Appl. Math. and Comp. Sci., 1998, Vol.8, No.1, 51–71

ADAPTIVE REDUCTION OF THE CONTROL EFFORT IN CHATTERING-FREE SLIDING-MODE CONTROL OF UNCERTAIN NONLINEAR SYSTEMS†

GIORGIO BARTOLINI*, ANTONELLA FERRARA**
ALESSANDRO PISANO*, ELIO USAI*

In previous papers, the authors presented a control procedure, based on second-order sliding modes, for the solution to the chattering problem in variable-structure control of uncertain systems. When the extremal values of the sliding variable are estimated by using a digital device with time delay δ, only a δ^2-vicinity of the sliding manifold can be reached. This fact implies that the resulting continuous control has residual oscillations which are the higher the larger the amplitude of the discontinuous derivative of the actual control plant is. In this paper, Utkin's concept of equivalent control is extended to second-order sliding modes and a method to evaluate their estimate by means of a proper high bandwidth filter is discussed. The knowledge of the estimate of the equivalent control is the basis of an adaptation mechanism which is able to modulate the amplitude of the discontinuous control so that a reduction in the boundary layer and in the corresponding oscillations of the plant input is attained. The proposed adaptive procedure is applied to a simple mechanical system as an example.

1. Introduction

The sliding-mode control methodology is characterized by a semi-group property (in time, one has an unreversible operator) when the trajectories are on the sliding manifold (Drakunov and Utkin, 1992). Any system, belonging to a proper set, during the sliding motion on a prespecified manifold is characterized by the same differential equation, and this property is regular to the sense that any motion close to the sliding manifold has phase trajectories close the ideal ones. This means that different systems have theoretically the same behaviour or, practically, very similar behaviour so that they are not distinguishable. This fact appears to prevent the use of sliding-mode control in identification of plant uncertainties, e.g. with the aim of reducing the control effort.

† This work was partially supported by the contract MAS3CT95-0024-AMADEUS of the European Community.

* Department of Electrical and Electronic Engineering, University of Cagliari, Piazza d'Armi, I–09123 Cagliari, Italy, e-mail: {giob,eusai}@diee.unica.it, giob@dist.unige.it.

** Department of Communication, Computer and System Sciences, University of Genova, Via Opera Pia 13, I–16145 Genova, Italy, e-mail: ferrara@dist.unige.it.

In contrast, Utkin (1992) has shown that discontinuous control acting on any system of the set, during the sliding motion, can be used to identify, with an *a-priori* evaluable error, an equivalent control. The latter is defined as a continuous control guaranteeing the condition $\dot{s}(t) = 0$ on the sliding manifold $s(t) = 0$. This condition has a solution which is strictly related to the r.h.s. of the differential equation of any system actually controlled. In other words, the state trajectories are not distinguishable, but discontinuous control contains hidden information regarding the actual system, which can be revealed by linear high bandwidth filters, with an approximation which is, more or less, the smaller the closer the system trajectories are to the ideal ones and the smaller the time constant of the filter is. Thus the practical availability of the equivalent control could be used to reduce the uncertainties of the system.

If the measured approximated equivalent control is suitably combined with discontinuous control, a reduction of the uncertainties is performed so that the control effort needed to maintain the system in sliding motion, i.e. discontinuous control amplitude, can be reduced, in principle. The reduction of discontinuous control amplitude generates a set of benefits in practical implementation and in simulation of real control systems, which ranges from the attenuation of chattering effect to the shortening of the simulation time.

The idea of using equivalent control for identification and adaptation purposes is obviously not new since it appears in Utkin's book (Utkin, 1992) and in the works of other researchers (Bartolini *et al.*, 1996b; Fu, 1991; Hsu and Costa, 1989). In this paper, this problem is faced for a particular second-order sliding mode approach recently introduced, with the aim of eliminating the chattering phenomenon (Bartolini *et al.*, 1998). In Sections 2 and 3 this approach is presented in the ideal and real cases, respectively. The ideal case means that an infinite-bandwidth peak detector is assumed to be available, while the real case is relevant to a practical implementation of this device. In Section 4, a procedure for achieving a reduction of the uncertainties' bound analogous to that presented in Utkin (1992) is described. In Section 5, an adaptive scheme exploiting this fact to reduce the control effort is presented, and, finally, in Section 6, a simulation example illustrating the performance improvement is provided.

2. Chattering Elimination Problem for Uncertain Nonlinear SISO Systems

When the sliding-mode approach to the control of a real plant is considered, the chattering problem, arising from nonidealities of real actuators, must be faced. The finite frequency control arising from various kinds of nonidealities could excite unmodelled oscillatory modes with unpredictable effect on the system behaviour. Any attempt to smooth the discontinuity of the control could even worsen such a situation. One approach to chattering reduction, by maintaining a very high commutation frequency, is based on the use of observers for the modelled part of the system (Utkin, 1992). The sliding mode is attained in the observer state space with a motion which is close to the ideal one. The resulting high-frequency control is filtered out by the fast dynamics of the plant so that a practical continuous control is fed to the slow dynamical subsystem. In the case of known nonlinear systems, a general framework has been proposed in (Sira-Ramirez, 1992) and then extended to uncertain systems in (Bartolini and Pydynowski, 1993; 1996). Recently, the authors presented a chattering-free control scheme based on second-order sliding modes (Bartolini *et al.*, 1998), reported here for the sake of clarity.

Given the system

$$\begin{cases} \dot{x}_i(t) = x_{i+1}(t), \qquad i = 1, \dots, n-1 \\ \dot{x}_n(t) = f\big[x(t)\big] + g\big[x(t)\big]u(t) \end{cases} \tag{1}$$

with $x(t) = [x_1, x_2, \dots, x_n]^T$ representing the completely available state, $f[x(t)]$ and $g[x(t)]$ being uncertain smooth functions satisfying the classical conditions for the existence of the solution, and the following inequalities:

$$0 < G_1 \leq g\big[x(t)\big] \leq G_2 \tag{2}$$

$$\big|f\big[x(t)\big]\big| \leq P_f + Q_f\big\|x(t)\big\| \tag{3}$$

$$\left\|\frac{\partial f\big[x(t)\big]}{\partial x}^T\right\| \leq P_{df} + Q_{df}\big\|x(t)\big\| \tag{4}$$

$$\left\|\frac{\partial g\big[x(t)\big]}{\partial x}^T\right\| \leq P_{dg} + Q_{dg}\big\|x(t)\big\| \tag{5}$$

where G_1, G_2, P_f, Q_f, P_{df}, Q_{df}, P_{dg}, Q_{dg} are known real positive constants, the problem is to find a continuous control $u(t)$ such that, in spite of the uncertainties (2)–(5), the state of (1) is steered exponentially to zero.

In order to determine the desired continuous control, the following procedure has to be followed (Bartolini and Pydynowski, 1996):

1. Differentiate the second equation of (1), setting

$$x_{n+1}(t) = f\big[x(t)\big] + g\big[x(t)\big]u(t)$$

and consider the augmented-order system

$$\begin{cases} \dot{x}_i(t) = x_{i+1}(t), \qquad i = 1, 2, \dots, n-1 \\ \dot{x}_n t = x_{n+1}(t) \\ \dot{x}_{n+1} = \dfrac{\mathrm{d}}{\mathrm{d}t} f\big[x(t)\big] + \dfrac{\mathrm{d}}{\mathrm{d}t} g\big[x(t)\big]u(t) + g\big[x(t)\big]\dfrac{\mathrm{d}}{\mathrm{d}t}u(t) \end{cases} \tag{6}$$

2. Choose an n-th order sliding manifold

$$s\big[x(t)\big] = x_n(t) + \sum_{i=1}^{n-1} c_i x_i(t) = 0 \tag{7}$$

with c_i, $i = 1, \dots, n-1$ being real positive constants such that the characteristic equation $z^{n-1} + \sum_{i=1}^{n-1} c_i z^{i-1} = 0$ has all roots with negative real parts.

3. Consider the first and second time derivatives of $s[\boldsymbol{x}(t)]$, namely

$$\dot{s}[\boldsymbol{x}(t)] = f[\boldsymbol{x}(t)] + g[\boldsymbol{x}(t)]u(t) + \sum_{i=1}^{n-1} c_i x_{i+1}(t) \tag{8}$$

$$\begin{aligned}\ddot{s}[\boldsymbol{x}(t)] &= \frac{\mathrm{d}}{\mathrm{d}t} f[\boldsymbol{x}(t)] + u(t)\frac{\mathrm{d}}{\mathrm{d}t} g[\boldsymbol{x}(t)] + c_{n-1}\Big\{ f[\boldsymbol{x}(t)] \\ &\quad + g[\boldsymbol{x}(t)]u(t)\Big\} + \sum_{i=1}^{n-2} c_i x_{i+2}(t) + g[\boldsymbol{x}(t)]\dot{u}(t)\end{aligned} \tag{9}$$

If it is possible to steer $s[\boldsymbol{x}(t)]$ to zero in a finite time by using a discontinuous control signal $\dot{u}(t)$, then the corresponding $u(t)$ is continuous, thereby eliminating the undesired high-frequency oscillations of $u(t)$ (the chattering effect) typical of the standard Variable-Structure Control (VSC) design. Once on $s[\boldsymbol{x}(t)] = 0$, the system performs like a reduced-order linear system with stable transfer function. Assume $y_1(t) = s[\boldsymbol{x}(t)]$ and $y_2(t) = \dot{s}[\boldsymbol{x}(t)]$. Then, relying on (7), the system dynamics (1) and the relevant uncertain dynamics (8), (9) can be rewritten as

$$\begin{cases} \dot{\hat{\boldsymbol{x}}}(t) = \boldsymbol{A}\hat{\boldsymbol{x}}(t) + \boldsymbol{B}y_1(t) \\ x_n(t) = -\boldsymbol{C}\hat{\boldsymbol{x}} + y_1(t) \\ \dot{y}_1(t) = y_2(t) \\ \dot{y}_2(t) = F[\boldsymbol{x}(t), u(t)] + g[\boldsymbol{x}(t)]v(t) \end{cases} \tag{10}$$

where $\hat{\boldsymbol{x}} = [x_1, x_2, \dots, x_{n-1}]^T$, $\boldsymbol{C} = [c_1, c_2, \dots, c_{n-1}]$, $\boldsymbol{A}$ is an $(n-1) \times (n-1)$-matrix in companion form whose last row coincides with the vector $-\boldsymbol{C}$, $\boldsymbol{B} = [0, \dots, 0, 1]^T \in \mathbb{R}^{n-1}$, $v(t) = \dot{u}(t)$ and $F[\cdot,\cdot]$ collects all the uncertainties not involving $v(t)$. The first two lines of (10) correspond to a linear system controlled by $y_1(t)$, and this system is stable by assumption. The second two equations of (10) correspond to a nonlinear uncertain second-order system ($y_2(t)$ is not available for measurement) with control $v(t)$. If the control $v(t)$ steers both $y_1(t)$ and $y_2(t)$ to zero, then the linear system becomes an autonomous system evolving on the manifold defined by (7). Note that the last two equations of (10) are coupled with the previous ones through the uncertainties $F[\boldsymbol{x}(t), u(t)]$, $g[\boldsymbol{x}(t)]$.

2.1. The Auxiliary Problem

It is worth noticing that the system state $\boldsymbol{x}(t)$ in (10) is an implicit function of the sliding variables $\boldsymbol{y}(t) = [y_1(t), y_2(t)]^T$. Furthermore, the system input $u(t)$ can be represented as a function of time, so that, in general, $F[\boldsymbol{x}(t), u(t)] \equiv F[\boldsymbol{y}(t), t]$ and

$g[x(t)] \equiv g[y(t),t]$. As a preliminary step of our treatment, we assume that, instead of bounds (2)–(5), the following particular bounds:

$$\left|F[y(t),t]\right| < \bar{F} \tag{11}$$

$$0 < G_1 \leq g\left[y(t),t\right] \leq G_2 \tag{12}$$

are considered. With this assumption, which will be dealt with in the next section, the dynamics relevant to $y_1(t)$ and $y_2(t)$ can be isolated and the following auxiliary problem can be solved separately.

Problem 1. Given a second-order system

$$\begin{cases} \dot{y}_1(t) = y_2(t) \\ \dot{y}_2(t) = F\left[y(t),t\right] + g\left[y(t),t\right]v(t) \end{cases} \tag{13}$$

with unmeasurable $y_2(t)$, and $F[y(t),t]$, $g[y(t),t]$ being uncertain functions with bounds (11), (12), respectively, find a control law $v(t)$ such that $y_1(t)$ and $y_2(t)$ are steered to zero in a finite time in spite of the uncertainties.

Since $y_2(t)$ is not available and $F[y(t),t]$, $g[y(t),t]$ are uncertain, this problem is not easily solvable by any consolidated theory.

A possible solution is derived from a suboptimal version of the well-known bang-bang time-optimal control for a double integrator in which, instead of defining the commutation line as the line on which the quantity $y_1(t) - \frac{1}{2}y_2(t)|y_2(t)|$ changes its sign, it is equivalently defined as the line on which the difference between the current value of $y_1(t)$ and half of its last extremal value y_{1_M} changes its sign. The corresponding suboptimal control algorithm can be obtained by setting $\alpha^* = 1$ in the following.

Algorithm 1.

i) Set $\alpha^* \in (0,1] \cap (0, 3G_1/G_2)$.

ii) Set $y_{1_M} = y_1(0)$.

Repeat, for any $t > 0$, the following steps:

iii) If $[y_1(t) - \frac{1}{2}y_{1_M}][y_{1_M} - y_1(t)] > 0$, then set $\alpha = \alpha^*$; else set $\alpha = 1$.

iv) If $y_1(t)$ is extremal, then set $y_{1_M} = y_1(t)$.

v) Apply the control law

$$v(t) = -\alpha V_M \text{sign}\left\{y_1(t) - \frac{1}{2}y_{1_M}\right\} \tag{14}$$

until the end of the control time interval.

This algorithm is equivalent to the traditional one if $y_1(0)y_2(0) \geq 0$, while it has only one more commutation if $y_1(0)y_2(0) < 0$, in the case when a double integrator is considered. In the case of the uncertain second-order systems under consideration, it is still possible to reach the origin of the y_1Oy_2-plane in a finite time provided that some slight modifications to the algorithm are introduced. To this end, the following lemma was proved (Bartolini *et al.*, 1997).

Lemma 1. *Consider the state equation (13) with bound as in (11)–(12) and $y_2(t)$ not available for measurements. If the extremal value of $y_1(t)$ is evaluated with ideal precision, for any $y_1(0)$ and $y_2(0)$, the sub-optimal control strategy defined by Algorithm 1 with the additional constraint*

$$\begin{aligned} &\alpha^* \in (0,1] \cap \left(0, \frac{3G_1}{G_2}\right) \\ &V_M > \max\left(\frac{\bar{F}}{\alpha^* G_1}; \frac{4\bar{F}}{3G_1 - \alpha^* G_2}\right) \end{aligned} \tag{15}$$

causes the generation of a sequence of states with coordinates $(y_{1_{M_i}}, 0)$ which has the following contraction property:

$$|y_{1_{M_{i+1}}}| < |y_{1_{M_i}}|, \qquad i = 1, 2, \dots \tag{16}$$

Moreover, the convergence of the system trajectory to the origin of the error state plane takes place in a finite time.

The results of Lemma 1 are also valid in case the bound of the uncertain function $F[y(t), t]$ is not constant, e.g. $|F[y(t), t]| \leq N + k(|y_1| + |y_2|)$, provided that $\bar{F}$ overestimates the maximum of $|F[y(t), t]|$ between two subsequent extremal values of the available variable $y_1(t)$, that (15) holds and that a proper initialization phase is implemented (Bartolini *et al.*, 1996a; 1997).

2.2. Chattering Elimination

Now, it will be shown that it is possible to solve the chattering elimination problem by relying on the results obtained with reference to the auxiliary problem of the previous section. To this end, consider eqn. (10) which can be viewed as the connection of two systems coupled through the signal $y_1(t)$ and the nonlinear term $F[x(t), u(t)] + g[x(t)]v(t)$. Simply assume that (2) holds, which is reasonable in many practical situations, while $F[x(t), u(t)]$, though bounded in any bounded domain, cannot be *a priori* assumed to be bounded, since proving the boundedness of its arguments is an objective of this treatment. Thus, the aim of the following analysis is to prove that, after an initialization phase, the state trajectories reach regions of the state space including the origin. Once such regions are reached, the application of Algorithm 1, with minor modifications, leads to a contractive process steering $y_1(t)$ and $y_2(t)$ to zero in a finite time. After that time, the further evolution of the system states is that of an autonomous linear exponentially stable system. In order to describe formally this procedure, the following lemma has been proved (Bartolini *et al.*, 1998).

Lemma 2. *Given the state vector $x(t)$, its norm can be bounded in any bounded interval by a function of its initial values and of the maximum value assumed by $y_1(t)$, whose evolution is represented by (10), in such an interval. In other words,*

$$\|x(t)\| \leq P_x \|x(t_i)\| + Q_x Y_{1(t_1,t)} \tag{17}$$

with

$$Y_{1(t_1,t)} = \max_{t_1 \leq \tau \leq t} |y_1(\tau)|$$

On the basis of previous relationships and lemmas, $F[x(t), u(t)]$ can be written as

$$F[x(t), u(t)] = \Theta_1[x(t)] + \Theta_2[x(t)]u(t) + \Theta_3[x(t)]u^2(t) \tag{18}$$

Using (2)–(5) and (17), we express the upper bounds of $|\Theta_i[\cdot]|$, $i = 1,2,3$, in any finite interval (t_i, t_f), as

$$|\Theta_i[x(t)]| < F_i[Y_{1(t_i,t_f)}], \qquad i = 1,\ldots,3 \tag{19}$$

$F_i[\cdot]$ being an increasing positive function of its argument, i.e. of the maximum value of $y_1(t)$ in the interval (t_i, t_f). Hence, in any finite interval, one can define an upper bound of $|F[x(t), u(t)]|$ as

$$F^*[Y_{1(t_i,t_f)}] = F_1[Y_{1(t_i,t_f)}] + F_2[Y_{1(t_i,t_f)}]|u(t)| + F_3[Y_{1(t_i,t_f)}]u^2(t) \tag{20}$$

Note that the term depending on $u^2(t)$ would not appear in case $g[x(t)]$ were not dependent on $x_n(t)$.

From the previous analysis, to go on with the treatment it must be proved that the chattering elimination problem, after an easily implementable initialization procedure, has the same feature as Problem 1 provided that the control $\dot{u}(t)$ is modified according to the following lemma (Bartolini *et al.*, 1998).

Lemma 3. *Consider the system (10), provided that for $t \in [t_{Max_i}, t_{Max_{i+1}}]$, $(i = 1, 2, \ldots)$, where t_{Max_i} and $t_{Max_{i+1}}$ are the time instant corresponding to two subsequent extremal values of $y_1(t)$, y_{1M_i} and $y_{1M_{i+1}}$, the control signal $\dot{u}(t)$ is chosen as*

$$\begin{aligned}\dot{u}(t) &= -\alpha V_{Max}\operatorname{sign}\Big\{y_1(t) - \frac{1}{2}y_{1M_i}\Big\} \\ &\quad - \Big\{F_2[y_{1M_i}]\,\big|u(t) - u(t_{Max_i})\big| \\ &\quad + F_3[y_{1M_i}]\big|u^2(t) - u^2(t_{Max_i})\big|\Big\}\operatorname{sign}\{y_{1M_i}\}\end{aligned} \tag{21}$$

where α is defined according to Algorithm 1, and V_{Max} is chosen as specified in (15) with $\bar{F}$ given by

$$\begin{aligned}\bar{F} &= F^*[y_{1M_i}] \\ &= F_1[y_{1M_i}] + F_2[y_{1M_i}]\,\big|u(t_{Max_i})\big| + F_3[y_{1M_i}]u^2(t_{Max_i})\end{aligned} \tag{22}$$

Then the trajectories of $y_1(t)$ and $y_2(t)$ in the considered time interval lie between the abscissa axis and the "external" limiting curve defined by Algorithm 1.

Previous results cover a large class of uncertain systems and are semi-global. An effort to extend the developed theory to more general uncertain cases and to exploit some form of learning and adaptation to reduce the control efforts is in progress. In (Bartolini *et al.*, 1996a) some steps in this direction are reported.

3. Approximate Second-Order Sliding-Mode Control

Algorithm 1 relies on the availability of a device capable of detecting the maximum or minimum value of a function of time. This device is obviously an idealization with infinite bandwidth of peak detectors with very high bandwidth which are available on the market. In practical situations, the extremal values of the function can also be evaluated by an approximate differentiator

$$\Delta(t) = \big[y_1(t-\delta) - y_1(t)\big]y_1(t) \tag{23}$$

where δ is an arbitrarily small time delay, and evaluating the time instant when $\Delta(t)$ changes its sign. In particular, the following algorithm, which is a modified version of Algorithm 1, and the related theorem can be regarded as an extension to this approach of the approximability property offered by standard sliding-mode control.

Algorithm 2.

i) Set $\alpha^* \in (0,1] \cap (0, 3G_1/G_2)$.

ii) Set $\delta > 0$, $y_{1_M} = x_1(0)$.

Repeat, for any $t > 0$, the following steps:

iii) If $[y_1(t) - \frac{1}{2}y_{1_M}][y_{1_M} - y_1(t)] > 0$ then set $\alpha = \alpha^*$ else set $\alpha = 1$.

iv) If $(t-\delta) < 0$ then set $y_1(t-\delta) = 0$.

v) If $\Delta(t) < 0$ then $y_{1_{\rm mem}} = y_1(t)$ else $y_{1_{\rm mem}} = y_{1_{\rm mem}}$.

vi) If $\Delta(t) \le 0$ then
if $\{y_{1_{\rm mem}} y_{1_M} > 0\}$ and $\{|y_{1_{\rm mem}}| < |y_{1_M}|\}$ then $y_{1_M} = y_{1_{\rm mem}}$ else $y_{1_M} = y_{1_M}$
else $y_{1_M} = y_{1_{\rm mem}}$.

vii) Apply the control law (14)

until the end of the control time interval.

Theorem 1. *Consider the state equation (13) with bounds as in (11)–(12), $y_2(t)$ not available for measurements, and evaluation of the extremal values of $y_1(t)$ corresponding to the time instants when $\Delta(t)$ in (23) changes its sign according to Algorithm 2.*

Then, for any $y_1(0)$ and $y_2(0)$, the control strategy defined by Algorithm 2 with the additional constraints

$$\begin{aligned}
&\alpha^* \in (0,1] \cap \left(0, \frac{3G_1}{G_2}\right)\\
&V_M \in \left]\max\left(\frac{\bar{F}}{\alpha^* G_1}; V_{M_1}(\delta; y_{1_M})\right); V_{M_2}(\delta; y_{1_M})\right[\\
&\qquad \subseteq \left]\max\left(\frac{\bar{F}}{\alpha^* G_1}; \frac{4\bar{F}}{3G_1 - \alpha^* G_2}\right); +\infty\right[\\
&V_{M_i}: \left[(3G_1 - \alpha^* G_2)\frac{V_{M_i}}{\bar{F}} - 4\right]\frac{y_{1_M}}{\bar{F}\delta^2} - \frac{V_{M_i}}{8\bar{F}}\left[G_1 + G_2(2-\alpha^*)\right]\left(G_2\frac{V_{M_i}}{\bar{F}} + 1\right) = 0
\end{aligned} \tag{24}$$

causes the finite-time convergence of the system trajectory to a δ-vicinity of the origin. Moreover, a sequence of states with coordinates $(x_{Max_i}, 0)$ is generated with the following contraction property:

$$\exists N \in \mathbb{N}^+ : \quad |y_{1_{M_{i+1}}}| < |y_{1_{M_i}}|, \qquad i = 1, 2, \ldots, N-1 \tag{25}$$

and

$$|y_{1_{M_i}}| = O(\delta^2), \qquad i = N, N+1, \ldots \tag{26}$$

with

$$|y_{1_{M_i}} - \hat{y}_{1_{M_i}}| = O(\delta^2), \qquad i = 1, 2, \ldots \tag{27}$$

$y_{1_{M_i}}$ being the actual i-th extremal value of $y_1(t)$, and $\hat{y}_{1_{M_i}}$ being the corresponding value determined by storing the value of $y_1(t)$ when (23) changes its sign according to Algorithm 2.

Proof. The sketch of the proof is reported in the Appendix. ∎

Actually, in this case, inequality (15) is slightly modified in the sense that V_M in (24) does not belong to a semi-infinite but to a finite interval depending on the time delay δ, nevertheless the following limit properties are verified:

$$\lim_{\delta \to 0} V_{M_1}(\delta; y_{1_M}) = \frac{4\bar{F}}{3G_1 - \alpha^* G_2}$$

$$\lim_{\delta \to 0} V_{M_2}(\delta; y_{1_M}) = +\infty$$

The effect of the approximated evaluation of the extremal point of $y_1(t)$ due to the time delay δ in (23) is similar to that of real commutation devices which introduce a delay in the commutation of the control. Yet, by using the proposed procedure, the dimension of the vicinity of the sliding manifold within which the state trajectories are constrained when the real sliding motion is reached is $O(\delta^2)$ instead of $O(\delta)$.

4. The Equivalent Control in Second-Order Sliding-Modes

Second-order sliding modes are characterized by the fact that the control input of the plant $u(t)$ is such that, once the system reaches the sliding manifold $s[x(t)] = 0$ at time instant T_1, the following relationships hold (Levant and Fridman, 1996):

$$s(t) = \dot{s}(t) = 0, \qquad t \geq T_1 \tag{28}$$

If the system (1) is considered, the dynamics of the sliding variable $s(t)$ is characterized by the second-order uncertain system (13) in which $s(t) = y_1(t)$ and $\dot{s}(t) = y_2(t)$, and when the system is in a second-order sliding mode, the switching frequency of the control $v(t) = \dot{u}(t) =$ tends to infinity. The concept of *equivalent control* introduced by Utkin for the first-order sliding modes (Utkin, 1992) can be extended easily to the considered case of second-order sliding modes. Such an extension and the related properties are reported here for the sake of clarity.

Definition 1. Given the system (1), the equivalent control generating a second-order sliding mode on the sliding manifold (7) is the continuous control $v_{\text{eq}}(t) = \dot{u}(t)$ such that the condition (28) is verified with $\ddot{s}(t) = 0, \quad t \geq T_1$.

In the case considered, by means of (2)–(13), the equivalent control can be trivially determined as

$$v_{\text{eq}}(t) = -g\big[y(t), t\big]^{-1} F\big[y(t), t\big] \tag{29}$$

In Section 3, it has been shown that, by applying Algorithm 2 with the conditions defined by Theorem 1, it is possible to reach in a finite time T_1 a boundary layer of the sliding manifold (7) whose dimensions are such that

$$\begin{aligned} &\big|s(t)\big| \leq O(\delta^2) \\ &\big|\dot{s}(t)\big| \leq O(\delta) \end{aligned} \qquad t \geq T_1 \tag{30}$$

The switching frequency of the control $v(t)$ defined by Algorithm 2 tends to infinity as the time delay δ in (23) tends to zero, i.e. the switching imperfections vanish and the system motion tends to an ideal sliding mode. Actually, the time delay δ can be very small when using proper electronic devices, but only a "real" sliding motion can be attained, so that, in practice, the effect of the control $v(t)$ always differs from that of the equivalent control $v_{\text{eq}}(t)$.

Now, define the *average control* $v_{\text{av}}(t)$ as the output of a first-order filter whose input is the discontinuous control $\hat{v}(t)$ defined by Algorithm 2, i.e.

$$\tau\dot{v}_{\text{av}}(t) + v_{\text{av}}(t) = \hat{v}(t) \tag{31}$$

with the time constant τ small enough compared with the slow component of $\hat{v}(t)$, and yet large enough to filter out the high-rate component. By means of the results of Theorem 1 and following the approach in (Utkin, 1992), from which the following is true:

$$\hat{v}(t) = v_{\text{eq}}(t) + g\big[y(t), t\big]^{-1} \ddot{s}(t) \tag{32}$$

we can prove the following.

Theorem 2. *Given a real second-order sliding motion (13) within a boundary layer such that (30) holds, and subject to (2)–(5) and to the additional constraint*

$$\left|\frac{\mathrm{d}F(t)}{\mathrm{d}t}\right| \le \dot{\bar{F}} \tag{33}$$

$\dot{\bar{F}}$ *being a positive constant, it is possible to find a first-order filter of type (31) such that its output is an* $O(\sqrt{\delta})$ *approximation of the ideal equivalent control, i.e.*

$$\begin{aligned} &v_{\mathrm{av}}(t) = v_{\mathrm{eq}}(t) + \Delta v(t) \\ &|\Delta v(t)| = O(\sqrt{\delta}) \end{aligned} \qquad t \gg T_1 \tag{34}$$

Proof. Consider the output of the filter (31) from the time instant $t_i \ge T_1$ on,

$$v_{\mathrm{av}}(t) = v_{\mathrm{av}}(t_i)e^{-\frac{t}{\tau}} + \frac{1}{\tau}e^{-\frac{t}{\tau}} \int_{t_i}^{t} e^{\frac{\theta}{\tau}} \hat{v}(\theta)\,\mathrm{d}\theta$$

By substituting the value of $\hat{v}(t)$ in (32), the following relationship defines $v_{\mathrm{av}}(t)$:

$$v_{\mathrm{av}}(t) = v_{\mathrm{av}}(t_i)e^{-\frac{t}{\tau}} + \frac{1}{\tau}e^{-\frac{t}{\tau}} \int_{t_i}^{t} e^{\frac{\theta}{\tau}} v_{\mathrm{eq}}(\theta)\,\mathrm{d}\theta + \frac{1}{\tau}e^{-\frac{t}{\tau}} \int_{t_i}^{t} e^{\frac{\theta}{\tau}} \frac{\ddot{s}(\theta)}{g[y(\theta),\theta]}\,\mathrm{d}\theta$$

The two integrals can be integrated by parts so that

$$\begin{aligned} v_{\mathrm{av}}(t) \;=\; & v_{\mathrm{eq}}(t) + \left[v_{\mathrm{av}}(t_i) - v_{\mathrm{eq}}(t_i) - \frac{1}{\tau}\frac{\dot{s}(t_i)}{g[y(t_i),t_i]}\right] e^{-\frac{t}{\tau}} + \frac{1}{\tau}\frac{\dot{s}(t)}{g[y(t),t]} \\ & - e^{-\frac{t}{\tau}} \int_{t_i}^{t} e^{\frac{\theta}{\tau}} \left\{ \dot{v}_{\mathrm{eq}}(\theta) + \frac{\dot{s}(\theta)}{\tau}\left[\frac{1}{\theta g[y(\theta),\theta]} - \frac{\dot{g}[y(\theta),\theta]}{g[y(\theta),\theta]}\right]\right\} \mathrm{d}\theta \end{aligned}$$

The uncertain terms of the integral argument can be upper bounded by means of (2)–(5) and (33), and it is possible to define three positive constants A_i, $(i = 1,2,3)$ such that the following relationship holds:

$$|v_{\mathrm{av}}(t) - v_{\mathrm{eq}}(t)| \le \left[v_{\mathrm{av}}(t_i) - v_{\mathrm{eq}}(t_i) - \frac{1}{\tau}\frac{1}{G_1}\right] e^{\frac{-t}{\tau}} + A_1\frac{\delta}{\tau} + A_2\tau + A_3\delta, \quad t \ge t_i \tag{35}$$

This difference is characterized by a time decreasing exponential term and by a constant term; the latter is minimized by the following choice of the time constant τ of the filter:

$$\tau = \sqrt{\delta\frac{A_1}{A_2}} \tag{36}$$

and, by substitution into (35), the following relationship holds:

$$|v_{\mathrm{av}}(t) - v_{\mathrm{eq}}(t)| \le \left[v_{\mathrm{av}}(t_i) - v_{\mathrm{eq}}(t_i) - \frac{1}{\tau}\frac{1}{G_1}\right] e^{\frac{-t}{\tau}} + 2\sqrt{\delta A_1 A_2} + A_3\delta, \quad t \ge t_i \tag{37}$$

Hence there exists a time instant $t_a > t_i \ge T_1$ such that (34) is true. ■

Remark. The choice of the filter time constant as in (36) is such that the condition

$$\lim_{\substack{\tau \to 0 \\ \delta/\tau \to 0}} v_{\text{av}}(t) = v_{\text{eq}}(t)$$

still holds, i.e. the real sliding mode tends to the ideal sliding mode as the nonideality vanishes.

5. Adaptive Reduction of the Discontinuous Control

In the previous section it has been proved that the discontinuous control causing an approximate second-order sliding mode yields, at the output of a linear filter, the same effect of the equivalent control plus an error depending on the dimension of the neighborhood of the sliding manifold to which the system motion is confined (in our case $O(\delta^2)$). In this section this fact is exploited to adaptively reduce the amplitude of the discontinuous control needed to maintain the system in sliding motion on the chosen surface. Indeed, an approximate cancellation of the uncertain " drift" term $F[y(t), t]$ can be performed if the signal v_{av} is part of the derivative of the input of the plant, $v(t)$. This cancellation, in turn, reduces the uncertainty bounds and, consequently, the amplitude of the discontinuous control deriving from the solution of the inequalities (15) or (24) could be reduced. Indeed, the upper bound $\bar{F}$ appearing in such inequalities could be substituted by a new upper bound, *a-priori* evaluable from (35) and from the knowledge of G_1 which can be made arbitrarily small by suitably choosing τ and δ/τ.

The problem is to identify a way to reduce the control amplitude according to the new bound without reducing the robustness of the controlled system whose trajectories need to remain on or close to the sliding manifold.

Assume that the derivative of the plant control input, $v(t)$, can be expressed as

$$v(t) = \lambda(t)\hat{v}(t) + \theta(t)v_{\text{av}}(t) \tag{38}$$

where $\hat{v}(t)$ is the discontinuous control whose amplitude is evaluated from (15) or (24) and is related to the *a-priori* known bounds of $F[y(t), t]$ and $g[y(t), t]$, and $v_{\text{av}}(t)$ is the output of a filter of the type (31) which will be defined later on. The actual control is therefore the weighted sum of $\hat{v}(t)$ and $v_{\text{av}}(t)$ with weights $\lambda(t)$ and $\theta(t)$, respectively, to be adapted during the control phase. From the previous section, when the system is in sliding motion or it evolves within the $O(\delta^2)$-vicinity of the sliding manifold, the equivalent control method can be applied to the system with the new control (38). The equivalent control $\hat{v}_{\text{eq}}(t)$ is evaluated from

$$\ddot{s}(t) = F\big[y(t), t\big] + g\big[y(t), t\big]\Big[\lambda(t)\hat{v}_{\text{eq}}(t) + \theta(t)v_{\text{av}}(t)\Big] = 0$$

so that

$$\hat{v}_{\text{eq}}(t) = -\frac{g^{-1}[y(t), t]F[y(t), t] + \theta(t)v_{\text{av}}(t)}{\lambda(t)} \tag{39}$$

The equivalence between Filippov's solution concept and Utkin's equivalent control method for systems affine in the control law, and the approximability property of systems whose trajectories are close to the sliding manifold, guarantee that the new $v_{\text{av}}(t)$ evaluated from

$$\tau\dot{v}_{\text{av}}(t) + v_{\text{av}}(t) = \hat{v}(t)$$

can be derived from

$$\tau\dot{v}_{\text{av}}(t) + v_{\text{av}}(t) = \hat{v}_{\text{eq}}(t)$$

since the discontinuous control $\hat{v}(t)$ has, on any differential equation coupled with the original system, the same effect of $\hat{v}_{\text{eq}}(t)$. From (39), the equivalent filter representation is

$$\tau\dot{v}_{\text{av}}(t) + \frac{\lambda(t)+\theta(t)}{\lambda(t)} v_{\text{av}}(t) = -\frac{g^{-1}[y(t),t]F[y(t),t]}{\lambda(t)} \tag{40}$$

If $\lambda(t)$ and $\theta(t)$ are time-varying functions satisfying, at any time instant, the conditions $\lambda(t) + \theta(t) = 1$ and $\lambda_{\text{min}} \leq \lambda(t) \leq 1$, with λ_{min} being a lower bound to be determined, the equivalent filter representation is

$$\tau\lambda(t)\dot{v}_{\text{av}}(t) + v_{\text{av}}(t) = -g^{-1}\big[y(t),t\big]F\big[y(t),t\big]$$

i.e.

$$v_{\text{av}}(t) = -g^{-1}\big[y(t),t\big]F\big[y(t),t\big] + O\big(\tau\lambda(t)\big) \tag{41}$$

in the ideal case, $\delta = 0$, or

$$v_{\text{av}}(t) = -g^{-1}\big[y(t),t\big]F\big[y(t),t\big] + O\big(\tau\lambda(t)\big) + O\left(\frac{\delta}{\tau\lambda(t)}\right) \tag{42}$$

in the real case $\delta \neq 0$.

The foregoing derivations are valid under the assumption that with the change of the control from $\dot{u}(t) = \hat{v}(t)$, which resulted from Algorithm 1 or 2 and the related theorem and lemmas, to $\dot{u}(t) = \lambda(t)\hat{v}(t) + \theta(t)v_{\text{av}}(t)$ the system is able to reach the sliding manifold in a finite time and to remain there until a new perturbation occurs. This property is obviously dependent on the choice of the adaptation mechanism used to modify $\lambda(t)$ which must be driven by an on-line evaluation of the discrepancy between the effect of the discontinuous control and its filtered value.

The following treatment is aimed at identifying a proper procedure for achieving the goal of reducing the amplitude of the discontinuous component of the control derivative without loosing the property of counteracting disturbances and uncertainties acting on the system at any unpredictable time instant.

Let us consider now the discontinuous signal

$$v(t) = \hat{v}(t) - v_{\text{av}}(t) \tag{43}$$

In the ideal situation $\delta \to 0$. By applying Filippov's theory, since in the ideal motion $\dot{s}(t) = 0$ and $\ddot{s}(t) = 0$, the effect of the discontinuous signal (43) on any other first-order system

$$\zeta \dot{z}(t) = z(t) + m(t)\left[\hat{v}(t) - v_{\mathrm{av}}(t)\right] \tag{44}$$

is equivalent to that obtained by substituting $\hat{v}(t)$ with $v_{\mathrm{eq}}(t)$ as defined in (39)

$$\zeta \dot{z}(t) = -z(t) + m(t)\left\{-\frac{g^{-1}[y(t),t]F[y(t),t] + \theta(t)v_{\mathrm{av}}(t)}{\lambda(t)} - v_{\mathrm{av}}(t)\right\}$$

Taking into account the condition $\lambda(t) + \theta(t) = 1$, we have

$$\zeta \dot{z}(t) = -z(t) + \frac{m(t)}{\lambda(t)}\left\{-g^{-1}\left[y(t),t\right]F\left[y(t),t\right] - v_{\mathrm{av}}(t)\right\}$$

By introducing $\varepsilon(t)$ as the difference between the ideal equivalent control (29) and the output of the filter (41), the above relationship changes into the following:

$$\zeta \dot{z}(t) = -z(t) - \frac{m(t)}{\lambda(t)}\varepsilon(t)$$

Once within the boundary layer, after setting $m(t) = \lambda(t)$ in (44), a reasonable way to choose $\lambda(t)$ is as follows (Fig. 1):

$$\begin{aligned}
&\zeta \dot{z}(t) = -z(t) + \lambda(t)\left[\hat{v}(t) - v_{\mathrm{av}}(t)\right] \\
&z(t_i) = 1 \\
&\lambda(t) = \begin{cases} 1 & \text{if} \quad |z(t)| \geq 1 \\ |z(t)| & \text{if} \quad \lambda_{\min} < |z(t)| < 1 \\ \lambda_{\min} & \text{if} \quad |z(t)| \leq \lambda_{\min} \end{cases} \\
&\theta(t) = 1 - \lambda(t)
\end{aligned} \tag{45}$$

Here $\lambda_{\min}$ and the saturation level at $\lambda(t) = 1$ are justified by the fact that the various errors sources in filtering can be evaluated *a priori* so that a minimum control weight $\lambda_{\min}$ is evaluable in sliding motion. The saturation level $\lambda = 1$ corresponds to the fact that the discontinuous control signal $\hat{v}(t)$ is always able to guarantee the reaching of the desired sliding manifold in finite time.

6. Example

We consider the position control of a trolley with a time-varying mass, connected to a rigid structure by means of an active suspension, and subjected to an external disturbing force $d(t)$ (Figs. 2 and 3). The control aim is to constrain the system in a fixed position in spite of external disturbances and system uncertainties, i.e. mass variations.

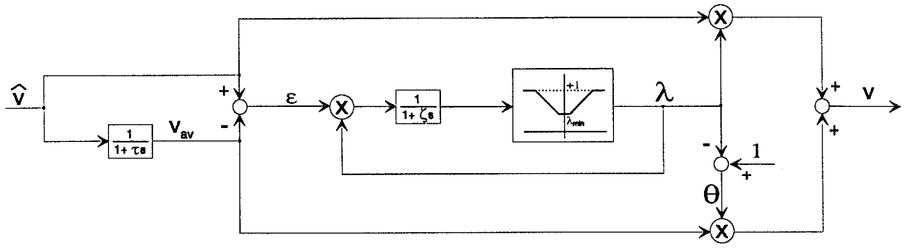

Fig. 1. Adaptation scheme.

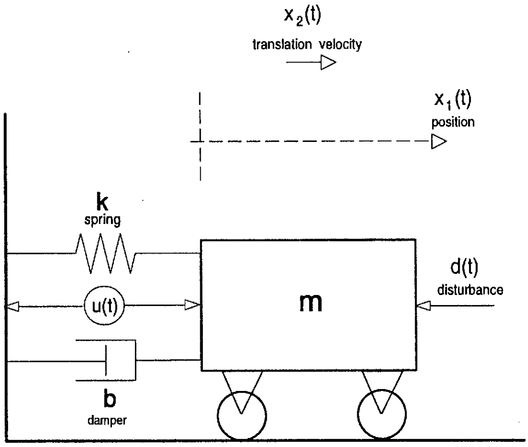

Fig. 2. Controlled trolley.

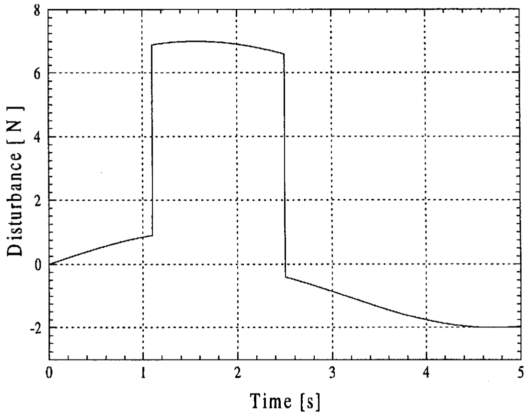

Fig. 3. Disturbing external force.

The dynamics of the system is defined by the following differential system:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = \dfrac{1}{m(t)}\Big[-kx_1(t) + bx_2(t)\Big] + \dfrac{u(t)}{m(t)} - \dfrac{d(t)}{m(t)} \end{cases} \tag{46}$$

where $x_1(t)$ is the variation of the trolley position with respect to the undisturbed one.

The control objective is attained by defining the sliding manifold as $s[\boldsymbol{x}(t)] = x_2(t) + c_1x_1(t) = 0$, so that, once the system is in sliding motion, it performs like a first-order system, and by applying the proposed control procedure. Note that in this example the function $g[\boldsymbol{x}(t)] = 1/m(t)$ is not actually dependent on $\boldsymbol{x}(t)$, and then $F[\boldsymbol{x}(t), u(t)]$ assumes a simpler form than that in (17). Indeed, the system can be expressed as

$$\begin{cases} \dot{y}_1(t) = y_2(t) \\ \dot{y}_2(t) = \Theta_0\big[\boldsymbol{x}(t)\big] + \Theta_1\big[\boldsymbol{x}(t)\big]y_1(t) \\ \qquad +\Theta_2\big[\boldsymbol{x}(t)\big]y_2(t) - \dfrac{1}{m(t)}\dot{u}(t) \end{cases} \tag{47}$$

and this fact implies that the second term of the control signal $\dot{u}(t)$ in (21) is not present.

The simulations have been carried out using Matlab Simulink v.1.3c with the following values: $m = 10 \pm 1\,\mathrm{kg}$, $b = 10\,\mathrm{N\,s\,m^{-1}}$, $k = 500\,\mathrm{N\,m^{-1}}$, $x_1(0) = 0.1$, $x_2(0) = 0$, $c_1 = 5\,\mathrm{s^{-1}}$, $\delta = 10^{-2}\,\mathrm{s}$, $V_M = 500\,\mathrm{N\,s^{-1}}$ and $\alpha^* = 1$.

In the first case Algorithm 2 has been applied directly (without any adaptation mechanism) and the results are depicted in Figs. 4–6. While the control signal $\dot{u}(t)$ is obviously discontinuous with a constant amplitude equal to V_M, the control force actually applied $u(t)$ is continuous but with some oscillations due to the approximate evaluation of the extremal values of the sliding variable $s(t)$. The position of the trolley remains constant after a transient phase. The convergence of $s[\boldsymbol{x}(t)]$ and $\dot{s}[\boldsymbol{x}(t)]$ to a vicinity of the origin of the sliding plane is achieved with a steady-state error less than 10^{-4} and less than 4×10^{-2} as far as $s(t)$ and $\dot{s}(t)$ are concerned, respectively.

If the estimation of the equivalent control and the adaptation, with $\tau = 0.5$ s and $\zeta = 10$ s, is used once the boundary layer is reached, the derivative of the plant input is discontinuous but with reduced amplitude (Fig. 7) so that the actual plant input is continuous with a reduced amplitude of the residual oscillations (Fig. 8). The trolley position and velocity, as well as the trajectory on the sliding plane are depicted in Figs. 9 and 10, respectively. When the adaptation mechanism is used, the steady-state error on the sliding variable $s[\boldsymbol{x}(t)]$ is less than 5×10^{-6} and less than 2×10^{-3} as far as $s(t)$ and $\dot{s}(t)$ are concerned, respectively.

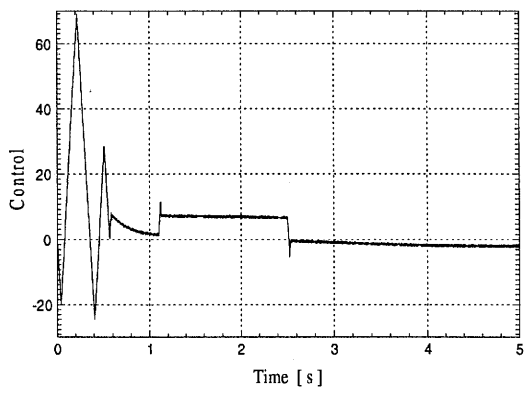

Fig. 4. Plant control input (without adaptation).

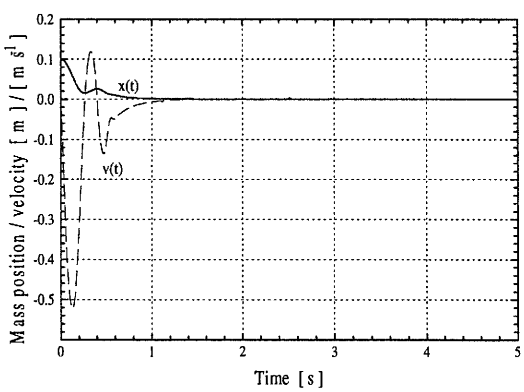

Fig. 5. Mass position and velocity (without adaptation).

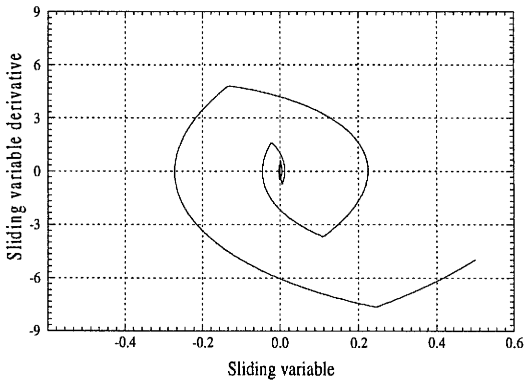

Fig. 6. Trajectory on the sliding plane (without adaptation).

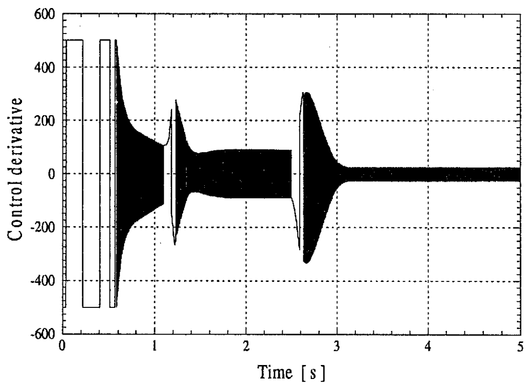

Fig. 7. Derivative of the plant control input (with adaptation).

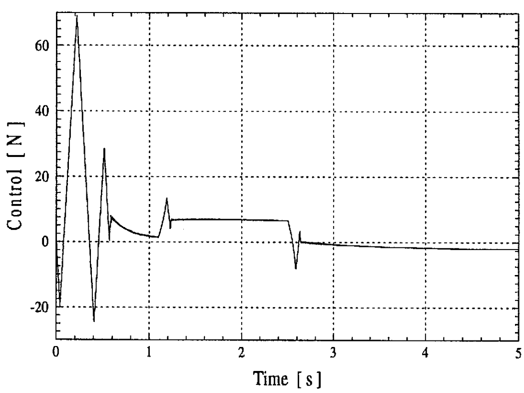

Fig. 8. Control input (with adaptation).

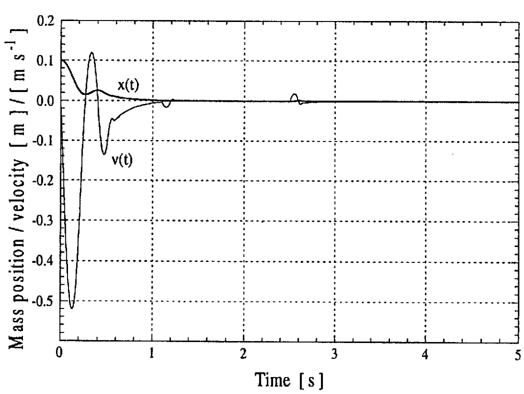

Fig. 9. Mass position and velocity (with adaptation).

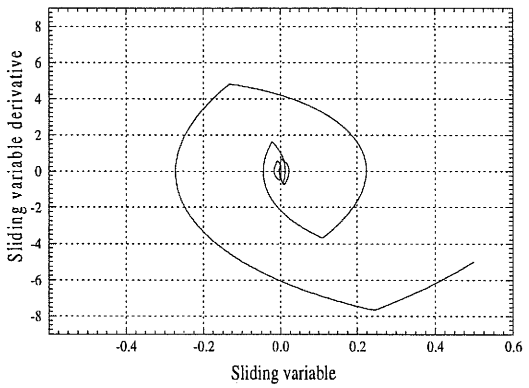

Fig. 10. Trajectory on the sliding plane (with adaptation)

7. Conclusions

The main criticism that sliding mode control detractors usually attribute to this approach when real applications are concerned, is related to the chattering phenomenon and the high control authority, i.e. a high amplitude of the discontinuous control evaluated *a priori* to counteract uncertainties. Both these aspects have been addressed in this paper and a feasible solution has been identified in the spirit of Utkin's work adapted to a second-order sliding mode control approach recently proposed by the authors. Approximability and practical availability of the equivalent control at the output of a high-bandwidth filter are the tools which allow for a satisfactory solution of the problem under consideration. This philosophy paves the way for practical implementations of real control devices applicable in many fields ranging from the control of mechanical systems and electronic drives to the control of power systems.

Appendix

Sketch of the proof of Theorem 1

Consider the k-th extremal value of $y_1(t)$, i.e. $y_{1_{M_k}}$, and assume, for the sake of simplicity, that it is positive. By applying Algorithm 2, its estimate will be evaluated as soon as the device (23) changes its sign from negative to positive with a time delay at most equal to $\delta/2$. Taking into account the uncertainties of the system dynamics it is possible to show that the estimated extremal point $\hat{P}_{M_k} = \left(\hat{y}_{1_{M_k}}\, ; \hat{y}_{2_{M_k}}\right)$ is such that

$$\begin{aligned} \hat{y}_{1_{M_k}} &\in \left[y_{1_{M_k}} - \frac{1}{8}\left(\bar{F} + G_2 V_M\right)\delta^2 \; ; \; y_{1_{M_k}} \right] \\ \hat{y}_{2_{M_k}} &\in \left[0 \; ; \; -\frac{1}{2}\left(\bar{F} + G_2 V_M\right)\delta \right] \end{aligned} \tag{48}$$

At the same time instant at which the extremal point is estimated, the control modulation parameter α switches to α^*, and it remains constant until the switching of the control occurs at a point $P_{c_k} = \left(y_{1_{c_k}}; y_{2_{c_k}}\right)$ such that, for $y_{1_{c_k}} = \hat{y}_{1_{M_k}}/2$,

$$
\begin{aligned}
y_{1_{c_k}} &\in \left[\frac{1}{2} y_{1_{M_k}} - \frac{1}{16}\left(\bar{F} + G_2 V_M\right)\delta^2 \; ; \; \frac{1}{2} y_{1_{M_k}}\right] \\
\hat{y}_{2_{M_k}} &\in \Bigg[-\sqrt{\left(\alpha^* G_1 V_M - \bar{F}\right) y_{1_{M_k}}} \; ; \\
& \qquad -\sqrt{\left(\alpha^* G_2 V_M + \bar{F}\right) y_{1_{M_k}} + \frac{\delta^2}{8}\left(G_2 V_M + \bar{F}\right)\left[\bar{F} + (2-\alpha^*) G_2 V_M\right]} \Bigg]
\end{aligned} \tag{49}
$$

Considering the worst-case trajectory on the $y_1 O y_2$ plane, the condition which assures the decreasing of the modulus of the extremal values of $y_1(t)$ is defined by the following relationship:

$$
\frac{(\alpha^* G_2 - G_1) V_M + 2\bar{F}}{2(G_1 V_M - \bar{F})} y_{1_{M_k}} + \frac{\delta^2}{8} V_M (G_2 V_M + \bar{F}) \frac{G_1 + G_2(2-\alpha)}{2(G_1 V_M - \bar{F})} < y_{1_{M_k}} \tag{50}
$$

Taking into account the dominance condition, i.e. $\alpha^* G_1 V_M > \bar{F}$, by simple algebraic computations, it is possible to show that inequality (50) possesses a solution with respect to V_M within a real interval whose boundary is defined by the solution of the second-order equation in (24). Given $\bar{F}$ and δ, such an interval depends on the current extremal value of $y_1(t)$ and it exists if the equation in (24) has distinct real roots.

As for the value of $|y_{1_{M_k}}|$ which implies the existence of a double real root for the third relationship in (24), it is possible to compute the control amplitude V_M^* which assures the convergence of $y_1(t)$ to the zero-neighbourhood of minimum width ε_1, i.e.

$$
V_M^* = \frac{4\bar{F}}{3G_1 - \alpha^* G_2}\left[1 + \sqrt{1 + \frac{3G_1 - \alpha^* G_2}{4G_2}}\right] \tag{51}
$$

with α^* satifying the first constraint of (24) and such that $V_M^* \geq \bar{F}/\alpha^* G_1$, so that

$$
\begin{aligned}
& |y_{1_{M_i}}| \leq \varepsilon_1, \qquad i = N, N+1, \ldots \\
& \varepsilon_1 \approx \bar{F}\delta^2 \frac{\beta}{8\gamma^2}\left(\gamma + 8G_2 + 4\sqrt{G_2(\gamma + 4G_2)}\right] \\
& \beta = G_1 + G_2(2 - \alpha^*) \\
& \gamma = 3G_1 - \alpha^* G_2
\end{aligned} \tag{52}
$$

The finite-time convergence is a straightforward consequence of the finite time evolution between subsequent extremal values of the available variable $y_1(t)$.

References

Bartolini G. and Pydynowski P. (1993): *Asymptotic linearization of uncertain nonlinear systems by means of continuous control.* — Int. J. Robust Nonlin. Contr., Vol.3, No.2, pp.87–103.

Bartolini G. and Pydynowski P. (1996): *An improved, chattering free, V.S.C. scheme for uncertain dynamical systems.* — IEEE Trans. Automat. Contr., Vol.41, No.8, pp.1220–1226.

Bartolini G., Ferrara A. and Usai E. (1996a): *Second order VSC for non linear systems subjected to a wide class of uncertainty conditions.* — Proc. IEEE Int. Workshop *Variable Structure Systems VSS'96*, Seiken Symp. No.19, Tokyo, pp.49–54.

Bartolini G., Ferrara A., Giacomini L. and Usai E. (1996b): *A combined backstepping/second order sliding mode approach to control a class of nonlinear systems.* — Proc. IEEE Int. Workshop *Variable Structure Systems VSS'96*, Seiken Symp. No.19, Tokyo, pp.205–210

Bartolini G., Ferrara A. and Usai E. (1997a): *Applications of a sub-optimal discontinuous control algorithm for uncertain second order systems.* — Int. J. Robust Nonlin. Contr., Vol.7, No.4, pp.299–319.

Bartolini G., Ferrara A. and Usai E. (1997b): *Chattering avoidance by second order sliding mode control.* — IEEE Trans. Automat. Contr., Vol.43, No.2.

Drakunov S.V. and Utkin V.I. (1990): *Sliding mode control in dynamic systems.* — Int. J. Contr., Vol.55, No.4, pp.1029–1037.

Fu L.C. (1991): *A robust model reference adaptive control using variable structure adaptation for a class of plants.* — Int. J. Contr., Vol.53, pp.1359–1375.

Hsu L. and Costa R.R. (1989): *Variable structure model reference adaptive control using only input and output measurement: Part 1.* — Int. J. Contr., Vol.49, No.2, pp.399–419.

Levant A. and Fridman L. (1996): *Higher order sliding modes as a natural phenomenon in control theory,* In: Robust Control via Variable Structure and Lyapunov Techniques (F. Garofalo and L. Glielmo, Eds.). — LNCIS, Vol.217, London: Springer, pp.302–309.

Sira-Ramirez H. (1992): *On the sliding mode control of nonlinear systems.* — Syst. Contr. Lett., Vol.19, pp.303–312.

Utkin V.I. (1992): *Sliding Modes in Control and Optimization.* — Berlin: Springer.