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OPTIMALITY CONDITIONS FOR LINEAR 2-D CONTROL SYSTEMS WITH CONSTRAINTS

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Optimal control problems for linear two-dimensional (2-D) discrete and continuous-discrete systems with mixed constraints are investigated. The problems under consideration are reduced to linear-programming ones in appropriate Hilbert spaces. The main duality relations are derived such that optimality conditions are specified by using methods of linear operator theory. The optimality conditions are expressed in terms of solutions of adjoint systems. A simple illustrative example is also given.

1. Introduction

The most popular models of two-dimensional (2-D) discrete systems have been introduced by Attasi (1973), Fornasini and Marchesini (1976), Givone and Roesser (1973), Roesser (1975). In particular, these systems have been studied in relation to multidimensional digital filtering, analysis of satellite photographs and videoinformation, recent advances in microprocessor technology and other modern engineering problems. Some aspects of these questions are given e.g. in (Bose, 1977). Detailed investigations of various models have been published in many papers (cf. Kaczorek, 1985). In this context, only few works have been devoted to the continuous-discrete systems. However, such systems provide an appropriate mathematical tool for simulation of a number of real physical processes.

For at least two decades a great deal of attention has been paid to the linear control problem for 2-D systems. Among the most important contributions we can cite controllability, observability and stabilizability (Gaishun, 1991; Gaishun and Quang, 1992; Kaczorek, 1985; Kurek, 1987; Rogers and Owens, 1992). For some other results we refer the reader to (Bisiacco, 1995; Gałkowski, 1991; Kaczorek, 1995; Kaczorek and Klamka, 1987; Kurek and Zaremba, 1993; Lewis, 1992). We also mention the new monograph by Gaishun (1996) where traditional control problems for abstract multidimensional systems are treated.

Some 2-D optimal control problems, such as LQP and minimum energy control problems have been considered by several researchers (Bisiacco and Fornasini, 1990; Klamka, 1994; Sebek, 1989; Dymkov, 1993a; 1993b; 1994). On the other hand, few

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Proof. Let Ω be a bounded set. Therefore Ω is weakly compact. Suppose that η_k is a minimizing sequence such that $\lim_{k\to\infty} \mathrm{Re}(\eta, b) = \sup_{\eta\in\Omega} \mathrm{Re}(\eta, b)$. We conclude from the inequality stated above that this limit is finite. Since Ω is weakly compact, there exists a weakly convergent subsequence denoted again by η_k. Let $\eta_k \to \eta^*$ as $k \to \infty$. It is evident that $\mathrm{Re}(\eta^*, b) = \sup_{\eta\in\Omega} \mathrm{Re}(\eta, b)$. Since $\eta_k \in \mathcal{K}_1^*$ for all k and $\mathcal{K}_1^*$ is closed, we have $\lim_{k\to\infty} \eta_k = \eta^* \in \mathcal{K}_1^*$.

The proof for the other case follows in much the same way and is omitted. ■

Let us introduce the Lagrangian $L : W_E \times l_2(W_E) \times l_2(W_V) \to \mathbb{R}$ as follows:

$$L(\xi, \eta) = \mathrm{Re}(\tilde{p}, \xi) + \mathrm{Re}(\eta, T\xi - b)$$

A pair (ξ^0, η^0) is called the saddle point if $\xi \in \mathcal{K}_0$, $\eta^0 \in \mathcal{K}_1^*$ and

$$L(\xi, \eta^0) \leq L(\xi^0, \eta^0) \leq L(\xi^0, \eta) \tag{15}$$

for all $\xi \in \mathcal{K}_0$, $\eta \in \mathcal{K}_1^*$.

Lemma 3. *Suppose that the pair* (ξ^0, η^0) *is a saddle point. Then* ξ^0 *and* η^0 *are optimal solutions to (11), (12) and (13), (14), respectively. Moreover, the complementary slackness conditions* $\mathrm{Re}(\eta^0, T\xi^0 - b) = 0$, $\mathrm{Re}(\xi^0, \tilde{p} - T^*\eta^0) = 0$ *are satisfied.*

Proof. From the right-hand side of (15) it follows that $\mathrm{Re}(\tilde{p}, \xi^0) + \mathrm{Re}(\eta^0, T\xi^0 - b) \leq \mathrm{Re}(\tilde{p}, \xi^0) + \mathrm{Re}(\eta, T\xi^0 - b)$ for all $\eta \in \mathcal{K}_1^*$. This inequality is possible if $T\xi^0 - b \in \mathcal{K}_1$, $\mathrm{Re}(\eta^0, T\xi^0 - b) = 0$. Now, for all $\xi \in \mathcal{K}_0$, $T\xi - b \in \mathcal{K}_1$ from the left-hand side of (15) we deduce that

$$\mathrm{Re}(\tilde{p}, \xi) \leq \mathrm{Re}(\tilde{p}, \xi) + \mathrm{Re}(\eta^0, T\xi - b) \leq \mathrm{Re}(\tilde{p}, \xi) + \mathrm{Re}(\eta^0, T\xi^0 - b) = \mathrm{Re}(\tilde{p}, \xi^0)$$

which amounts to the optimality of ξ^0. Similar considerations apply to the optimality of η^0 and the equality $\mathrm{Re}(\xi^0, \tilde{p} - T^*\eta^0) = 0$. ■

Corollary 1. *Suppose that* $\mathrm{Re}(\eta^*, T\xi^* - b) = 0$ *and* $\mathrm{Re}(\xi, \tilde{p} - T^*\eta^*) = 0$ *for some admissible elements* ξ^* *and* η^*. *Then* ξ^* *and* η^* *are optimal solutions to problems (11), (12) and (13), (14), respectively.*

Proof. It is sufficient to show that the pair (ξ^*, η^*) is a saddle point for the Lagrange function. It is easily seen that

$$\mathrm{Re}(\tilde{p}, \xi^*) + \mathrm{Re}(\eta, T\xi^* - b) \geq \mathrm{Re}(\tilde{p}, \xi^*) + \mathrm{Re}(\eta^*, T\xi^* - b)$$

for all $\eta \in \mathcal{K}_1^*$. Hence $L(\xi^*, \eta^*) \leq L(\xi^*, \eta)$ for all $\eta \in \mathcal{K}_1^*$. The other inequality in (15) may be handled in much the same way. ■

Theorem 1. *Let* $\eta^0 \in l_2(W_E)$ *be an optimal solution to (13), (14). Then there is an optimal solution* $\xi^0 \in W_E \times l_2(W_V)$ *to (11), (12) such that* $\mathrm{Re}(\tilde{p}, \xi^0) = -\mathrm{Re}(\eta^0, b)$ *and the complementary slackness conditions* $\mathrm{Re}(\tilde{p} - T^*\eta^0, \xi^0) = 0$, $\mathrm{Re}(\eta^0, T\xi^0 - b) = 0$ *are valid.*

Proof. It is sufficient to find and admissible element ξ^0 such that the pair (ξ^0, η^0) is a saddle point for the Lagrange function.

Write $\alpha = -\text{Re}(\eta^0, b)$. Since η^0 is an optimal solution to (13), (14), we have $-\text{Re}(b,\eta) - \alpha \geq 0$ for any η such that $\eta \in \mathcal{K}_1^*$ and $\tilde{p} - T^*\eta \in \mathcal{K}_0^*$. As in (Ter-Kricorov, 1977), we show that if $\text{Re}\rho \geq 0$, $\eta \in \mathcal{K}_1^*$, and $\rho\tilde{p} - T^*\eta \in \mathcal{K}_0^*$, then $-\text{Re}(\eta, b) - \text{Re}\rho\alpha \geq 0$.

Conversely, suppose that this were false. Then there are $\bar{\eta} \in \mathcal{K}_1^*$ and $\bar{\rho}$, $\text{Re}\bar{\rho} \geq 0$ such that $\bar{\rho}\tilde{p} - T^*\bar{\eta} \in \mathcal{K}_0^*$, but $-\text{Re}(\bar{\eta}, b) - \alpha\text{Re}\bar{\rho} < 0$. If $\text{Re}\bar{\rho} < 0$, then setting $\tilde{\eta} = \bar{\eta}/\bar{\rho}$ we have $\tilde{\eta} \in \mathcal{K}_1^*$ (since $\mathcal{K}_1^*$ is a cone) and $\tilde{p} - T^*\tilde{\eta} = (1/\rho)(\bar{\rho}\tilde{p} - T^*\bar{\eta}) \in \mathcal{K}_1^*$, but $-\text{Re}(\tilde{\eta}, b) - \alpha < 0$, a contradiction. If $\text{Re}\bar{\rho} = 0$, then $-\text{Re}(\bar{\eta}, b) < 0$, $\bar{\eta} \in \mathcal{K}_1^*$, $T^*\bar{\eta} \in \mathcal{K}_0^*$. Therefore, for any ρ, $\text{Re}\rho > 0$ and $[-T^*\bar{\eta} + \rho(\tilde{p} - T^*\eta^0)] \in \mathcal{K}_0^*$ or $[\rho\tilde{p} - T^*(\bar{\eta} + \rho\eta^0)] \in \mathcal{K}_0^*$. It is clear that $\bar{\eta} + \rho\eta^0 \in \mathcal{K}_1^*$. According to the previous case, we have $-\text{Re}(\bar{\eta} + \rho\eta^0, b) - \alpha\text{Re}\rho \geq 0$. Passing to the limit as $\rho \to 0$, we have $-\text{Re}(\bar{\eta}, b) \geq 0$ and $\bar{\eta} \in \mathcal{K}_1^*$ because $\mathcal{K}_1$ is a closed cone. This contradicts our hypothesis.

Thus the following proposition is correct:

$$\text{If } \text{Re}\rho \geq 0,\ \eta \in \mathcal{K}_1^*,\ \rho\tilde{p} - T^*\eta \in \mathcal{K}_0^*, \text{ then } -\text{Re}(\eta, b) - \alpha\text{Re}\rho \geq 0. \tag{16}$$

Now, we prove the following assertion: If (16) is valid, then there exists an element $\xi^0 \in \mathcal{K}_0$ such that

$$\text{Re}(\xi^0, \tilde{p} - T^*\eta) \leq -\text{Re}(b, \eta) - \alpha \quad \text{for all } \eta \in \mathcal{K}_1 \tag{17}$$

Define the operator $\tilde{T} : W \times l_2(W_E) \to l_2(W_E) \times \mathbb{C}$ as $\tilde{T}\xi = (-T\xi, (\tilde{p}, \xi))$. The adjoint operator $\tilde{T}^* : l_2(W_E) \times \mathbb{C} \to W_E \times l_2(W_E)$ has the form $\tilde{T}^*\tilde{y} = -T^*\eta + \rho\tilde{p}$, where $\tilde{y} = (\eta, \rho) \in l_2(W_E) \times \mathbb{C}$.

Set $Q_1^* = \mathcal{K}_1^* \times \mathbb{C}_+$, where $\mathbb{C}_+ = \{z \in \mathbb{C},\ \text{Re}\, z \geq 0\}$. It is easy to verify that $Q_1 = \mathcal{K}_1 \times \mathbb{R}_+$, where $\mathbb{R}_+$ is the set of nonnegative real numbers.

Let $Q_0^* = \{\tilde{y} \in l_2(W_E) \times \mathbb{C},\ \tilde{T}^*\tilde{y} \in \mathcal{K}_0^*\}$. Clearly, $Q_0 = (Q_0^*)^* = \{z \in l_2(W_E) \times W_E,\ z = \tilde{T}\tilde{x},\ \tilde{x} \in \mathcal{K}_0\}$. Now, (16) can be expressed as follows:

$$\text{If } \tilde{y} \in Q_1^* \cap Q_0^*, \text{ then } -\text{Re}(\tilde{y}, \tilde{b}) \geq 0,$$

where $\tilde{b} = (b, \alpha)$, $\tilde{y} = (y, \rho) \in l_2(W_E) \times \mathbb{C}$. This means that $-\tilde{b} \in (Q_1^* \bigcap Q_0^*)^*$. Since Q_0 and Q_1 are closed, $(Q_1^* \bigcap Q_0^*)^* = ((Q_1 + Q_0)^*)^* = Q_1 + Q_0$. Therefore $-\tilde{b} \in Q_1 + Q_0$. This means that there are $\xi^0 \in \mathcal{K}_0$ and $\tilde{z} \in Q_1$ such that $-\tilde{b} = \tilde{T}\xi^0 + \tilde{z}$. Hence, for any $\tilde{\eta} \in Q_1^*$ we have $-\text{Re}(\tilde{b}, \tilde{\eta}) = \text{Re}(\tilde{T}\xi^0, \tilde{\eta}) + \text{Re}(\tilde{z}, \tilde{\eta}) \geq \text{Re}(\xi^0, \tilde{T}^*\tilde{\eta})$. From this it follows that $-\text{Re}(b, \eta) - \alpha\text{Re}\rho \geq \text{Re}(\xi^0, -T^*\eta) + \text{Re}(\rho(\tilde{p}, \xi^0))$ or $\text{Re}(\xi^0, \rho\tilde{p} - T^*\eta) \leq -\text{Re}(b, \eta) - \text{Re}\rho\alpha$ for all $\eta \in \mathcal{K}_1$ and ρ, $\text{Re}\rho \geq 0$.

Setting $\rho = 1$ in the last inequality, we have $\text{Re}(\xi^0, \tilde{p} - T^*\eta) \leq -\text{Re}(b, \eta) - \alpha$ for all $\eta \in \mathcal{K}_1$ as was required for (17).

Now, in (17) we set $\eta = \eta^0$. Then $\text{Re}(\xi^0, \tilde{p} - T^*\eta^0) \leq -\text{Re}(b, \eta^0) + \text{Re}(b, \eta^0) = 0$, or $\text{Re}(\xi^0, \tilde{p} - T^*\eta^0) \leq 0$. Since $\tilde{p} - T^*\eta^0 \in \mathcal{K}_0^*$ and $\xi^0 \in \mathcal{K}_0$, we have $\text{Re}(\xi^0, \tilde{p} - T^*\eta^0) \geq 0$. Hence $\text{Re}(\xi^0, \tilde{p} - T^*\eta^0) = 0$. On account of this fact we have from (17) $\text{Re}(b, \eta) + \text{Re}(\xi^0, \tilde{p} - T^*\eta) \leq \text{Re}(b, \eta^0) + \text{Re}(\xi^0, \tilde{p} - T^*\eta^0)$. The obtained inequality implies $L(\xi^0, \eta) \geq L(\xi^0, \eta^0)$ for all $\eta \in \mathcal{K}_1^*$.

Since $\mathrm{Re}(\xi^*, \tilde{p} - T^*\eta^0) = 0$ for $\xi^0 \in \mathcal{K}_0$, $\tilde{p} - T^*\eta^0 \in \mathcal{K}_0^*$, we have $\mathrm{Re}(b, \eta^0) = \mathrm{Re}(b, \eta^0) + \mathrm{Re}(\xi^0, \tilde{p} - T^*\eta^0) \leq \mathrm{Re}(b, \eta^0) + \mathrm{Re}(\xi, \tilde{p} - T^*\eta^0)$ for all $\xi \in \mathcal{K}_0$. Hence $L(\xi^0, \eta^0) \geq L(\xi, \eta^0)$ for any $\xi \in \mathcal{K}_0$.

Thus we have proved that the pair (ξ^0, η^0) is a saddle point for the Lagrangian. The proof of the theorem is now completed by using Lemma 3. ■

Remark 3. It is easy to check that the dual theorem is valid. In other words, if there is an optimal solution to (11), (12), then there exists an optimal solution to (13), (14) such that the conclusion of Theorem 1 is valid.

2.3. The Main Result for Continuous-Discrete 2-D Systems

It is well-known that the duality theory is a commonly-used tool to obtain optimality conditions for linear extremal problems. Below this method is used in an analysis of the original problem (1)–(3).

Introduce a new variable $\lambda = (\psi, y) \in W_E \times l_2(W_E)$ given by $\lambda = (\mathcal{A})^{-1*}(\mathcal{P}^*\eta + p)$. Using the explicit form of the adjoint operator $(\mathcal{A}^{-1})^*$, we obtain

$$\psi = \mathcal{P}^*\eta(0) + p(0) + a^*y(0), \quad y(t) = \mathcal{P}^*\eta(t+1) + p(t+1) + a^*y(t+1)$$

where $t \in \mathbb{Z}_+$, $\|y(t)\| \to 0$ as $t \to \infty$. Here $\psi : \mathbb{C} \to E$ is a function from W_E, $y : \mathbb{Z}_+ \times \mathbb{C} \to E$ belongs to $l_2(W_E)$. Set $\eta = (v(0), v(1), \ldots)$, where $v : \mathbb{Z}_+ \times \mathbb{C} \to E$ is a function from $l_2(W_E)$.

Now, based on the form of a^* and $\mathcal{P}^*$, the above relations can be expressed as follows:

$$\begin{aligned} y(t,s) &= A^*y(t+1,s) - D^*\frac{\partial y(t+1,s)}{\partial s} \\ &\quad + P^*v(t+1,s) + p(t+1,s), \quad \|y(t,\cdot)\|_W \to 0, \quad t \to \infty \\ \psi(s) &= A^*y(0,s) - D^*\frac{\partial y(0,s)}{\partial s} + P^*v(0,s) + p(0,s), \quad t \in \mathbb{Z}, \quad s \in \mathbb{R} \end{aligned} \tag{18}$$

In this case the constraints (15) and the cost functional (13) are written down in the form

$$\begin{cases} -B^*y(t,s) - Q^*v(t,s) \in K_2^*, \ v(t,s) \in K_1^* \\ -q(s) - \psi(s) \in K_3^*, \ t \in \mathbb{Z}_+, \ s \in \mathbb{R} \end{cases} \tag{19}$$

$$r(v) = -\sum_{t \in \mathbb{Z}_+} \int_{\mathbb{R}} \mathrm{Re}(b(t,s), \overline{v(t,s)})_E \, \mathrm{d}s \tag{20}$$

Thus the dual problem to (1)–(3) is to minimize the functional (20) on the solutions to (18) under the conditions (19).

Theorem 2. *Let $u^0(t,s)$, $\varphi^0(s)$, $t \in \mathbb{Z}$, $s \in \mathbb{R}$ be an optimal solution to the problem (1)–(3). Then there exists an optimal solution $v^0(t,s)$, $t \in \mathbb{Z}$, $s \in \mathbb{R}$ to the dual problem (18)–(20) such that $J(u^0,\varphi^0) = r(v^0)$ and $\mathrm{Re}((Px^0(t,s) + Qu^0(t,s) - b(t,s)), v^0(t,s))_E = 0$, $\mathrm{Re}(\psi^0(s) + q(s), \varphi^0(s))_E = 0$, $\mathrm{Re}((B^*y^0(t,s) + Q^*v^0(t,s)), u^0(t,s))_E = 0, t \in \mathbb{Z}_+$, $s \in \mathbb{R}$, where $x^0(t,s)$ and $y^0(t,s)$ are solutions to (2) and (18) corresponding to $u^0(t,s), \varphi(s)$ and $v^0(t,s), t \in \mathbb{Z}_+, s \in \mathbb{R}$, respectively.*

Proof. According to the previous reasoning, an optimal solution to (1)–(3) generates an optimal solution $\xi^0 \in W_E \times l_2(W_V)$ to (11)–(12) , which is obviously defined by u^0 and φ^0. Using Remark 3 we obtain that the dual problem (13), (14) has an optimal solution $\eta^0 \in l_2(W_E)$ such that $J(u^0,\varphi^0) = l(\xi^0) = r(\eta^0)$ and the complementary slackness conditions are valid. The desired functions $v^0(t,s)$ are constructed by setting $v^0(t,\cdot) = \eta^0(t)$ for any $t \in \mathbb{Z}_+$. It is easily seen that the required relations follow from the complementary slackness conditions. For example, $\sum_{t\in\mathbb{Z}_+} \int_{\mathbb{R}} \mathrm{Re}(v^0(t,s),\ ((Px^0(t,s) + Qu^0(t,s) - b(t,s))_E \,\mathrm{d}s = 0$ follows from the condition $\mathrm{Re}(\eta^0, T\xi^0 - b)_{l_2(W_E)} = 0$. Since $\eta^0 \in \mathcal{K}_1^*$, $T\xi^0 - b \in \mathcal{K}_1$, we have $\mathrm{Re}(v^0(t,s), Px^0(t,s) + Qu^0(t,s) - b(t,s))_E \geq 0$ for all $t \in \mathbb{Z}_+$ $s \in \mathbb{R}$. From the previous equality we have $\mathrm{Re}(v^0(t,s), Px^0(t,s) + Qu^0(t,s) - b(t,s))_E = 0$, $t \in \mathbb{Z}_+$ $s \in \mathbb{R}$, which was to be proved. Other relations are proved in much the same way. ∎

3. Problem Formulation for Discrete 2-D Systems

Let E and V be finite-imensional Hilbert spaces over the real field $\mathbb{R}$, as opposed to the above case. The replacement of $\mathbb{C}$ by $\mathbb{R}$ is made only to simplify the presentation. The sets of square-summable functions $x : \mathbb{Z}_+^2 \to E, u : \mathbb{Z}_+^2 \to V, \varphi : \mathbb{Z}_+ \to E$ are denoted by $X(\mathbb{Z}_+^2, E), U(\mathbb{Z}_+^2, V), F(\mathbb{Z}_+, E)$, respectively, where $\mathbb{Z}_+^2 = \mathbb{Z}_+ \times \mathbb{Z}_+$. In this case our optimal control problem in a discrete version can be formulated as follows: Maximize the linear functional

$$J(u) = \sum_{(t,s)\in\mathbb{Z}_+^2} p^*(t,s)x(t,s) + \sum_{s\in\mathbb{Z}_+} q^*(s)\varphi(s) \tag{21}$$

on the solutions to the system

$$x(t+1,s) = Ax(t,s+1) + Dx(t,s) + Bu(t,s), \quad x(0,s) = \varphi(s), \quad (t,s) \in \mathbb{Z}_+^2 \tag{22}$$

subject to

$$Qx(t,s) + Gu(t,s) - b(t,s) \in K_1, \quad \varphi(s) \in K_3, \quad u(t,s) \in K_2, \quad (t,s) \in \mathbb{Z}_+^2 \tag{23}$$

Here A, D, Q and G, B are linear operators acting from E into E and from V into E, respectively; $p^*(t,s)$ and $q^*(s)$ are given square-summable functions.

For given $u(t,s)$, $\varphi(s)$, $(t,s) \in \mathbb{Z}_+^2$ we say that a function $x(t,s)$ from $X(\mathbb{Z}_+^2, E)$ is a solution to the system (22) if x satisfies (22) for all $(t,s) \in \mathbb{Z}_+^2$.

A control $u \in U(\mathbb{Z}_+^2, V)$ and an initial function $\varphi \in F(\mathbb{Z}_+^2, E)$ are called admissible if they and the corresponding solution $x(t,s)$ satisfy (23) for all $(t,s) \in \mathbb{Z}_+^2$.

In the sequel, we suppose that the assumptions made in Section 2 are fulfilled here, as well.

3.1. Preliminaries and the Adjoint System

Consider the nonhomogeneous system in the form

$$x(t+1,s) = Ax(t,s+1) + Dx(t,s) + g(t,s), \quad x(0,s) = \varphi(s), \quad (t,s) \in \mathbb{Z}_+^2 \quad (24)$$

where $g : Z_+^2 \to E, \varphi : \mathbb{Z}_+ \to E$ are given square-summable functions. Denote by $l^2(E)$ the Hilbert space of square-summable sequences of elements from E. $\mathcal{B}(\mathbb{Z}_+, l^2(E))$ stands for the Hilbert space of square-summable maps $\xi : \mathbb{Z}_+ \to l^2(E)$ equipped with the standard inner product.

Define the operator $a : l^2(E) \to l^2(E)$ by $(ax)_s = Ax_{s+1} + Dx_s, x \in l^2(E)$. Obviously, a is a bounded linear operator and $\|a\| \leq \|A\| + \|D\|$. In the sequel, we shall assume that $\|A\| + \|D\| < 1$.

Thus (24) can be represented in the form

$$\alpha(t+1) = a\alpha(t) + g(t), \quad \alpha(0) = \varphi, \quad t \in \mathbb{Z}_+ \quad (25)$$

where $\alpha(t) = (x(t,0), x(t,1), \ldots) \in l^2(E)$, $\varphi = (\varphi(0), \ \varphi(1), \ldots) \in l^2(E)$, $g(t) = (g(t,0), g(t,1), \ldots) \in l^2(E)$.

Define the bounded linear operator $\mathcal{A} : \mathcal{B}(\mathbb{Z}_+, l^2(E)) \to l^2(E) \times \mathcal{B}(\mathbb{Z}_+, l^2(E))$ by $\mathcal{A} : (\alpha(0), \alpha(1), \ldots) \to (\alpha(0), (\alpha(1) - a\alpha(0), \alpha(2) - a\alpha(1), \ldots))$. Then eqn. (25) can be written as the following linear equation in $\mathcal{B}(\mathbb{Z}_+, l^2(E))$:

$$\mathcal{A}\alpha = f, \quad f = (\varphi, g) \in l^2(E) \times \mathcal{B}(\mathbb{Z}_+, l^2(E)) \quad (26)$$

It is easy to check that $\mathcal{A}$ has the bounded inverse operator $\mathcal{A}^{-1}$: $(\varphi, (\eta(0), \eta(1), \ldots)) \to (\varphi, \eta(0) + a\varphi, \eta(1) + a\eta(0), \ldots)$ for any $\varphi \in l^2(E), \eta = (\eta(0), \eta(1), \ldots) \in \mathcal{B}(\mathbb{Z}_+, l^2(E))$. Hence, if $\|A\| + \|D\| < 1$, then (26) has a unique square-summable solution given by $\alpha = \mathcal{A}^{-1}f$ for any $f \in l^2(E) \times \mathcal{B}(\mathbb{Z}_+, l^2(E))$.

Define now the adjoint equation associated with (26):

$$\mathcal{A}^*\xi = g, \quad \mathcal{A}^* : l^2(E) \times \mathcal{B}(\mathbb{Z}_+, l^2(E)) \to \mathcal{B}(\mathbb{Z}_+, l^2(E)) \quad (27)$$

where $g \in \mathcal{B}(\mathbb{Z}_+, l^2(E))$, $\mathcal{A}^*$ denotes the adjoint operator of $\mathcal{A}$, given by $\mathcal{A}^*$: $(\beta, (\eta(0), \eta(1), \ldots)) \to (\beta - a^*\eta(0), \eta(0) - a^*\eta(1), \ldots)$. Here $a^* : l^2(E) \to l^2(E)$ is the adjoint operator of a, defined by the formula $a^* : (x(0), x(1), \ldots) \to (D^*x(0), A^*x(0) + D^*x(1), \ldots)$, where A^* and D^* are adjoint operators of A and D, respectively. Using this representation of $\mathcal{A}^*$, we may rewrite (27) as

$$\eta(t) = a^*\eta(t+1) + g(t+1), \quad \beta = a^*\eta(0) + g(0), \quad t \in \mathbb{Z}_+ \quad (28)$$

with respect to the unknown variables $\beta \in l^2(E)$, $\eta(t) \in l^2(E)$, $t \in \mathbb{Z}+$. The system (28) is called the adjoint system of (25).

In turn, setting $y(t,s) = [\eta(t)](s)$ from (28) we have

$$\begin{aligned} &y(t,s) = A^*y(t+1,s-1) + D^*y(t+1,s) + g(t+1,s) \\ &y(t,0) = D^*y(t+1,0) + g(t+1,0), \quad (t,s) \in \mathbb{Z}_+^2 \\ &\beta(s) = A^*y(0,s-1) + D^*y(0,s), \beta(0) = D^*y(0,0), \quad s \in \mathbb{Z}_+ \end{aligned} \tag{29}$$

We say that eqn. (29) is the adjoint system of (24).

As is well-known, (26) possesses a unique solution for any $f \in l^2(E) \times \mathcal{B}(\mathbb{Z}_+, l^2(E))$ if and only if the adjoint equation (27) has a unique solution for any $g \in \mathcal{B}(\mathbb{Z}_+, l^2(E))$. In addition, in this case there exist continuous inverse operators $\mathcal{A}^{-1}$ and $(\mathcal{A}^*)^{-1}$. Moreover, $(\mathcal{A}^*)^{-1} = (\mathcal{A}^{-1})^*$. Since for the case $\|A\| + \|D\| < 1$ eqn. (26) is solved, the adjoint system (28) has a unique square-summable solution for any $g \in X(\mathbb{Z}+^2, E)$.

It is easy to prove that the operator $(\mathcal{A}^{-1})^* : \mathcal{B}(\mathbb{Z}_+, l^2(E)) \rightarrow l^2(E) \times \mathcal{B}(\mathbb{Z}_+, l^2(E))$ can be expressed in the form

$$(\mathcal{A}^{-1})^* : (\alpha(0), \alpha(1), \dots) \rightarrow \Big\{ \sum_{s\in\mathbb{Z}_+} a^{*s}\alpha(s), \big(\sum_{s\in\mathbb{Z}_+} a^{*s}\alpha(s+1), \sum_{s\in\mathbb{Z}_+} a^{*s}\alpha(s+2), \dots \big) \Big\}$$

Therefore, the solution to the adjoint system (28) can be written down as

$$\beta = \sum_{s\in\mathbb{Z}_+} a^{*t}g(t), \eta(t) = \sum_{s\in\mathbb{Z}_+} a^{*s}g(t+1+s), \quad t \in \mathbb{Z}_+$$

Thus we have proved the following result.

Lemma 4. *Let* $\|A\| + \|D\| < 1$. *Then systems (24) and (29) have unique square-summable solutions for any* $\varphi \in F(\mathbb{Z}+, E)$, $f, g \in X(\mathbb{Z}+^2, E)$, *respectively.*

3.2. The Main Results for the Discrete Case

Similarly to $\mathcal{B}(\mathbb{Z}_+, l^2(E))$, we define the space $\mathcal{B}(\mathbb{Z}_+, l^2(V))$ of square-summable functions $w : \mathbb{Z}_+ \rightarrow l^2(V)$. Define the operator $\hat{B} : \mathcal{B}(\mathbb{Z}_+, l^2(V)) \rightarrow \mathcal{B}(\mathbb{Z}_+, l^2(E))$ as $(\hat{B}\xi)(t) = \tilde{B}\xi(t)$, $t \in \mathbb{Z}_+$, where the map $\tilde{B} : l^2(E) \rightarrow l^2(E)$ is given by $(\tilde{B}x)_t = Bx_t$, $t \in \mathbb{Z}_+$. In this case we say that B produces $\hat{B}$.

In a similar way, $B^* : E \rightarrow V$ produces $\hat{B}^* : \mathcal{B}(\mathbb{Z}+, l^2(E) \rightarrow \mathcal{B}(\mathbb{Z}+, l^2(V))$. Below it will be supposed that $\hat{Q} : \mathcal{B}(\mathbb{Z}_+, l^2(E)) \rightarrow \mathcal{B}(\mathbb{Z}_+, l^2(E))$, $\hat{G} : \mathcal{B}(\mathbb{Z}_+, l^2(V)) \rightarrow \mathcal{B}(\mathbb{Z}_+, l^2(E))$ and their adjoint operators are defined by analogy.

According to Section 3.1, the solution $x \in \mathcal{B}(\mathbb{Z}_+, l^2(E))$ to eqn. (22) is of the form

$$\begin{cases} x = \mathcal{A}^{-1}f, \;\; f = (\varphi, \hat{B}u), \;\; u \in \mathcal{B}\big(\mathbb{Z}+, l^2(V)\big) \\ u(t) = \big(u(t,0), u(t,1), \dots\big), \quad t \in \mathbb{Z}_+ \end{cases} \tag{30}$$

Define the operator $\hat{T} : l^2(E) \times \mathcal{B}(\mathbb{Z}_+, l^2(V)) \to \mathcal{B}(\mathbb{Z}_+, l^2(E))$ by $\hat{T}(\varphi, u) = \hat{Q}\mathcal{A}^{-1}(\varphi, \hat{B}u) + \hat{G}u$ for $(\varphi, u) \in l^2(E) \times \mathcal{B}(\mathbb{Z}_+, l^2(V))$. It is easily seen that $\hat{T}$ is a bounded linear operator. Denote by $\mathcal{K}_1$ the cone in $\mathcal{B}(\mathbb{Z}_+, l^2(E))$ which is produced by the cone K_1 according to the formula $\mathcal{K}_1 = \{\alpha \in \mathcal{B}(\mathbb{Z}_+, l^2(E)) : \alpha(t) \in K_1(l^2(E)), t \in \mathbb{Z}_+\}$, where $K_1(l^2(E)) = \{\lambda \in l^2(E), \lambda_s \in K_1, s \in \mathbb{Z}_+\}$. Analogously, we define the cone $\mathcal{K}_2$ in $\mathcal{B}(\mathbb{Z}_+, l^2(V))$.

Let $K_1^* = \{x^* \in E, (x^*, x)_E \geq 0, x \in K_1\}$ be the dual cone of K_1 in E. This cone produces dual cones in $l^2(E)$ and $\mathcal{B}(\mathbb{Z}_+, l^2(E))$ as follows:

$$K_1^*(l^2(E)) = \{\lambda \in l^2(E), \lambda_s \in K_1^*, s \in \mathbb{Z}_+\}$$

$$\mathcal{K}_1^* = \left\{\eta \in \mathcal{B}(\mathbb{Z}+, l^2(E)) : \eta(t) \in K_1^*(l^2(E)), t \in \mathbb{Z}+\right\}$$

Similarly, we define dual cones $K_2^*(l^2(E))$, $K_3^*(l^2(V))$ and $\mathcal{K}_3^*$ in appropriate spaces.

For brevity, we write $N = K_2(l^2(E)) \times \mathcal{K}_3$ and $N^* = K_2^*(l^2(E)) \times \mathcal{K}_3^*$. Thus the original optimal control problem (21)–(23) can be formulated as the following linear programming problem in the Hilbert space $l^2(E) \times \mathcal{B}(\mathbb{Z}_+, l^2(V))$:

$$g(\xi) = (\hat{p}^*, \xi) \to \max, \quad \hat{T}\xi - b \in \mathcal{K}_1, \quad \xi \in N \tag{31}$$

where $\hat{p}^* = (q + \sum_{t\in\mathbb{Z}_+} a^{*t}p(t), \tilde{B}\sum_{t\in\mathbb{Z}_+} a^{*t}p(t+1), \ldots)$, $b = (b(0), b(1), \ldots) \in \mathcal{B}(\mathbb{Z}_+, l^2(E))$, $b(t) = (b(t,0), b(t,1), \ldots)$, $t \in \mathbb{Z}_+$.

The dual problem of (31) is the following:

$$r(\eta) = -(b, \eta) \to \min, \quad \hat{p}^* - \hat{T}\eta \in N^*, \quad \eta \in \mathcal{K}_1^*, \quad \eta \in \mathcal{B}(\mathbb{Z}+, l^2(E)) \tag{32}$$

Theorem 3 below gives a necessary optimality condition for the problem (21)–(23) in terms of solutions to the adjoint system. To this end we provide the following result stated here without proof since it follows immediately from the results of Section 2.2.

Lemma 5. *Let $\xi^0 \in l^2(E) \times \mathcal{B}(\mathbb{Z}_+, l^2(V))$ be an optimal solution to (31). Then there exists an optimal solution $\eta^0 \in \mathcal{B}(\mathbb{Z}_+, l^2(E))$ to the problem (32) such that $(\hat{p}^*, \xi^0) = -(b, \eta^0)$ and the complementary slackness conditions $(\hat{T}\xi^0 - b, \eta^0) = 0$ and $(\hat{T}^*\eta^0 - \hat{p}^*, \xi^0) = 0$ are valid.*

It is a simple matter to prove that the adjoint operator $\hat{T}^* : \mathcal{B}(\mathbb{Z}_+, l^2(E)) \to \mathcal{B}(\mathbb{Z}_+, l^2(V))$ can be expressed in the form

$$\hat{T}^* : (\eta(0), \eta(1), \ldots) \to \left(\sum_{s\in\mathbb{Z}_+} a^{*s}\tilde{Q}^*\eta(s), \sum_{s\in\mathbb{Z}_+} \tilde{B}^* a^{*s} Q^*\eta(s+1) + \tilde{G}^*\eta(1), \ldots\right)$$

We introduce a new variable $\xi = (\beta, y) \in \mathcal{B}(\mathbb{Z}_+, l^2(E)) \times \mathcal{B}(\mathbb{Z}_+, l^2(V))$ by the formula $\xi = (\mathcal{A}^{-1})^*(\tilde{Q}^*\eta - p^*)$. From the representation of $(\mathcal{A}^{-1})^*$ we have

$$\beta = \tilde{Q}^*\eta(0) - p^*(0) + a^*y(0), \quad y(t) = \tilde{Q}^*\eta - p^*(t+1) + a^*y(t+1), \quad t \in \mathbb{Z}_+$$

Let $\beta = (\psi(0), \psi(1), \ldots)$, $y(t) = (y(t,0), y(t,1), \ldots)$, and $\eta(t) = (v(t,0), v(t,1), \ldots)$.

Taking into account the representations of a^* and $\tilde{Q}^*$, we obtain the following equations:

$$\begin{aligned} &y(t,s) = A^*y(t+1,s-1) + D^*y(t+1,s) + Q^*v(t+1,s) - p^*(t+1,s) \\ &y(t,0) = D^*y(t+1,0) + Q^*v(t+1,0) - p^*(t+1,0) \\ &\psi(s) = A^*y(0,s-1) + D^*y(0,s) + Q^*v(0,s) - p^*(0,s), \quad (t,s) \in \mathbb{Z}_+^2 \\ &\psi(0) = D^*y(0,0) + Q^*v(0,0) - p^*(0,0), \quad s \in \mathbb{Z}_+ \end{aligned} \tag{33}$$

From (32) it follows that

$$-B^*y(t,s) - G^*v(t,s) \in K_2^*, \quad -v(t,s) \in K_1^*, \quad -\psi(s) - q^*(s) \in K_3^*, \quad (t,s) \in \mathbb{Z}_+^2 \tag{34}$$

The cost functional for the optimal problem (32) is written as

$$r(y) = -\sum_{(t,s)\in\mathbb{Z}_+^2} b(t,s)y(t,s) \tag{35}$$

Hence, the dual optimal control problem of (21)–(23) is to minimize $r(y)$ on the solutions to (33) under conditions (34).

Theorem 3. *Let* $\|A\| + \|D\| < 1$ *and* $u^0(t,s)$, $\varphi^0(s)$, $(t,s) \in \mathbb{Z}_+^2$ *be an optimal solution to the problem (21)–(23). Then the dual problem (33), (34) has an optimal solution* $v^0(t,s), (t,s) \in \mathbb{Z}+^2$ *such that* $[Qx^0(t,s) + Gu^0(t,s) - b(t,s)]v^0(t,s) = 0$, $[B^*y^0(t,s) + G^*v^0(t,s)]u^0(t,s) = 0$, $[\psi^0(s) - q^*(s)]\varphi^0(s) = 0, (t,s) \in \mathbb{Z}_+^2$, *where* $x^0(t,s), \varphi^0(t,s)$ *and* $y^0(t,s)$, $(t,s) \in \mathbb{Z}_+^2$ *are solutions to systems (22) and (33) corresponding to* $u^0(t,s), \varphi^0(t,s)$ *and* $v^0(t,s), (t,s) \in \mathbb{Z}_+^2$, *respectively.*

The proof follows immediately from Lemmas 4 and 5, and Theorem 2.

4. A Simple Example and Concluding Remarks

The purpose of this section is to illustrate the foregoing theoretical considerations based on a simple scalar example .

Let us consider the discrete scalar linear 2-D system

$$x(t+1,s) = \frac{1}{8}(x(t,s) + u(t,s)), \quad x(0,s) = \varphi(s) = \frac{1}{2\cdot 4^{s+1}}, \quad (t,s) \in \mathbb{Z}_+^2 \tag{36}$$

with constraints

$$x(t,s) + u(t,s) \geq \frac{1}{4^{t+s+1}}, \quad u(t,s) \geq 0, \quad (t,s) \in \mathbb{Z}_+^2 \tag{37}$$

Our objective is to find a control sequence $u^0(t,s)$ in $l^2(\mathbb{R})$, $(t,s) \in \mathbb{Z}_+^2$, such that the cost functional

$$J(u) = -\sum_{(t,s)\in\mathbb{Z}_+^2} \frac{1}{2^{t+s}} x(t,s) \tag{38}$$

is maximized subject to (36) and (37).

In our case the dual optimization problem is to minimize the functional

$$r(v) = -\sum_{(t,s)\in\mathbb{Z}+^2} \frac{1}{4^{t+s+1}} y(t,s)$$

in $l^2(\mathbb{R})$ subject to

$$y(t,s) = \frac{1}{8} y(t+1,s) + v(t+1,s) + \frac{1}{2^{t+s+1}}, \quad (t,s) \in \mathbb{Z}_+^2 \tag{39}$$

$$\psi(s) = \frac{1}{8} y(0,s) + v(0,s) + \frac{1}{2^s}, \quad s \in \mathbb{Z}_+$$

such that

$$y(t,s) + v(t,s) \leq 0, \quad v(t,s) \leq 0, \quad (t,s) \in \mathbb{Z}_+^2$$

It follows from Theorem 3 that the optimal solutions satisfy the conditions

$$(x^0(t,s) + u^0(t,s) - \frac{1}{4^{t+s+1}}) v^0(t,s) = 0, \quad (t,s) \in \mathbb{Z}_+^2 \tag{40}$$

$$(\frac{1}{8} y^0(t,s) + v^0(t,s)) u^0(t,s) = 0, \quad (t,s) \in \mathbb{Z}_+^2 \tag{41}$$

From (41) we get $v^0(t,s) = -\frac{1}{8} y^0(t,s)$, $(t,s) \in \mathbb{Z}_+^2$. Otherwise, $u^0(t,s) = 0$, $(t,s) \in \mathbb{Z}_+^2$. This yields $x^0(t,s) = 2^{-(t+1)} 4^{-(t+s+1)}$, $(t,s) \in \mathbb{Z}+^2$. Then $x^0(t,s) + u^0(t,s) = 2^{-(t+1)} 4^{-(t+s+1)} < 4^{-(t+s+1)}$, $(t,s) \in \mathbb{Z}_+^2$ which contradicts the constraints (37). Now from (40) we obtain $u^0(t,s) = 4^{-(t+s+1)} - x^0(t,s)$, $(t,s) \in \mathbb{Z}_+^2$. Applying this to (36) yields $x^0(t,s) = 2^{-1} 4^{-(t+s+1)}$, $(t,s) \in \mathbb{Z}_+^2$. Hence $u^0(t,s) = 2^{-1} 4^{-(t+s+1)}$, $(t,s) \in \mathbb{Z}_+^2$. Similarly, it is easy to check that $y^0(t,s) = 2^{-(t+s+1)}$ and $v^0(t,s) = -2^{-(t+s+4)}$, $(t,s) \in \mathbb{Z}+^2$. Finally, we get

$$J(u^0) = r(v^0) = -\sum_{(t,s)\in\mathbb{Z}_+^2} \frac{1}{4^{t+s+1}} \frac{1}{2^{t+s+1}} = -\sum_{s\in\mathbb{Z}_+} \frac{1}{8^s} \sum_{t\in\mathbb{Z}_+} \frac{1}{8^{t+1}} = -\frac{8}{49}$$

As was already mentioned, this example illustrates the theoretical considerations. However, the practical usefulness of the obtained results should be tested on examples which are more representative of multidimensional systems than our scalar example.

The method described above has been used in solving other optimal problems, e.g. the linear-quadratic regulator problem and the minimum-energy control problem for discrete and differential-difference 2-D systems. Moreover, it should be pointed out that the results given in the present paper can be extended to the case of abstract 2-D systems with mixed constraints.

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