}{1+\beta e^{\alpha\,\mathrm{sgn}(\dot{e})e}} + k_0, \qquad b(e,\dot{e}) = \frac{b_1}{1+\beta e^{\alpha\,\mathrm{sgn}(\dot{e})e}} + b_0 \tag{2}$$

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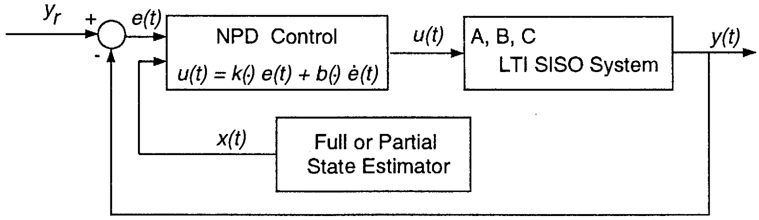


Fig. 1. A block diagram illustrating NPD control. This study is limited to linear, time-invariant, SISO systems.

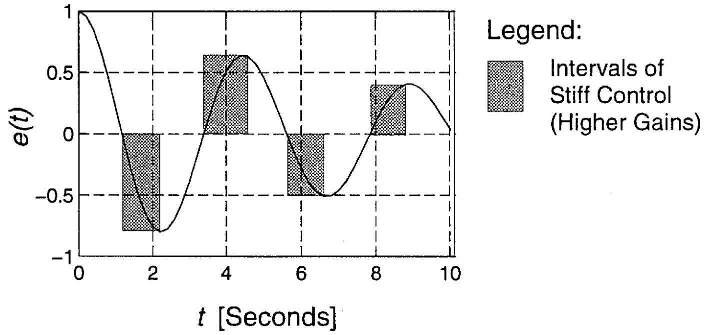


Fig. 2. NPD control applies high stiffness and damping gains during periods when the output is receding from the goal point.

where k_0, k_1, b_0 and b_1 are constants which determine the magnitudes of the feedback gains, and α and β are constants which determine the width of the transition between $k(\cdot) \simeq k_0$ and $k(\cdot) \simeq k_0 + k_1$, and between $b(\cdot) \simeq b_0$ and $b(\cdot) \simeq b_0 + b_1$. Xu *et al.* (1993; 1994; 1995) demonstrate by simulation and experiment that NPD control can achieve damping that is difficult to achieve by standard PD control.

Modulation of the proportional gain, $k(\cdot)$, can increase damping, while modulation of the derivative gain, $b(\cdot)$, can shorten rise time. The stabilizing effect of NPD control can be qualitatively understood in this way: consider a second-order system with $k(\cdot)$ chosen so that the larger control gain, $k(\cdot) = k_0 + k_1$, operates as a stiff spring while the system output is moving away from the goal point; the smaller control gain, $k(\cdot) = k_0$, operates as a softer spring while the output is moving toward the goal point. With each half cycle, the stiff spring is compressed. At the point of greatest compression, $\bar{e} \equiv \sup\{e(t)\}$, the stiff spring is removed from the system and the soft spring applied ($k(\cdot)$ is switched from $k_0 + k_1$ to k_0). In doing so, the energy difference between the springs, $\Delta E = (1/2)\, k_1\, \bar{e}^2$, is removed from the system, thus providing dissipation.

If the rate gain, $b(\cdot)$, is modulated with the smaller gain applied while the output is moving toward the desired value, the rise time may be shortened. Reduction of settling time will depend on the interaction of $k(\cdot)$, $b(\cdot)$ and system dynamics.

Xu *et al.* justify heuristically NPD gain selection such that the stiff gain, $k_0 + k_1$ is chosen so as not to destabilize the system if the stiff gains were continuously applied (i.e., standard PD control, $k = k_0 + k_1$, $b = b_0 + b_1$) (Xu *et al.* , 1994). This is a conservative value which should not result in instability. *But simulations presented below show that increasing k_1 beyond this limit continues to yield increased damping.* Thus, achieving a tighter determination of the stability limit for NPD control will make possible increased performance for these systems. An example of NPD control showing instability in this system and treatment of third-order systems may be found in (Armstrong *et al.*, 1996).

The object of this paper is rigorous determination of bounds within which the larger gains of NPD control may be applied. To support derivation of a Lyapunov function, a discontinuous control is introduced in Section 2. In Section 2.3 a Lyapunov function is presented and stability established. In Section 3 the design of NPD control is addressed and simulations are presented. The conclusions follow in Section 4.

2. Stability of NPD Control

2.1. Model

A linear, time-invariant, single-input single-output, strictly proper state-space system is considered:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{3}$$

where A is $[n \times n]$, B is $[n \times 1]$, C is $[1 \times n]$. The system $\{A,\, B,\, C\}$ must be stabilizable by PD control.

The NPD control law considered is given by

$$\begin{aligned} u(t) &= -k(x)y(t) - b(x)\dot{y}(t) \\ k(x) &= \begin{cases} k_0 & \text{if } s_k(x) == 0 \\ k_0 + k_1 & \text{if } s_k(x) == 1 \end{cases} \\ b(x) &= \begin{cases} b_0 & \text{if } s_b(x) == 0 \\ b_0 + b_1 & \text{if } s_b(x) == 1 \end{cases} \end{aligned} \tag{4}$$

where k_0, $b_0 \geq 0$ are constants which determine the smaller control gains; k_1, $b_1 \geq 0$ determine the larger control gains; and $s_k(x)$, $s_b(x) : \mathbb{R}^n \rightarrow \{0, 1\}$ are switch functions which control the application of the larger control gains. Regulation to $y(t) = y_r$ can be represented in eqns. (3) and (4) by a suitable shift of coordinates or a feed-forward term; reference tracking is not considered. The NPD control law can also be expressed as

$$k(x) = k_0 + k_1 s_k(x), \qquad b(x) = b_0 + b_1 s_b(x) \tag{5}$$

In this paper, the term *stiff control* will refer to applications of a larger gain, $k(\cdot) = k_0 + k_1$ and/or $b(\cdot) = b_0 + b_1$. *Soft control* will refer to application of the smaller gains, $k(\cdot) = k_0$ and $b(\cdot) = b_0$. The term *high gain* will refer to NPD control with values of the stiff gain, $k(\cdot) = k_0 + k_1$ and/or $b(\cdot) = b_0 + b_1$ sufficiently large that instability would result if the stiff control were applied continuously in linear PD control. With these definitions, the design guideline of Xu *et al.* (1994) may be stated: "choose stiff gains which are not high gains."

Writing

$$\dot{y} = C\,\dot{x} = C\,A\,x + C\,B\,u \tag{6}$$

it is possible to solve for $u(t)$ in terms of $x(t)$, giving

$$\begin{aligned} u(t) &= -k(\cdot)\,C\,x(t) - b(\cdot)\,C\,A\,x(t) - b(\cdot)\,C\,B\,u(t) \\ &= \frac{1}{1 + b(\cdot)\,C\,B}\Big(-k(\cdot)\,C - b(\cdot)\,C\,A\Big)x(t) \end{aligned} \tag{7}$$

Folding the control into the state derivative matrix gives the autonomous state space system:

$$\dot{x}(t) = \widehat{A}(t)\,x(t) \tag{8}$$

where

$$\widehat{A}(t) = A + B\,\frac{1}{1 + b(\cdot)\,C\,B}\Big(-k(\cdot)\,C - b(\cdot)\,C\,A\Big) \tag{9}$$

Feedback matrix $\widehat{A}(t)$ takes four possible values, depending on the values of $s_k(x)$ and $s_b(x)$.

2.2. Treatment of the Discontinuity: Differential Inclusions

Because of the switch functions in control law (5), the right-hand side of differential eqn. (8) may be discontinuous. The classical existence and uniqueness results for differential equations require the right-hand side be at least Lipschitz continuous and are not immediately applicable to differential equations which are discontinuous with respect to $x(t)$ (Filippov, 1988; Schevitz and Paden, 1994). Among others, Filippov and Roxin have established a framework which gives meaning to the solutions of differential equations with discontinuous right-hand sides and provides a generalized notion of Lyapunov stability (Filippov, 1964; 1988; Roxin, 1965a; 1965b; 1966). More recently, the Clarke generalized gradient (Clarke, 1983) has been employed by Paden and Sastry (1987) and Schevitz and Paden (1994) to develop the mechanics of establishing the existence and stability of solutions for systems which arise with variable structure control. The calculus of differential inclusions presented by Paden and Sastry (1987) has been applied by many authors (e.g., Chiacchiarini *et al.*, 1995; Heck, 1991; Hsu, 1990; Oh and Khalil, 1995; Subbarao and Iyer 1993).

Following Paden and Sastry (1987), given

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,t) \tag{10}$$

where

(i) $f(\cdot)$ is defined almost everywhere (which is to say that $f(\cdot)$ may be discontinuous on a region of Lebesgue measure 0),

(ii) $f(\cdot)$ is measurable in an open region $Q \subset \mathbb{R}^{n+1}$,

(iii) for all compact $D \subset Q$ there exists an integrable function $A(t)$ such that $\|f(t)\| \leq A(t)$ almost everywhere in D,

a vector function $x(t)$ on $[t_0\ t_1]$ is a solution to (10) in the sense of Filippov when:

(iv) $x(t)$ is absolutely continuous on $[t_0\ t_1]$;

(v) for almost all $t \in [t_0\ t_1]$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} \in K[f]\big(x(t)\big) \tag{11}$$

where

$$K[f](x) \equiv \bigcap_{\delta>0} \bigcap_{\mu N=0} \overline{\mathrm{co}}\, f\Big(B(x,\delta) - N, t\Big) \tag{12}$$

and $B(x, \delta)$ defines a ball in $\mathbb{R}^n$ around x of radius δ; $\overline{\mathrm{co}}$ indicates the convex closure over the set of values generated by applying f on the ball $B(\cdot)$ with subsets N removed. The intersection $\bigcap_{\mu N=0}$ denotes the intersection over all sets N of Lebesgue measure zero.

Equation (11) is a *differential inclusion* (Filippov, 1988). It is a differential equation with a set-valued right-hand side, and existence and uniqueness properties which are quite distinct from ODE's with Lipschitz-continuous right-hand sides. The term $K[f](x)$ is a set-valued function returning a single value at any point x at which $f(\cdot)$ is continuous, and the convex closure over limits of the derivative at points where $f(\cdot)$ is discontinuous. The set of points on which $f(x,t)$ is evaluated to form the convex closure is a ball of diminishing radius around x, less regions of measure 0, which, by condition (i), includes the region of discontinuity. The calculus of differential inclusions (Paden and Sastry, 1987) provides a means of calculating $K[f](x)$ for a broad class of functions $f(x,\, t)$, including those common to variable-structure control.

Many constructions of discontinuous control, including that studied here, give rise to ODE's with piecewise continuous right-hand sides. Such systems may be studied within the framework of eqns. (10)–(12) and conditions (i)–(v). The derivative $f(\cdot)$ is undefined at points of discontinuity, and condition (i) requires that the regions of discontinuity occupy zero volume in $\mathbb{R}^n$. This condition is satisfied for a piecewise continuous control where the discontinuity exists on finitely-many boundary regions of dimension $n-1$.

Conditions (ii) and (iv) serve to insure that the derivative in (10) is meaningful. The requirement of condition (iii), that $f(\cdot)$ be dominated by an integrable function, assures that the integral of $f(\cdot)$ is well-defined. Conditions (ii)–(iv) are satisfied for piecewise continuous control systems with bounded control action and physically meaningful $x(t)$. Finally, condition (v) is satisfied when $x(t)$ is a solution to the differential inclusion (Filippov, 1988).

A Lyapunov stability theory for differential inclusions is demonstrated by Filippov (1988). For a function $V(t,x) \in C^1$ the upper and lower derivatives due to the differential inclusion (11) are defined by (following (Filippov, 1988)):

$$\begin{aligned} \dot{V}^*(t,x) &\equiv \left(\frac{\mathrm{d}V}{\mathrm{d}t}\right)^* = \sup_{y \in k[f](x)} (V_t + \Delta V \cdot y) \\ \dot{V}_*(t,x) &\equiv \left(\frac{\mathrm{d}V}{\mathrm{d}t}\right)_* = \inf_{y \in k[f](x)} (V_t + \Delta V \cdot y) \end{aligned} \tag{13}$$

where V_t is the partial derivative $\partial V/\partial t$; ΔV is the gradient of v (recall that $V(\cdot)$ is not discontinuous) and y is an element from the set of possible derivatives of x given by the differential inclusion, $y \in K[f](x)$.

Theorem 1. (Filippov, 1988) Generalized Lyapunov stability.

Given:

(i) a differential inclusion, as described by eqn. (11);

(ii) the inclusion is defined on a closed domain $D(t_0 \le t < \infty,\ |x| \le \epsilon_0)$*;*

(iii) on D *are defined functions* $V(t,x) \in C^1$, $V_0(x) \in C$ *for which* $V(t,0) = 0$ *and* $0 < V_0(x) \le V(t,x)$.

Then:

1. *If* $\dot{V}^*(t,x) \le 0$ *in* D*, the solution* $x(t) \equiv 0$ *of the inclusion is stable;*
2. *If there exist functions* $V_1(x) \in C$, $w(x) \in C$ *defined on* D *and* $0 < V_0(x) \le V(t,x) \le V_1(x)$, $\dot{V}^*(t,x) \le -w(x) < 0$, $V_1(0) = 0$*, then the solution* $x(t) \equiv 0$ *is asymptotically stable.*

Using Filippov's generalized notion of Lyapunov stability, conditions for the assured asymptotic stability of NPD control will be established.

2.3. Lyapunov Stability of NPD Control

Let us define the standard quadratic Lyapunov function:

$$V(t,x) = V(x) = x^T(t)Px(t) \tag{14}$$

where P is a constant, symmetric, positive-definite matrix. Then the Lyapunov derivative is given by

$$\dot{V}(t) = x^T(t)\left(\widehat{A}^T(t)P + P\widehat{A}(t)\right)x(t) \tag{15}$$

If we require that k_0 and b_0 be chosen so as to stabilize $\{A, B, C\}$, then during intervals when $s_k(\cdot) = s_b(\cdot) = 0$, the Lyapunov derivative is given by

$$\dot{V}(t) = x^T(t)\left(\widehat{A}_L^T P + P\widehat{A}_L\right)x(t) = -x^T(t)Q_L x(t) \tag{16}$$

where

$$\widehat{A}_L = A + B\frac{1}{1+b_0CB}(-k_0C - b_0CA) \tag{17}$$

and

$$-Q_L = \widehat{A}_L^T P + P\widehat{A}_L \tag{18}$$

Standard results for linear systems assure that given $\widehat{A}_L$ stable, for any constant, symmetric, positive-definite matrix Q_L, a constant, symmetric, positive-definite matrix P can be found which is the solution to eqn. (18) (see e.g. De Carlo, 1989).

Expanding $\widehat{A}(t)$ in (15) using (8) gives

$$\widehat{A}(t) = \widehat{A}_L + \left(\frac{1}{1+b(\cdot)CB}\right)\left(-s_k(\cdot)k_1BC - s_b(\cdot)b_1BCA\right) \tag{19}$$

Folding $\widehat{A}(t)$ in (15) gives

$$\dot{V}(t) = -x^T(t)\left\{Q_L + s_k(\cdot)k_1\left(\frac{1}{1+b(\cdot)CB}\right)(C^TB^TP + PBC) + s_b(\cdot)b_1\left(\frac{1}{1+b(\cdot)C,B}\right)(A^TC^TB^TP + PBCA)\right\}x(t)$$

which may be written as

$$\dot{V}(t) = -x^T(t)\left\{Q_L + s_k(\cdot)Q_{k_1} + s_b(\cdot)Q_{b_1}\right\}x(t) \tag{20}$$

where

$$\begin{aligned} Q_{k_1}(\cdot) &= k_1\left(\frac{1}{1+b(\cdot)CB}\right)(C^TB^TP + PBC) \\ Q_{b_1}(\cdot) &= b_1\left(\frac{1}{1+b(\cdot)CB}\right)(A^TC^TB^TP + PBCA) \end{aligned} \tag{21}$$

Theorem 2. Stability of NPD control.

Consider a linear, time-invariant, SISO, strictly-proper system, as given by (3), and NPD control, as given by (5), and choose the switch functions so that

$$s_k(\cdot) = \begin{cases} 0 & \text{if} \quad (x^TQ_{k_1}x) < 0 \\ 1 \text{ or } 0 & \text{if} \quad (x^TQ_{k_1}x) \geq 0 \end{cases} \tag{22}$$

and

$$s_b(\cdot) = \begin{cases} 0 & \textit{if} \quad (x^T Q_{b_1} x) < 0 \\ 1 \ \textit{or} \ 0 & \textit{if} \quad (x^T Q_{b_1} x) \geq 0 \end{cases} \tag{23}$$

Then the system with NPD control will be globally asymptotically stable for any choice of $k_1 \geq 0$ and $b_1 \geq 0$.

Proof. The proof follows directly from Theorem 1. With $V(t, x)$ as given in (14), $V_0(x) = V_1(x) = V(x)$ and the domain D taken to be a large but finite cylinder in $\mathbb{R}^{n+1}$, we obtain, by the above choice of $s_k(\cdot)$ and $s_b(\cdot)$

$$\dot{V}^*(t, x) \leq -x^T(t) Q_L x(t) \equiv -w(x) < 0, \quad \|x\| \neq 0 \tag{24}$$

■

Theorem 2 establishes the stability of NPD control. It has the remarkable implication that we may chose k_1 and b_1 to be arbitrarily large positive values and that $s_k(t)$ and $s_b(t)$ may be chosen with great freedom, including finitely-many switches during the interval $(x^T Q_{k_1} x) > 0$. Simulations presented below demonstrate that very large values may be chosen.

2.4. Properties of $Q_{k_1}(\cdot)$ and $Q_{b_1}(\cdot)$

The following discussion addresses properties of $Q_{k_1}(\cdot)$ and selection of $s_k(x)$. The corresponding properties carry through directly for $Q_{b_1}(\cdot)$ and selection of $s_b(x)$. Following eqn. (21) and considering the restriction to SISO systems, $Q_{k_1}(x)$ may be written in the form

$$Q_{k_1}(x) = \alpha(x)(\Phi^T + \Phi) \tag{25}$$

where $\alpha(x) = k_1/(1 + b(x)CB)$ is a scalar and $\Phi = PBC$ is a rank-one matrix.

Lemma 1. Rank and eigenvalues of matrices of the form $(\Phi^T + \Phi)$.
A matrix of the form $Q = (\Phi^T + \Phi)$ where Φ is a square, rank-one matrix, will:

(i) be symmetric,

(ii) have rank 2, except for the special case that Φ is symmetric, in this case Q is of rank 1,

(iii) when Φ is non-symmetric, Q will have one positive and one negative eigenvalue.

Proof. Q is evidently symmetric. That the rank of Q for the general case (Φ not symmetric) will be 2 can be seen from the singular-value decomposition (SVD) of Φ. When $[U, S, V] = \text{svd}(\Phi)$,

$$\Phi = USV^T = \begin{bmatrix} u_1 & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & 0 & \\ & & 0 \end{bmatrix} \begin{bmatrix} v_1^T \\ \cdots \\ \cdots \end{bmatrix} = \sigma_1 u_1 v_1^T \tag{26}$$

The vector v_1 is the row space of Φ, u_1 is the row space of Φ^T, and $[v_1, u_1]$ spans the row space of Q. If $\Phi \neq \Phi^T$, then $v_1 \neq u_1$ and the rank of Q is two. Matrix Q is evidently of rank 1 when Φ is symmetric.

To establish proposition (iii), we may write

$$x^T Q_{k_1} x = \alpha(x)\left(x^T \Phi^T x + x^T \Phi x\right) = 2\alpha(x) x^T \Phi x$$

$$= 2\alpha(x)\sigma_1 x^T u_1 v_1^T x = 2\alpha(x)\sigma_1 (u_1^T x)(v_1^T x) \tag{27}$$

where u_1 and v_1 arise with the singular value decomposition, eqn. (26). When $u_1 \neq v_1$ (Φ not symmetric), x may be chosen such that the terms $(u_1^T x)$ and $(v_1^T x)$ are same or opposite signed. Choosing x such that the terms are same-signed establishes that one eigenvalue is positive; choosing x such that the terms are opposite-signed establishes the other eigenvalue to be negative. ■

Lemma 1 establishes that, when Φ is not symmetric, the state space is partitioned into three subspaces: two with nonzero measure corresponding to $(x^T Q_{k_1} x) > 0$ and $(x^T Q_{k_1} x) < 0$, and the zero-measure null space of Q_{k1}. These subspaces of $\mathbb{R}^n$ are respectively the region on which stiff control may be applied, the region on which it may not, and the boundary between the two.

3. Implementation

To implement nonlinear PD control in the form of eqn. (5) requires that switch function $s_k(\cdot)$ be evaluated. To assure negative definiteness of the Lyapunov derivative (21), application of stiff control must be restricted to the region of state space on which $(x^T Q_{k_1} x) > 0$. Determining and maximizing the region in which stiff control may be applied, specification of switch functions which do not depend on the full state vector and issues of practical implementation are addressed in this section. In what follows, we assume that Φ in (25) is not symmetric. If B, C and P are such that Φ is symmetric, a different choice of Q_L, resulting in a different P, will break the symmetry.

3.1. Choosing Q_L to Maximize $\{x : (x^T Q_{k_1} x) > 0\}$

For stability as established by Theorem 2, $s_k(x)$ must be chosen so that

$$\left\{x : s_k(x) = 1\right\} \subseteq \left\{x : (x^T Q_{k_1} x) > 0\right\} \tag{28}$$

The subspace in which $(x^T Q_{k_1} x) > 0$ is determined by the Q_{k_1} matrix, which is determined in (21) by the P matrix, which itself is determined by the designer-chosen Q_L matrix. To determine the largest volume in which stiff control can be applied,

Q_L should be chosen to maximize $\{x : (x^T Q_{k_1} x) > 0\}$. This can be done by a gradient search:

1. Define ϕ: the half-angle subtended by $\{x : (x^T Q_{k_1} x) < 0\}$,

$$\phi = \tan^{-1}(-\lambda_1/\lambda_2) \tag{29}$$

   where λ_1 and λ_2 are the negative and positive eigenvalues of Q_{k_1}, respectively.
2. Set $Q_L = W^T W$, where $W \in \mathbb{R}^{n \times n}$ will be adjusted in the gradient search.
3. Since ϕ is a scalar, $\mathrm{d}\phi/\mathrm{d}W$ is an $[n \times n]$ array, which can be approximated using the numeric first difference, and evaluating $\phi(Q_{k_1}(P(Q_L(W))))$, according to (29).
4. With $\mathrm{d}\phi/\mathrm{d}W$ in hand, any suitable gradient technique will find W which minimizes locally ϕ.

3.2. Control in the Eigenspace of Q_{k_1}

The eigenspace of Q_{k_1} is guaranteed to be no more than two-dimensional. An interesting possibility arising with the rank deficiency of Q_{k_1} is that the null space of Q_{k_1} could be aligned with the x-coordinate axes, such that some elements of x would not be required to compute $(x^T Q_{k_1} x)$.

When A is the feedback matrix of a stable system $\dot{x} = Ax$, a unique PD matrix P solving the matrix Lyapunov equation, (18), will correspond to each PD choice of Q_L. Though numerically better alternatives exist, conceptually P is given by

$$-\operatorname{vec} Q_L = [I \otimes A^T + A^T \otimes I] \operatorname{vec} P \tag{30}$$

where n is the order of the system, P is an $[n \times n]$ matrix, $\operatorname{vec} P$ is the $[n^2 \times 1]$ column vector of the columns of P, and $\otimes$ is the Kronecker product operator (De Carlo, 1989).

If we define

$$\begin{aligned} K_A &= [I \otimes A^T + A^T \otimes I] && \in \mathbb{R}^{n^2 \times n^2} \\ K_B &= [I \otimes (BC)^T + (BC)^T \otimes I] && \in \mathbb{R}^{n^2 \times n^2} \end{aligned} \tag{31}$$

then

$$\begin{aligned} -\operatorname{vec} Q_L &= K_A \operatorname{vec} P \\ -\operatorname{vec} Q_{k_1} &= K_B \operatorname{vec} P \end{aligned} \tag{32}$$

which gives

$$\operatorname{vec} Q_{k_1} = K_B K_A^{-1} \operatorname{vec} Q_L \tag{33}$$

Recalling that Q_{k_1} is not directly chosen but is determined by Q_L, the matrix $(K_B K_A^{-1})$ may be used in at least two ways:

1. If a proposed $\operatorname{vec} Q_{k_1}$ does not lie in the column space of $K_B K_A^{-1}$, no Q_L will exist which will establish the corresponding controller to be Lyapunov stable.
2. By translating specified directions in the space of Q_{k_1} into constraints on the space of $\operatorname{vec} Q_{k_1}$ and projecting these onto the space of $\operatorname{vec} Q_L$, it is possible to select Q_L which will give Q_{k_1} orthogonal to specified directions.

Given a proposed switching rule which can be represented by $s_k(\cdot) = ((x^T Q_{k_1} x) \geq 0)$, the first use is straightforward.

The second use is carried out by selecting certain directions, X_p, in state space to which $(x^T Q_{k_1} x)$ is to be made insensitive. The matrix Q_L must be chosen so that each row (and therefore column) of Q_{k_1} is orthogonal to X_p. The orthogonality requirement can be translated to the space of $\operatorname{vec} Q_{k_1}$ by creating matrices $M_i = [0, \cdots, X_p, \cdots, 0]$ which have X_p as their i-th column. For each row of Q_{k_1} to be orthogonal to X_p, we must have

$$\operatorname{vec} Q_{k_1} \perp V_i, \qquad i \in \{1, \ldots, N\} \tag{34}$$

where $V_i = \operatorname{vec}(M_i + M_i^T) \in \mathbb{R}^{n^2 \times 1}$ are the constraint vectors in the $[n^2 \times 1]$ space of $\operatorname{vec} Q_{k_1}$. Equation (34) will be satisfied if

$$(\operatorname{vec} Q_L)^T (K_B K_A^{-1})^T V_i = 0, \qquad i \in \{1, \ldots, N\} \tag{35}$$

which is given by orthogonalizing $\operatorname{vec} Q_L$ with respect to $\{(K_B K_A^{-1})^T V_i\}$. This can be accomplished by first determining a set of orthogonal vectors spanning the subspace of $\{(K_B K_A^{-1})^T V_i\}$ using Gram-Schmidt orthogonalization, and then orthogonalizing $\operatorname{vec} Q_L$ with respect to these spanning vectors.

The matrix Q_L given by this process is not guaranteed to be PD. However, in an example described in Section 3.4 below, by search on randomly selected Q_L, candidates which can satisfy (35) and remain PD have been found. It is interesting to note that if a $\operatorname{vec} Q_L^o$ is known which is orthogonal to the vectors $(K_B K_A^{-1})^T V_i$, a larger class of orthogonal Q_L is given by

$$\operatorname{vec} Q_L = \operatorname{vec} Q_L^o + \operatorname{vec} Q_L^N \tag{36}$$

where $\operatorname{vec} Q_L^N$ is an element of the null space of $(K_B K_A^{-1})$. Unfortunately, the space of PD matrices is not a vector space, and so eqn. (36) does not directly provide a means to produce a PD matrix Q_L.

3.3. An Example Application

Xu *et al.* (1993; 1994; 1995) demonstrate NPD control of a robotic hand to quench oscillations arising with the transition from non-contact to contact and force control. Their application is used here to demonstrate the present results. The manipulator with force sensor, low pass filtering of the force error signal and force rate feedback may be represented by the state space model:

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{k_r}{m_1} & \frac{k_r}{m_1} & -\frac{b_r}{m_1} & \frac{(b_r - b_0 k_s)}{m_1} & \frac{1}{m_1 \beta_l} \\ \frac{k_r}{m_2} & -\frac{(k_r + k_s)}{m_2} & \frac{b_r}{m_2} & -\frac{(b_r + b_s)}{m_2} & 0 \\ 0 & 0 & 0 & 0 & \alpha_l \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \alpha_l \beta_l \end{bmatrix} u(t) \tag{37}$$

$$y(t) = \begin{bmatrix} 0 & k_s & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \end{bmatrix} u(t)$$

where the states are as follows:

$$x(t) = \begin{bmatrix} x_1 \; x_2 \; \dot{x}_1 \; \dot{x}_2 \; x_3 \end{bmatrix}^T, \qquad \begin{cases} x_1 : & \text{position of the acutator} \\ x_2 : & \text{position of the output} \\ x_3 : & \text{state of the low-pass filter} \end{cases} \tag{38}$$

and where the model parameters are given in Table 1. The model is fifth-order because the low-pass filter, described in (Xu *et al.*, 1994), has been folded into the state space description. The parameter α_{lpf} determines the low-pass filter pole location, and the scaling parameter β_{lpf} has been added to balance the A matrix of eqn. (37). The parameter b_0 in eqn. (37) and Table 1 is the force rate feedback gain used by the authors.

Table. 1. Parameters of the example NPD application (from (Xu *et al.*, 1995), parameter β_l added).

m_1	119.4	[kg]	m_2	13.24	[kg]
k_r	110100	[N/m]	k_s	11010	[N/m]
b_r	10	[N-s/m]	b_s	10	[N-s/m]
α_l	40.0π	[1/s]	β_l	$0.1/\alpha_l$	$[\cdot]$
b_0	0.01	[N-s/m]			

The output, $y(t)$, is the contact force, and is given by $y(t) = k_s x_2(t)$. The force rate is given by $\dot{y}(t) = k_s \dot{x}_2(t)$. A detailed description of the system, with mechanical schematic and system block diagrams, can be found in (Xu *et al.*, 1994; 1995).

3.4. Three Example Controllers

Three NPD controllers are presented here in simulation. These controllers are represented by the Q_{k_1} matrices: $Q_{k_1}^{Q_L}$, $Q_{k_1}^{ZFR}$ and $Q_{k_1}^{XHM}$. Each of these Q_{k_1} matrices determines the switching function according to

$$s_k(x) = \left(\left(x^T Q_{k_1} x \right) \geq 0 \right) \tag{39}$$

$Q_{k_1}^{Q_L}$: A controller derived from $Q_L = I$

The first controller was derived directly from (21), with Q_L chosen to have balanced eigenvalues. Setting $Q_L = I$ and solving eqns. (18) and (21) gives the matrix

$$Q_{k_1}^{Q_L} = \begin{bmatrix} 0 & -0.4010 & 0 & 0 & 0 \\ -0.4010 & 0.8384 & 0.0276 & -0.0030 & 0.0032 \\ 0 & 0.0276 & 0 & 0 & 0 \\ 0 & -0.0030 & 0 & 0 & 0 \\ 0 & 0.0032 & 0 & 0 & 0 \end{bmatrix} \tag{40}$$

which determines the switching function via eqn. (39).

The balanced eigenvalues in Q_L was made a consideration when a controller optimized as described in Section 3.1 performed poorly. While the optimization increased the volume of stiff control by approximately 20%, it resulted in Q_L with eigenvalues ranging from 10^2 to 10^{-4}. The small eigenvalues correspond to components of the response for which $V(t)$ decays slowly (cf. (16)). Empirically, balanced eigenvalues in Q_L were observed to give better performance.

$Q_{k_1}^{ZFR}$: A controller which does not depend on force rate

An important liability of the proposed NPD control is that evaluation of the switch function, eqn. (39), may require knowledge of the full state vector. In the example application, actuator velocity and contact force rate are unsensed (corresponding to $\dot{x}_1$ and $\dot{x}_2$ in (38)). The force rate was chosen as the more difficult sensing or estimation challenge, and correspondingly the row space of Q_{k_1} was orthogonalized with respect to $\dot{x}_2$. The resulting controller is given by:

$$Q_L = \begin{bmatrix} 1.187 & 0.511 & 0.641 & 0.627 & 1.526 \\ 0.511 & 1.971 & 1.286 & 0.709 & 1.707 \\ 0.641 & 1.286 & 1.109 & 0.666 & 1.536 \\ 0.627 & 0.709 & 0.666 & 1.134 & 1.129 \\ 1.526 & 1.707 & 1.536 & 1.129 & 3.052 \end{bmatrix} \tag{41}$$

and

$$Q_{k_1}^{ZFR} = \begin{bmatrix} 0 & -0.413 & 0 & 0 & 0 \\ -0.413 & 0.826 & 0.056 & 0 & 0.008 \\ 0 & 0.056 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0.008 & 0 & 0 & 0 \end{bmatrix} \tag{42}$$

where the superscript ZFR (Zero Force Rate) indicates that this is the controller independent of force rate.

The controller of Xu *et al.* (1995)

The third controller is that presented in eqn. (2) and demonstrated by Xu *et al.* (1993; 1994; 1995). In the limit as $\alpha \to \infty$, the control law of Xu *et al.* takes the form of eqn. (5), with

$$s_k(\cdot) = \begin{cases} 1 & \text{if} \quad \text{sign}\big(e(t)\big) = \text{sign}\big(\dot{e}(t)\big) \\ 0 & \text{if} \quad \text{sign}\big(e(t)\big) \neq \text{sign}\big(\dot{e}(t)\big) \end{cases} \tag{43}$$

where $e(t) = y_r - y(t)$. This control law may be written in the form of eqn. (39), where the Q_{k_1} matrix is

$$Q_{k_1}^{XHM} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{44}$$

The possible existence of a function $V = x^T P x$ which would establish this control law to be Lyapunov stable can be explored by examining whether $\text{vec}\, Q_{k_1}^{XHM}$ lies in the column space of $(K_B K_A^{-1})$. It does not, and thus there is no quadratic function of state which establishes this controller to be Lyapunov stable. This result does not establish instability for this controller. Simulations presented below show stability for moderate values of k_1 and instability for large (high gain) k_1. The question remains open as to whether stability can be demonstrated for this system with moderate k_1 and the control law of Xu *et al.* (1993; 1994; 1995).

3.5. Simulations

The simulations presented below show response of the system to each of the three controllers of Section 3.4. The controller and gain combinations tested are listed in Table 2. In each of Figs. 3–9 two plots are shown; they are the contact force, $y(t)$, the applied control $u(t)$.

Table. 2. Table of simulations presented.

Controller	Stiff Control Gain $k_1 = 0$	$k_1 = 1$	$k_1 = 20$
$Q_{k_1}^{Q_L}$	Fig. 3	Fig. 4	Fig. 5
$Q_{k_1}^{ZFR}$	—	Fig. 6	Fig. 7
$Q_{k_1}^{XHM}$	—	Fig. 8	Fig. 9

In Fig. 3 the response of a system with $k_1 = 0$ is shown. The NPD control is turned off and standard PD control is demonstrated. This case is similar to that presented by Xu *et al.* (1995) and is presented as a control case.

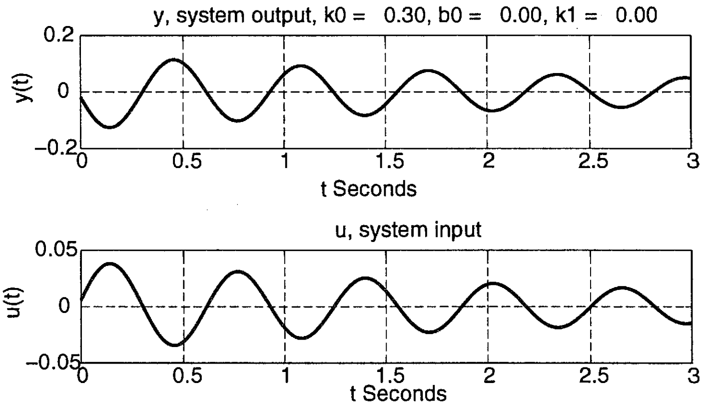


Fig. 3. System response for standard PD control.

Figures 4 and 5 illustrate the system response to the controller obtained by setting $Q_L = I$, eqn. (40). Of the three, this controller shows the greatest degree of damping. In Fig. 5, corresponding to $k_1 = 20$, it is seen that the slower mode of the system response is not visible after a single cycle and that the faster mode is also effectively damped by the NPD control. While damping of the faster mode is consistent with theory, it was anticipated that the discontinuous NPD control would excite rather than damp this mode.

Figures 6 and 7 illustrate the system response to the controller obtained by orthogonalizing Q_{k_1} with respect to the force rate. For $k_1 = 1$ the response is quite similar to that of the first controller. For $k_1 = 20$, it is seen that damping of the faster system mode is less effective than that of the first controller, partially offsetting the advantage of implementation with no estimate of force rate.

Figures 8 and 9 illustrate the system response to the controller of Xu *et al.* (1995), modified as described in Section 3.4. With $k_1 = 1$ the controller performs well, showing approximately the damping of the other NPD controllers. With $k_1 = 20$ the controller is unstable.

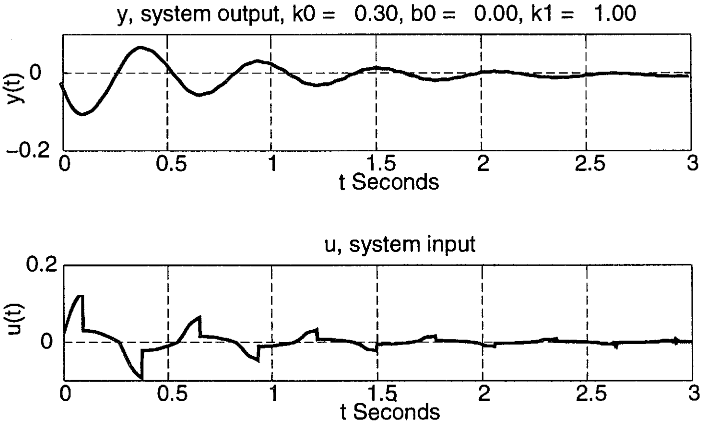


Fig. 4. System response with $Q_L = I,\ k_1 = 1$.

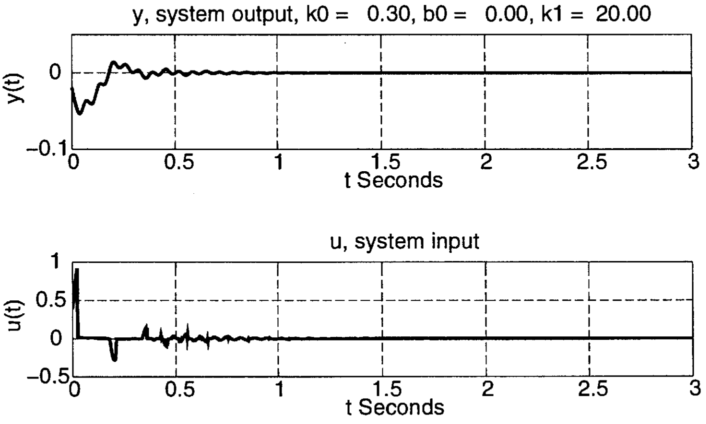


Fig. 5. System response with $Q_L = I,\ k_1 = 20$.

In Fig. 10 the damping rate demonstrated by five simulation runs of each of the three controllers is summarized. The damping rates presented in Fig. 10 were obtained by fitting an exponential curve through the peaks to the successive cycles of $y(t)$; damping rate values below zero correspond to instability. The controllers are identical for $k_1 = 0$ and have a damping rate of 0.142 [sec^{-1}]. The three controllers show comparable improvement for low and moderate values of k_1, but quite different performance as k_1 is increased. The controller given by $Q_L = I$ showed the greatest damping above $k_1 = 10$.

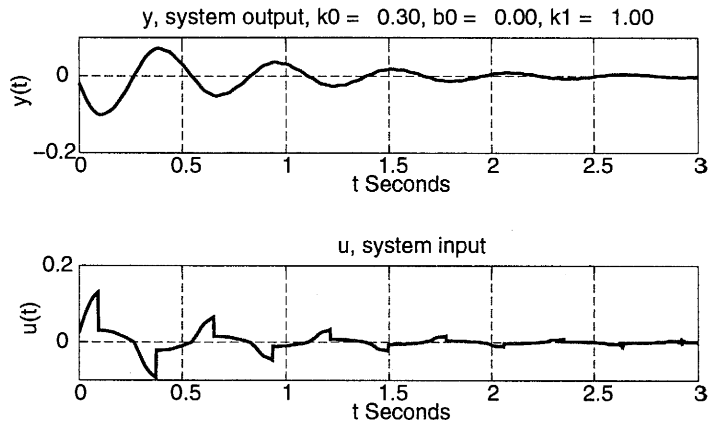


Fig. 6. System response with Q_{k_1} independent of force rate, $k_1 = 1$.

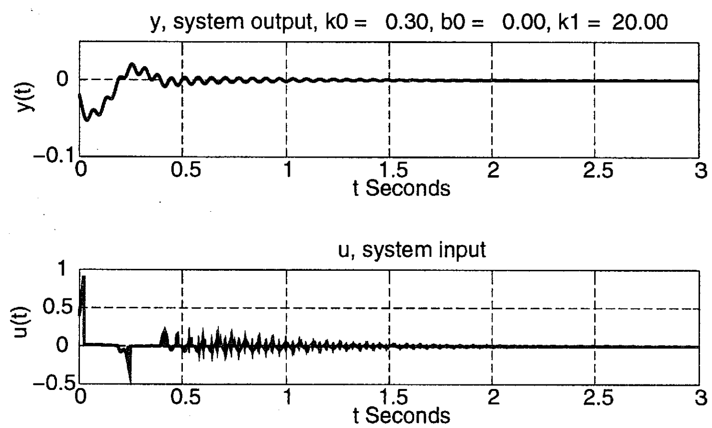


Fig. 7. System response with Q_{k_1} independent of force rate, $k_1 = 20$.

4. Conclusions

A Lyapunov proof of stability has been presented for NPD regulation of LTI SISO systems and a particular construction of NPD control. The result establishes that on predetermined subspaces of state space, arbitrarily large proportional and rate gains can be applied. Simulations, as well as experiments reported in the literature, show that application of the larger gains can provide increased damping.

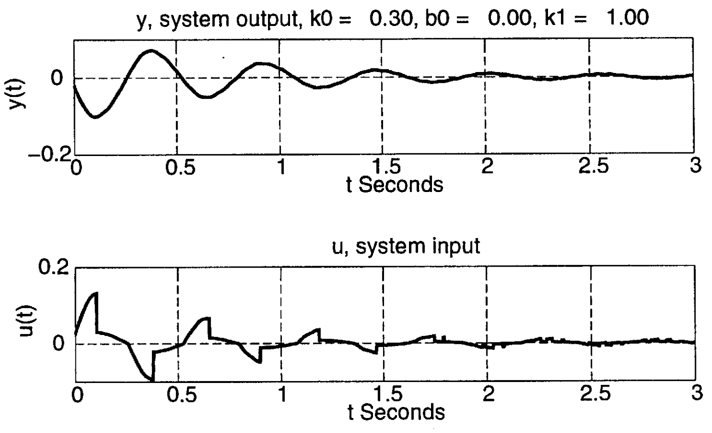


Fig. 8. System response with control of (Xu *et al.*, 1995), eqn. (2), in the limit as $\alpha \to \infty$, $k_1 = 1$ ($\alpha = 100$ and $k_1 = 1$ demonstrated by Xu *et al.* (1995)).

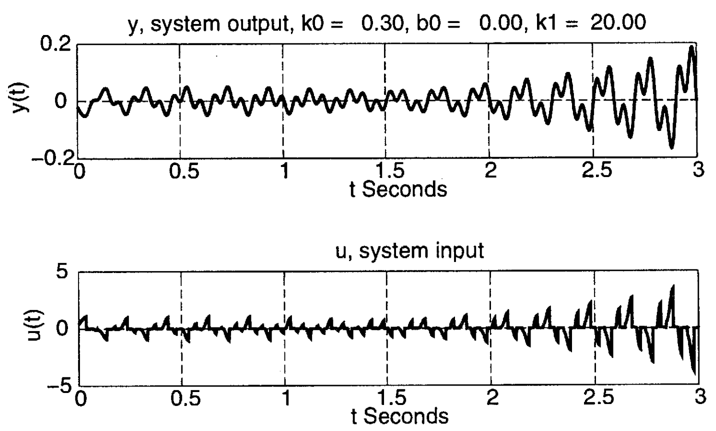
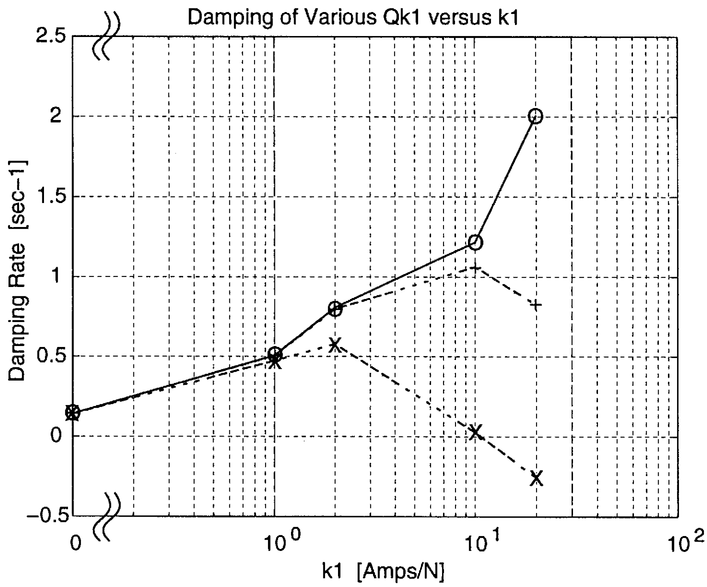


Fig. 9. System response with control of (Xu *et al.*, 1995), eqn. (2), in the limit as $\alpha \to \infty$, $k_1 = 20$.

The switching function which determines application of the stiff gains has been shown to be no more than rank two, even for higher-order systems. This opens the possibility that only two states might be required to compute the NPD control. A fifth-order application has been demonstrated for which only four states are required to compute the switch function.

The present study raises many questions regarding NPD control. The stability proof presented will carry over when the higher gains are allowed to vary during the interval of stiff control. The implications and utility of time varying stiff gains is the subject of continuing investigation. Questions of robustness are also central to practical application of this control strategy. As these and other questions are resolved, NPD control may enter the growing list of nonlinear controller structures which offer performance advantages when applied to approximately linear systems.



Legend:

- o Q_{k_1} from $Q_L = I$, $Q_{k_1}^{Q_L}$;
- + Q_{k_1} independent of force rate, $Q_{k_1}^{ZFR}$;
- × Q_{k_1} corresponding to control of (Xu *et al.*, 1995) $Q_{k_1}^{XHM}$.

Fig. 10. Damping rates achieved by three constructions of NPD control.

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