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TIME-OPTIMAL CONTROL OF REDUNDANT MANIPULATORS

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A method of finding optimal controls for both non-redundant and redundant manipulators is considered. It is based on a strict convexifying of the original control set with any desired accuracy in the Hausdorff sense. The optimal controls thus obtained are shown to converge (weakly) to a bang-bang solution. Furthermore, this method produces also controls which may be directly used to provide nominal inputs in on-line manipulator control. A simple numerical example involving a three-degree-of-freedom revolute kinematically redundant manipulator is presented.

1. Introduction

The time-optimal planning of motions of kinematically redundant manipulators operating in a manufacturing environment becomes very important to increase productivity. Several works have addressed this problem with hard constraints on controls. Numerical iterative algorithms using Pontryagin's maximum principle (Sakawa and Shindo, 1980; Weinreb and Bryson, 1985) and a method of (Mayne and Polak, 1975) have proposed strong variations based on the maximum principle to solve the time-optimal control problem. Another iterative method is the slack-variable method (Miele, 1975). It handles the inequality constraints as equality ones. This increases the number of unknown functions to be found by introducing the slack-variables. Another approach to minimum-time problems is to assume that the optimal controls are bang-bang. Then they are parametrized by their switching times and parameter optimization techniques are used to find them (Kahn and Roth, 1971; Vlassenbrok and van Dooren, 1988). A method involving joint-space tessellation, a dynamic time-scaling algorithm and a graph search has been used in (Sahar and Hollerbach, 1986) to find an optimal solution.

Most of the above works find time-optimal trajectories which are typically (by treating the torques/forces as the control variables) at the torque/force limit at least for one joint. Therefore there is no flexibility to take care of disturbances or modelling discrepancies. Moreover, such strategies are physically undesirable due to typical discontinuities at switching times and non-negligible actuator dynamics.

There are several approaches eliminating some of the above drawbacks. Asada and Slotine (1986) have reduced the assumed torque/force bounds to leave room

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for a closed-loop control action. In (Slotine and Spong, 1985) an on-line adjustment scheme is proposed, where the trajectory is modified on-line by changing the reference trajectory. The modified trajectory is executed at the same time as the reference one. Alternatively, in (Chen and Desrochers, 1988; Sakawa and Shindo 1980; Shiller, 1994) a quadratic energy term has been added to the performance index (or the Hamiltonian) to smooth the controls. However, modifying the Hamiltonian (or performance index) may result in conservative solutions since the controls thus obtained lose their bang-bang nature.

To avoid these shortcomings, a solution to the time-optimal control problem is proposed in this study, where the optimal controls are continuous and leave room for feedback actions. This involves strict convexifying the original control set with any accuracy in the Hausdorff sense. The optimal controls obtained by means of Pontryagin's maximum principle are shown to converge weakly to a bang-bang solution, as the above approximation approaches the original control set in the Hausdorff sense.

The paper is organized as follows. Section 2 formulates the problem of time-optimal control. The application of Pontryagin's maximum principle with a strictly convex control set is described in Section 3. General transversality conditions are given in Section 4, which reduces the considered control problem to a two-point boundary-value problem. Section 5 provides a computer example involving a three-degree-of-freedom planar manipulator.

2. Problem Formulation

Consider the following dynamic model of a redundant manipulator:

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{F}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{u} \tag{1}$$

where $\boldsymbol{q} \in \mathbb{R}^n$ is the vector of generalized coordinates, $n > 1$, $\boldsymbol{M}(\boldsymbol{q}) \in \mathbb{R}^{n\times n}$ denotes the inertia matrix which is symmetric positive definite, $\boldsymbol{F}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ stands for the n-dimensional vector of Coriolis, centrifugal and gravity forces, $\boldsymbol{u} = (u_1, u_2, \dots, u_n)^T$ is the n-dimensional vector of control inputs. Without loss of generality, the controls being functions of time are bounded in magnitude:

$$-1 \le u_i \le 1 \tag{2}$$

where $i = 1, \dots, n$.

The task of the manipulator is to transfer the end-effector in the m-dimensional work space, where $m < n$, from the initial position

$$\begin{cases} \boldsymbol{P}(\boldsymbol{q}(0)) - \boldsymbol{P}_0 = \boldsymbol{0} \\ \dot{\boldsymbol{q}}(0) = \boldsymbol{0} \end{cases} \tag{3}$$

where $\boldsymbol{P} : \mathbb{R}^n \to \mathbb{R}^m$, $\boldsymbol{P}(\boldsymbol{q})$ is a kinematic model of the manipulator consisting of m non-linear, scalar equations, $\boldsymbol{P}_0 \in \mathbb{R}^m$ stands for a given initial position of the end-effector in the work space, to the final position

$$\begin{cases} \boldsymbol{P}(\boldsymbol{q}(T)) - \boldsymbol{P}_T = \boldsymbol{0} \\ \dot{\boldsymbol{q}}(T) = \boldsymbol{0} \end{cases} \tag{4}$$

$\boldsymbol{P}_T \in \mathbb{R}^m$ being a given final location of the end-effector. Here T denotes the unknown final time of task execution. The manipulator motion should be realized in minimum time under constraints (2)–(4).

The above task may be expressed in state-space form as follows:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(\boldsymbol{x}) + \boldsymbol{B}(\boldsymbol{x})\boldsymbol{u} \tag{5}$$

where $\boldsymbol{x} = (\boldsymbol{q}\ \dot{\boldsymbol{q}})^T \in \mathbb{R}^{2n}$,

$$\boldsymbol{A}(\boldsymbol{x}) = \begin{pmatrix} \dot{\boldsymbol{q}} \\ -\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \end{pmatrix}, \quad \boldsymbol{B}(\boldsymbol{x}) = \begin{pmatrix} 0 \\ \boldsymbol{M}^{-1}(\boldsymbol{q}) \end{pmatrix}$$

with the boundary constraints

$$\boldsymbol{\Phi}_0\big(\boldsymbol{x}(0)\big) = \boldsymbol{0}, \quad \boldsymbol{\Phi}_T\big(\boldsymbol{x}(T)\big) = \boldsymbol{0} \tag{6}$$

where

$$\boldsymbol{\Phi}_0\big(\boldsymbol{x}(0)\big) = \begin{pmatrix} \boldsymbol{P}\big(\boldsymbol{q}(0)\big) - \boldsymbol{P}_0 \\ \dot{\boldsymbol{q}}(0) \end{pmatrix}, \quad \boldsymbol{\Phi}_T\big(\boldsymbol{x}(T)\big) = \begin{pmatrix} \boldsymbol{P}\big(\boldsymbol{q}(T)\big) - \boldsymbol{P}_T \\ \dot{\boldsymbol{q}}(T) \end{pmatrix}$$

The performance index to be minimized equals

$$I(\boldsymbol{u}) = \int_0^T 1\,\mathrm{d}t \tag{7}$$

The relations (2), (5)–(7) form a time-optimal control problem with hard control bounds and terminal state constraints. Its solution is proposed in the next section.

3. Application of Pontryagin's Maximum Principle

In order to find optimal controls for the problem defined by (2) and (5)–(7), the Pontryagin maximum principle can be used. However, due to the weak convexity of the control set (2), its direct application to find numerically the solution seems difficult. The accurate controls thus obtained are of bang-bang type. As a consequence, they leave no room for feedback actions to compensate for e.g. model errors and disturbances. On the other hand, the above controls are physically unrealizable due to discontinuities at the switching points and non-negligible actuator dynamics. This may then give rise to large tracking errors which lead to large trajectory deviations in on-line control.

The aim of this section is to eliminate the above-mentioned drawbacks by means of a strong convexifying of the control set (2). This is realized based on the following dependence:

$$\sum_{i=1}^{n} u_i^{2k} - 1 \leq 0 \tag{8}$$

where k is a fixed, positive integer. Note that the control set (8) may approximate the accurate one (given by inequalities (2)) with any desired accuracy in the Hausdorff sense.

The largest approximation error arises on the diameter of the n-dimensional hypercube (2). After simple calculations, omitted herein, it equals $\varepsilon = 1 - n^{-1/2k}$. Clearly, $\varepsilon \to 0$ as $k \to \infty$. The properties of that approximation become apparent in conjunction with Pontryagin's maximum principle. In order to apply it for the control problem defined by (5)–(7), the Hamiltonian is introduced:

$$H = -1 + \langle \boldsymbol{A}(\boldsymbol{x}), \boldsymbol{\Psi} \rangle + \sum_{i=1}^{n} S_i(\boldsymbol{x}, \boldsymbol{\Psi}) u_i \tag{9}$$

where $(S_1(\boldsymbol{x}, \boldsymbol{\Psi}), (S_2(\boldsymbol{x}, \boldsymbol{\Psi}), \dots, (S_n(\boldsymbol{x}, \boldsymbol{\Psi}))^T = \boldsymbol{B}^T(\boldsymbol{x})\boldsymbol{\Psi}$, $S_i(\boldsymbol{x}, \boldsymbol{\Psi})$ is the i-th switching function, $\boldsymbol{\Psi}$ denotes the $2n$-dimensional vector of adjoint variables, $\boldsymbol{\Psi} = (\Psi_1, \dots, \Psi_{2n})^T$. The optimal control must maximize H while taking into account the constraints (8). Due to the linearity with respect to $\boldsymbol{u}$ and the strong convexity of the control set (8), a unique maximum of the Hamiltonian is attained on the boundary of this set. Hence the problem to be solved is defined as follows:

$$\max_{(u_1, u_2, \dots, u_n)} \{H\} \tag{10}$$

subject to the constraints

$$1 - \sum_{i=1}^{n} u_i^{2k} = 0 \tag{11}$$

Introducing

$$H^\star = H + \lambda \left(1 - \sum_{i=1}^{n} u_i^{2k} \right) \tag{12}$$

where λ is a Lagrange multiplier, yields the necessary condition for a maximum of H in the form

$$\begin{cases} \dfrac{\partial H^\star}{\partial \boldsymbol{u}} = \boldsymbol{0} \\ \dfrac{\partial H^\star}{\partial \lambda} = 0 \end{cases} \tag{13}$$

Applying the coordinate description and omitting for simplicity the terms $\boldsymbol{x}$ and $\boldsymbol{\Psi}$ in the switching functions lead to

$$S_i - 2k\lambda u_i^{2k-1} = 0, \qquad 1 - \sum_{i=1}^{n} u_i^{2k} = 0 \tag{14}$$

where $i = 1 \dots, n$. Next, by using boundary constraints (11), the Lagrange multiplier in (14) is eliminated. There exist two possible solutions for λ, namely

$$\lambda = +\frac{1}{2k} \left(\sum_{i=1}^{n} S_i^{\frac{2k}{2k-1}} \right)^{\frac{2k-1}{2k}} \quad \text{and} \quad \lambda = -\frac{1}{2k} \left(\sum_{i=1}^{n} S_i^{\frac{2k}{2k-1}} \right)^{\frac{2k-1}{2k}}$$

It is easy to see that the Lagrange multiplier λ with a positive sign maximizes the Hamiltonian. Accordingly, the optimal controls may be expressed in the following explicit form:

$$u_i = \frac{(S_i)^{\frac{1}{2k-1}}}{\left(\sum_{i=1}^{n}(S_i)^{\frac{2k}{2k-1}}\right)^{\frac{1}{2k}}} \tag{15}$$

where $i = 1, \ldots, n$.

The results presented in (Sontag and Sussman, 1986) imply $\sum_{i=1}^{n}(S_i)^{\frac{2k}{2k-1}} > 0$. Hence all the controls (15) are well-defined. The next property is that they are continuous functions of time. This is a consequence of the fact that x and Ψ are continuous mappings with respect to time. Due to strong convexity, the controls (15) are also defined on singular arcs. It is interesting to consider the limit behaviour of these controls. Without loss of generality, all the switching functions are assumed to be non-zero for a fixed $t \in [0, T]$. If $k \to \infty$, then $(S_i)^{\frac{1}{2k-1}}$ converges to $\mathrm{sgn}(S_i)$. In order to find the limit of the denominator, it is easier to take its logarithm. Since $\log(\sum_{i=1}^{n}(S_i)^{\frac{2k}{2k-1}}) \xrightarrow[k\to\infty]{} \log(\sum_{i=1}^{n}|S_i|)$, this implies that $\log(\sum_{i=1}^{n}(S_i)^{\frac{2k}{2k-1}})/2k \xrightarrow[k\to\infty]{} 0$ and finally $u_i \xrightarrow[k\to\infty]{} \mathrm{sgn}(S_i)$. Hence, the following result has been established.

Lemma 1. *If $k \to \infty$, then the controls (15) retain the structure of the controls of the original problem.*

By the assumption of the non-zero switching functions for a fixed $t \in [0, T]$, the following inequality results:

$$|u_i| < 1$$

where $i = 1, \ldots, n$. Thus the optimal controls (15) leave room for feedback actions to compensate for e.g. model errors and disturbances in on-line control. In order to find a time-optimal trajectory of the redundant manipulator, the controls (15) are substituted into the Hamiltonian (9). The necessary conditions for optimality result in the complete system of differential equations:

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial \Psi} \\ \dot{\Psi} = -\dfrac{\partial H}{\partial x} \end{cases} \tag{16}$$

It is well-defined when $x(0)$, $\Psi(0)$ and T are known. Hence $4n + 1$ scalar dependencies relating these quantities are necessary in order to fully specify system (16). They are given in the next section.

4. Boundary and Transversality Conditions

As is known from Section 2, the boundary conditions (6) constitute $2(n+m)$ scalar nonlinear equations. The aim is to obtain other $2(n-m)+1$ scalar boundary dependencies which, together with (6), will fully specify the system (16). The application of the Hamiltonian results in one scalar equation

$$H_{t=T} = 0 \tag{17}$$

The use of the transversality conditions provides $2(n-m)$ dependencies. On account of (6), the transversality conditions assume the following form:

$$\begin{cases} \langle \Psi(0), \mathrm{d}\boldsymbol{x}(0) \rangle = 0 \\ \langle \Psi(T), \mathrm{d}\boldsymbol{x}(T) \rangle = 0 \end{cases} \tag{18}$$

where $\mathrm{d}\boldsymbol{x}(0)$ and $\mathrm{d}\boldsymbol{x}(T)$ are the vectors tangent to the hypersurfaces $\boldsymbol{\Phi}_0(\boldsymbol{x}(0)) = \boldsymbol{0}$ and $\boldsymbol{\Phi}_T(\boldsymbol{x}(T)) = \boldsymbol{0}$, respectively. The variations of the boundary conditions (6) at $\boldsymbol{x}(0)$ and $\boldsymbol{x}(T)$ equal

$$\begin{cases} \boldsymbol{J}_0 \, \mathrm{d}\boldsymbol{x}(0) = \boldsymbol{0} \\ \boldsymbol{J}_T \, \mathrm{d}\boldsymbol{x}(T) = \boldsymbol{0} \end{cases} \tag{19}$$

where $\boldsymbol{J}_0 = \partial \boldsymbol{\Phi}_0(\boldsymbol{x}(0))/\partial \boldsymbol{x}_0$, $\boldsymbol{J}_T = \partial \boldsymbol{\Phi}_T(\boldsymbol{x}(T))/\partial \boldsymbol{x}_T$. The matrices $\boldsymbol{J}_0$ and $\boldsymbol{J}_T$ are of dimensions $(m+n) \times 2n$. Let $\boldsymbol{J}_0$ and $\boldsymbol{J}_T$ have full rank, i.e. $\mathrm{rank}(\boldsymbol{J}_0) = m+n = \mathrm{rank}(\boldsymbol{J}_T)$. Hence it is possible to select from each of the above matrices $m+n$ linearly independent columns which are, without loss of generality, the first columns of $\boldsymbol{J}_0$ and $\boldsymbol{J}_T$, and to form respectively non-singular $(m+n)\times(m+n)$ matrices $\boldsymbol{J}_0^R$ and $\boldsymbol{J}_T^R$ (otherwise another set of independent columns should be chosen). The other columns of $\boldsymbol{J}_0$ and $\boldsymbol{J}_T$ constitute $(m+n)\times(m-n)$ matrices $\boldsymbol{J}_0^F$ and $\boldsymbol{J}_T^F$, respectively. Following the method presented in (Galicki, 1992), general transversality conditions are derived in the form

$$\begin{cases} \left[\left((\boldsymbol{J}_0^R)^{-1} \; \boldsymbol{J}_0^F \right)^T \; - \boldsymbol{I}_{n-m} \right] \Psi(0) = \boldsymbol{0} \\ \left[\left((\boldsymbol{J}_T^R)^{-1} \; \boldsymbol{J}_T^F \right)^T \; - \boldsymbol{I}_{n-m} \right] \Psi(T) = \boldsymbol{0} \end{cases} \tag{20}$$

where $\boldsymbol{I}_{n-m}$ denotes the $(m-n)\times(m-n)$ identity matrix. Taking into account eqns. (6), (17) and (20), a system of $4n+1$ independent boundary and transversality conditions is obtained:

$$\begin{pmatrix} \boldsymbol{\Phi}_0(x(0)) \\ \boldsymbol{\Phi}_T((T)) \\ H_{t=T} \\ \left[((\boldsymbol{J}_0^R)^{-1} \boldsymbol{J}_0^F) - \boldsymbol{I}_{n-m} \right] \Psi(0) \\ \left[((\boldsymbol{J}_T^R)^{-1} \boldsymbol{J}_T^F) - \boldsymbol{I}_{n-m} \right] \Psi(T) \end{pmatrix} = 0 \tag{21}$$

Summarizing, optimal controls (15) result from solving the two-point boundary-value problem specified by differential equations (16) and $4n + 1$ boundary transversality conditions (21). A numerical procedure has been proposed in (Galicki, 1992) to solve the system (21) in order to find its roots which uniquely determine the optimal controls (15).

5. Computer Example

A planar kinematically redundant manipulator of $n = 3$ revolute kinematic pairs, shown in Fig. 1 is considered. The data used in the numerical example are as follows:

- link lenghts $l_1 = 2.5$, $l_2 = 2.0$, $l_3 = 1.0$;
- link masses $m_1 = 5.0$, $m_2 = 4.0$, $m_3 = 2.0$;
- coefficients of friction $t_1 = 1.0$, $t_2 = 1.0$, $t_3 = 1.0$;
- lower and upper control limits:
  $u_1^l = -80.0$, $u_1^u = 80.0$, $u_2^l = -90.0$, $u_2^u = 90.0$, $u_3^l = -8.0$, $u_3^u = 8.0$.

The task is to transfer the end-effector from a given starting point $\boldsymbol{P}_s = (1.0, 1.0)$ to a target one $\boldsymbol{P}_f = (2.2, 1.5)$.

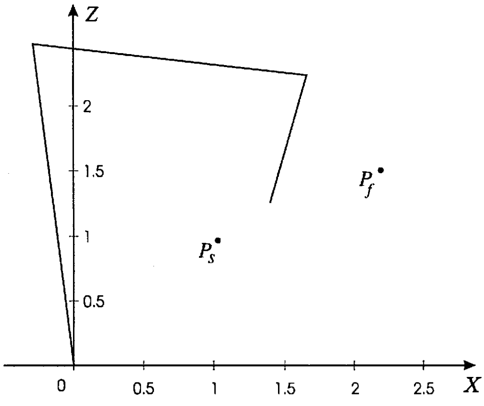

Fig. 1. Scheme of the manipulator operating in a two-dimensional workspace $0p_1p_2$ and the final end-effector location $\boldsymbol{P}_f$.

Figure 2 shows the results of computer simulation for $k = 1$ and Fig. 3 for $k = 4$. The time-optimal manipulator trajectory is depicted in Fig. 4 for $k = 4$. Let us note that the optimal controls from Fig. 3 leave room for an on-line control of the manipulator and retain a bang-bang structure. This result agrees with the theoretical considerations presented in Section 3.

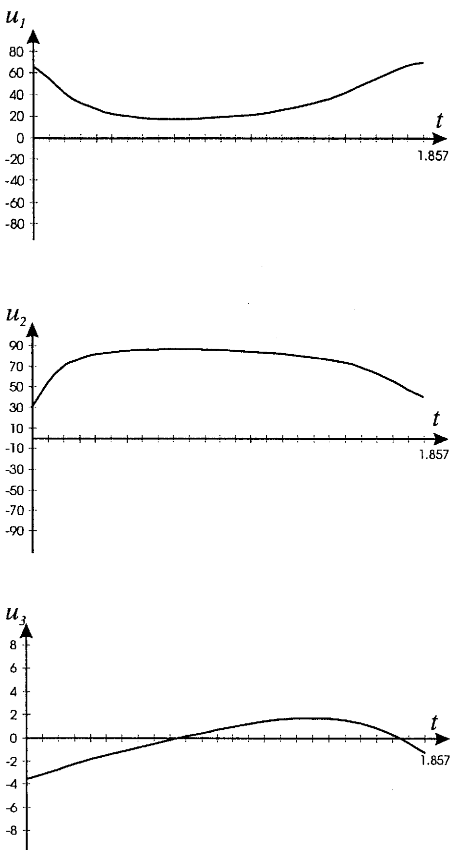

Fig. 2. Time-optimal controls v. time for $k = 1$.

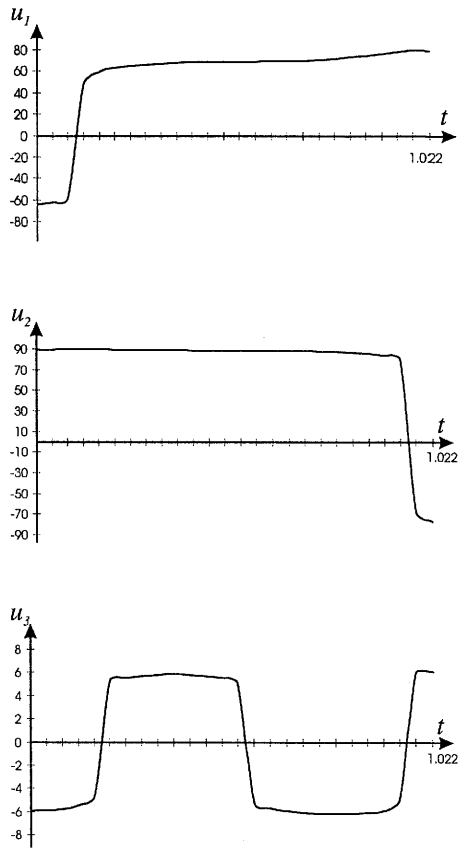

Fig. 3. Time-optimal controls v. time for $k = 4$.

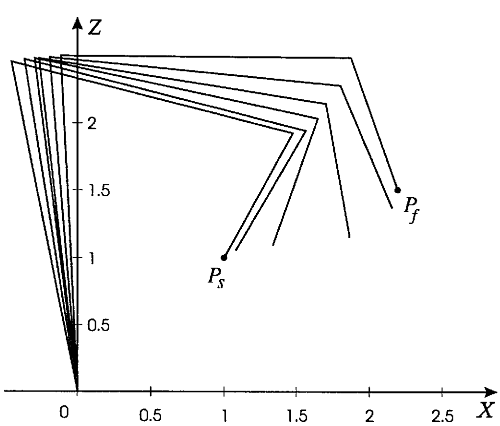

Fig. 4. Optimal manipulator motion for $k = 4$.

6. Conclusions

An application of the Pontryagin maximum principle to find time-optimal controls from a strictly convex set is presented. This approach produces continuous optimal controls which may be used to provide nominal inputs for an on-line manipulator control system. Furthermore, these controls leave room for closed-loop control actions. In contrast to the methods with a penalized performance index, the method presented here retains the structure of optimal controls and is not conservative. It may be directly applicable to very general mechanisms, including multiple manipulators performing tasks in a three-dimensional workspace.

The drawback of this method is its heavy computational burden which excludes its use in on-line control. But it can be used for planning long-term production cycles.

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