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ANALYTICAL DESIGN OF STABLE CONTINUOUS-TIME GENERALISED PREDICTIVE CONTROL

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With a recently renewed interest in the continuous-time approach to control system design the continuous-time generalised predictive control (CGPC) is also worth considering. The main objective of this presentation is the development of an analytical perspective that results in explicit design procedures for stable control of both minimum-phase and non-minimum-phase SISO systems. The basic project idea is founded on a set of closed-loop prototype characteristics with definite time-domain specifications.

Keywords: continuous-time systems, design specifications, non-minimum-phase systems, predictive control, stability, system design.

1. Introduction

In recent years the long-range model-based predictive control has been acknowledged as a significant and useful approach to adaptive control design (Pike *et al.*, 1996; Sánchez and Rodellar, 1996; Soeterboek, 1992). The corresponding control system synthesis procedures are based on the so-called 'emulator' paradigm, in which physically unrealisable operations, such as prediction or taking derivatives of output signals, are replaced (emulated) by means of non-parametric or parametric system models (Favier and Dubois, 1990; Gawthrop, 1987; Gawthrop *et al.*, 1996). The famous Minimum-Variance (MV) controller of Åström and Wittenmark (1989, 1997) as well as some extensions like the Generalised Minimum-Variance (GMV) approach by Clarke and Gawthrop (1975) and Peterka (1972), or the Generalised Pole-Placement (GPP) control by Lelič and Zarrop (1987) represent some original designs founded on the idea of the Emulator-Based Control (EBC). The essential relations between the EBC and the Internal Model Control (IMC) principle, established by Morari and Zafiriou (1989), have also been investigated by Gawthrop *et al.* (1996). The Generalised Predictive Control (GPC) proposed by Clarke *et al.* (1987) as a discrete-time parametric polynomial-based design methodology using a long-horizon quadratic cost function has attracted a great interest (Clarke and Mohtadi, 1989; Kouvaritakis *et al.*, 1992; Kowalczuk and Suchomski, 1995; 1996; 1997b; 1998a; 1999; Landau *et al.*, 1998; Suchomski and Kowalczuk, 1998; Wellstead and Zarrop, 1991), which is due to its general applicability as compared to other control strategies. It is also worth noticing

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that the GPC design approach, in which rational models of the controlled plant are used in an 'indirect' manner, is free from some fundamental hindrances restricting the area of applicability of other control algorithms, as it is, for instance, in the case of the basic IMC method 'directly' employing a complete model of the plant that suffers from the limitation to stable plants. What is more, nominal guaranteed stability of predictive algorithms can be obtained by imposing input and output constraints that results in monotonically non-increasing receding-horizon costs (Clarke and Scattolini, 1991; Kouvaritakis *et al.*, 1992; Mosca and Zhang, 1992).

The Continuous-time GPC approach (CGPC), which is an appropriate image of the discrete-time GPC paradigm in the continuous-time domain, has been introduced by Demircioglu and Gawthrop (1991, 1992). It can also be found out from the literature that CGPC is suitable for considering both in robust and adaptive control treatments (Demircioglu and Clarke, 1992; Demircioglu and Gawthrop, 1991; Gawthrop *et al.*, 1998; Kowalczuk *et al.*, 1996; Kowalczuk and Suchomski, 1997a; 1998b; Suchomski and Kowalczuk, 1997). There are three principles of the CGPC approach that can be briefly listed as follows:

(i) the controlled system output is predicted over a finite horizon by using a suitably defined time-domain polynomial functional basis,

(ii) as a future setpoint is known, such a future control (represented in the same functional basis) is evaluated that minimises a quadratic objective function of an anticipated error between the future output and future setpoint,

(iii) following the receding-horizon predictive control strategy, only the first element of the optimal control representation is used as the actual control input.

There are some stability results concerning the receding-horizon LQ control both in the discrete- and continuous-time domain (see e.g. Kleinman, 1970; 1974; Thomas, 1975; Kwon and Pearson, 1975; 1977; 1978; Longchamp, 1983). Such stability results have been employed in redesign of the discrete-time GPC controllers by Clarke and Scattolini (1991) and Mosca and Zhang (1992). In a quest of stability bonds appropriate for the continuous-time CGPC systems, two mutated versions of the generic approach of Demircioglu and Gawthrop (1991) were suggested by Demircioglu and Clarke (1992). These modifications exploiting the stability results of state-space receding-horizon LQ control laws are based on two formulations of the so-called 'end-point principle', namely: end-point state constraints (CGPC_C) and end-point state weighting (CGPC_W). The CGPC_C methodology uses the certainty adopted from the state-space receding-horizon LQ approach that if the state vector of the closed-loop control system at the end-point is constrained to be zero, then the stability of the system is guaranteed. As the system states are not available with the input-output transfer function model applied, they can be emulated by using the truncated Taylor-series expansion technique used for the system output prediction. It is stated (Demircioglu and Clarke, 1992) that for a suitably chosen order of state prediction, i.e. when the corresponding truncated Taylor-series approximation of the system (partial) state is good, the resulting closed-loop control system will be stable. Certain analytical results concerning the resulting closed-loop characteristic polynomial have also been given (*ibid.*) for a special selection of the CGPC_C tuning parameters. Within the CGPC_W approach to the CGPC control stability problem, a quadratic term of the weighted end-point state is included in the cost function. With the assumption that the truncated Taylor-series approximation of the system partial state is satisfactory, the corresponding CGPC_W design is also able to result in stable closed-loop systems. Apparently, this approach is of a general use, as the strategies CGPC and CGPC_C are special cases of the CGPC_W methodology.

The main objective of this contribution is a new development of the CGPC design resulting in an explicit stable CGPC control design procedure for both minimum-phase and non-minimum-phase SISO systems. The proposed completely analytical (ACGPC) method is based on a collection of closed-loop prototype characteristics (different from the proposition of Demircioglu and Clarke (1992)) with definite time-domain specifications. In the literature on predictive control the lack of explicit tuning rules is frequently highlighted as a drawback of the GPC methodology. This opinion also refers to the CGPC approach. In particular, to the best of our knowledge, the interplay between the design parameters and the stability and performance indices has not been satisfactorily explained yet. This work presents a constructive attempt to fill this gap by giving a closed-form design procedure that assures both the nominal closed-loop stability and nominal performance requirements at the same time. This approach represents a direct way of guaranteeing stability and an indirect way of assuring a limited control signal as opposed to the constrained receding-horizon control approach considered in the discrete-time domain.

The detailed contents of the paper is as follows. Minimal and non minimal (redundant) models of a scalar linear continuous-time plant are introduced in Section 2. The considered models take the forms of two rational transfer functions describing the controlled and the disturbed part of the plant, respectively. Procedures for both detection of a plant-model cancellation order and minimal-model reconstruction are also given in Section 2. Estimation of the true order of the plant model is established by examining the rank deficiency of a testing matrix composed of the coefficients of residual polynomials yielded by a properly defined set of Diophantine equations, resulting from the transfer function of the controlled part of the plant model. Two cases of model based prediction concerning the future output (Case α) and the future filtered output of the plant (Case $\bar{\alpha}$) are developed in Section 3. Suitable emulations of the output derivatives, performed by solving a set of coupled Diophantine equations, serve as a basis for the signal prediction. As shown in the next section, the prediction of type α is useful for the CGPC design in the case of minimum-phase models of the controlled plant. On the other hand, if the model is non-minimum phase, the emulation of derivatives of the output signal filtered by the numerator polynomial of the transfer function of the controlled part of the plant (Case $\bar{\alpha}$), is appropriate. The necessary emulation procedures are given in Section 3. The corresponding CGPC design algorithms are presented in Section 4. Namely, the generic CGPC scheme is given, and then the two introduced predictive control laws are described: the first control (α) based on the output emulation and restricted to minimum phase plant models, and the other control ($\bar{\alpha}$) utilising the filtered output emulation and applicable both for minimum phase and non-minimum phase plant models. The resulting closed-loop systems are thoroughly analysed. In particular, explicit and relatively simple formulae for closed-loop characteristic polynomials are given that can serve as an efficient basis for analytical CGPC design procedures (both ways, α and $\bar{\alpha}$), in which the principal CGPC 'tuning knobs', i.e. the output and control prediction orders as well as the horizon of observation, are directly related to common time-domain design specifications. Considerations indicating meaningful consequences of miscellaneous types of potential non-minimality of plant models are presented in Section 5, taking into account various aspects of the CGPC design. The analytical nature of the proposed methodology calls for an analysis of the emulation-based design from the viewpoint of pole placement. Such a deliberation is offered in Section 6. The paper is completed with a simple design example and conclusions presented in Section 7 and Section 8, respectively. There is also a collection of specific studies provided in Appendices (A–F), where, in particular, the prototype design characteristic polynomials are catalogued and certain computational aspects of the developed procedures are explained in detail.

2. Plant Modelling

Let a scalar linear continuous-time plant be described by the following model:

$$Y(s) = \frac{B(s)}{A(s)}U(s) + \frac{C(s)}{A(s)}V(s) \tag{1}$$

where $U(s)$ and $Y(s)$ are the input and output signals, $V(s)$ represents a disturbance function, $A(s)$, $B(s)$ and $C(s)$ are polynomials in the Laplace domain: $A(s) = \sum_{i=0}^{N_A} a_i s^i$, $a_{N_A} = 1$, $\deg A(s) = N_A \geq 2$; $B(s) = \sum_{i=0}^{N_B} b_i s^i$, $\deg B(s) = N_B$; $C(s) = \sum_{i=0}^{N_C} c_i s^i$, $\deg C(s) = N_C = N_A - 1$, with $\rho = N_A - N_B$, $\rho > 0$ being the relative plant order. Consider the first Diophantine equation (cf. Demircioglu and Gawthrop, 1991; Gawthrop, 1987)

$$(\mathrm{D1}): \qquad A(s)H_k(s) + L_k(s) = s^k B(s), \quad k \geq 0$$

where the quotient polynomials $H_k(s) = \sum_{i=0}^{k-\rho} h_{k-i} s^i$, $k \geq \rho$, are composed of the plant Markov parameters h_i's, $i \geq 0$, resulting from the power series expansion in s^{-1} of the controlled part of the plant model (1): $B(s)/A(s) = \sum_{i=0}^{\infty} h_i s^{-i}$, with $h_0 = \cdots = h_{\rho-1} = 0$, and $L_k(s) = \sum_{i=0}^{N_A-1} l_{k,i} s^i$, $k \geq 0$ standing for the residual polynomials. The two specific polynomials are additionally characterised by the following lemma:

Lemma 1. *Properties of the solutions to the first Diophantine equation (D1):*

$$\deg H_k(s) = k - \rho \quad \textit{if} \quad k \geq \rho \quad \textit{with} \quad H_k(s) = 0 \quad \textit{if} \quad k < \rho$$

$$\deg L_k(s) = \begin{cases} \leq N_A - 1 & \textit{if} \quad k \geq 0 \\ = N_B + k & \textit{if} \quad k < \rho \end{cases} \quad \textit{with} \quad L_k(s) = s^k B(s) \quad \textit{if} \quad 0 \leq k \leq \rho - 1$$

In order to facilitate further discussion, let us introduce two structured matrices. Let $\boldsymbol{L}_N$, $N \geq 0$ denote the first Diophantine residual polynomial-allied matrix defined as

$$\boldsymbol{L}_N = \begin{bmatrix} \boldsymbol{l}_0 & \cdots & \boldsymbol{l}_N \end{bmatrix}^T, \quad \boldsymbol{L}_N \in \mathbb{R}^{(N+1)\times N_A}$$

where the vectors $\boldsymbol{l}_k = [l_{k,0} \quad \cdots \quad l_{k,N_A-1}]^T$, $\boldsymbol{l}_k \in \mathbb{R}^{N_A}$, are composed of the coefficients of the residuals $L_k(s)$, $0 \leq k \leq N$. Moreover, let $\boldsymbol{T}_M^{N_\Gamma} \in \mathbb{R}^{(N_\Gamma+M)\times M}$, $M \geq 1$, denote the following M-column lower band-diagonal matrix associated with a given polynomial $\Gamma(s) = \sum_{i=0}^{N_\Gamma} \gamma_i s^i$, $\deg \Gamma(s) = N_\Gamma$:

$$\boldsymbol{T}_M^{N_\Gamma} = \begin{bmatrix} \gamma_0 & 0 & \cdots & 0 \\ \gamma_1 & \gamma_0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \gamma_0 \\ \gamma_{N_\Gamma} & \gamma_{N_\Gamma-1} & \cdots & \cdots \\ 0 & \gamma_{N_\Gamma} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \gamma_{N_\Gamma-1} \\ 0 & 0 & \cdots & \gamma_{N_\Gamma} \end{bmatrix}$$

For compactness of the presentation, the proofs of the following three lemmas are given in Appendix A.

Lemma 2. (Non-singularity of $\boldsymbol{L}_{N_A-1}^T$ for a coprime pair $(A(s), B(s))$) *Polynomials $A(s)$, $\deg A(s) = N_A$, and $B(s)$, $\deg B(s) = N_B$, are relatively prime if and only if $\boldsymbol{L}_{N_A-1}^T \in \mathbb{R}^{N_A \times N_A}$ is nonsingular.*

Consider now the case of a non-minimal model (1). Let $A(s) = A'(s)\Lambda(s)$, $\deg A'(s) = N_A - N_\Lambda$, and $B(s) = B'(s)\Lambda(s)$, $\deg B'(s) = N_B - N_\Lambda$, where $A'(s)$ and $B'(s)$ describing the true plant are relatively prime while $N_\Lambda = \deg \Lambda(s)$, $0 \leq N_\Lambda \leq N_B$, denotes the cancellation order of the plant model.

Lemma 3. (Rank deficiency of $\boldsymbol{L}_{N_A-1}^T$ for a reducible non-coprime pair $(A(s), B(s))$) *For polynomials $A(s)$, $\deg A(s) = N_A$, and $B(s)$, $\deg B(s) = N_B$, having the greatest common divisor $\Lambda(s)$, $\deg \Lambda(s) = N_\Lambda$, we have* $\operatorname{rank} \boldsymbol{L}_{N_A-1}^T = N_A - N_\Lambda$ *and the range space of $\boldsymbol{L}_{N_A-1}^T$ can be found as* $\mathcal{R}[\boldsymbol{L}_{N_A-1}^T] = \operatorname{span}\{\boldsymbol{l}_k\}_{k=0}^{N_A-N_\Lambda-1}$.

A robust procedure for determination of the cancellation order N_Λ is a crucial point of the CGPC design. As has been shown in Appendix B, for given $A(s)$ and $B(s)$ the cancellation order N_Λ can easily be obtained by examining the column rank of left column submatrices of $\boldsymbol{L}_{N_A-1}^T$, and if such an index i of minimal value $\rho \leq i \leq N_A - 1$ exists for which $\boldsymbol{l}_i \in \mathcal{R}[\boldsymbol{L}_{i-1}^T]$, it implies $N_\Lambda = N_A - i$. A complete recursive algorithm developed for this purpose is described in Appendix B. The cancellation order can also be determined in a standard way by examining rank deficiency of the resultant Sylvester matrix $\left[\boldsymbol{T}_{N_B}^{N_A} \,\vdots\, \boldsymbol{T}_{N_A}^{N_B}\right] \in \mathbb{R}^{(N_A+N_B)\times(N_A+N_B)}$ (Fuhrman, 1996;

Kailath, 1980; Landau *et al.*, 1998). The algorithm given in Appendix B has, however, an essential advantage that $\boldsymbol{L}_{N_A-1}^T$ has lower dimensions. Once we have determined $N_\Lambda > 0$, the minimal model, i.e. the coprime pair $(A'(s), B'(s))$, can be reconstructed.

Lemma 4. (Reconstruction of the minimal model $(A'(s), B'(s))$) *Let* $A(s) = A'(s)\Lambda(s)$, $\deg A(s) = N_A$, *and* $B(s) = B'(s)\Lambda(s)$, $\deg B(s) = N_B$, *are reducible with the greatest common divisor* $\Lambda(s)$ *of* $\deg \Lambda(s) = N_\Lambda > 0$. *The coefficients of the minimal polynomials* $A'(s) = \sum_{i=0}^{N_A-N_\Lambda} a_i' s^i$, $a'_{N_A-N_\Lambda} = 1$, *and* $B'(s) = \sum_{i=0}^{N_B-N_\Lambda} b_i' s^i$ *satisfy*

$$b'_{N_B-N_\Lambda} = b_{N_B}$$

$$\begin{bmatrix} \boldsymbol{T}_{N_A-N_\Lambda}^{N_B} & \vdots & -\boldsymbol{T}_{N_B-N_\Lambda}^{N_A} \end{bmatrix} \begin{bmatrix} \boldsymbol{a}' \\ \cdots \\ \boldsymbol{b}' \end{bmatrix} = \boldsymbol{t}_{ab} \quad \textit{if} \quad N_\Lambda < N_B \tag{2}$$

$$\boldsymbol{T}_{N_A-N_\Lambda}^{N_B} \boldsymbol{a}' = \bar{\boldsymbol{t}}_{ab} \quad \textit{if} \quad N_\Lambda = N_B \tag{3}$$

where

$$\boldsymbol{a}' = \begin{bmatrix} a_0' & \cdots & a'_{N_A-N_\Lambda-1} \end{bmatrix}^T, \quad \boldsymbol{a}' \in \mathbb{R}^{N_A-N_\Lambda}$$

$$\boldsymbol{b}' = \begin{bmatrix} b_0' & \cdots & b'_{N_B-N_\Lambda-1} \end{bmatrix}^T, \quad \boldsymbol{b}' \in \mathbb{R}^{N_B-N_\Lambda} \quad \textit{if} \quad N_\Lambda < N_B$$

$$\boldsymbol{t}_{ab} = -\begin{bmatrix} \boldsymbol{0}_{N_A-N_\Lambda} \\ \cdots\cdots \\ \boldsymbol{b} \end{bmatrix} + b_{N_B} \begin{bmatrix} \boldsymbol{0}_{N_B-N_\Lambda} \\ \cdots\cdots \\ \boldsymbol{a} \end{bmatrix}, \quad \boldsymbol{t}_{ab} \in \mathbb{R}^{N_A+N_B-N_\Lambda} \quad \textit{if} \quad N_\Lambda < N_B$$

$$\bar{\boldsymbol{t}}_{ab} = -\begin{bmatrix} \boldsymbol{0}_{N_A-N_\Lambda} \\ \cdots\cdots \\ \boldsymbol{b} \end{bmatrix} + b_{N_B}\boldsymbol{a}, \quad \bar{\boldsymbol{t}}_{ab} \in \mathbb{R}^{N_A} \quad \textit{if} \quad N_\Lambda = N_B$$

$$\boldsymbol{a} = [a_0 \ \cdots \ a_{N_A-1}]^T, \quad \boldsymbol{a} \in \mathbb{R}^{N_A}, \quad \boldsymbol{b} = [b_0 \ \cdots \ b_{N_B-1}]^T, \quad \boldsymbol{b} \in \mathbb{R}^{N_B}$$

3. Model Based Prediction

Two cases of the model-based emulation are considered in this section. In particular, two cases of prediction concerning the future output (Case α) and the future filtered output (Case $\bar{\alpha}$) of the plant are developed. The algebraically obtained estimates (emulations) of both the output derivatives (α) and the filtered output derivatives ($\bar{\alpha}$) lay foundations for signal prediction. The emulations are provided by solving properly defined sets of two coupled Diophantine equations. The above distinction ($\alpha/\bar{\alpha}$) is of consequence and serves as a basis for the CGPC design for minimum phase and non-minimum phase plants, respectively. As will be shown in the next section, the proposed analytical CGPC design method based on prediction of the output of the controlled plant (Case α) is suitable solely for minimum phase plants. In order to expand the area of applicability of the analytical ACGPC method, additional filtering of the plant output by an auxiliary all-pole filter employing the numerator of the transfer function describing the controlled part of the plant model (Case $\bar{\alpha}$) is introduced. In both the distinguished cases, computations are performed by considering a pair of coupled Diophantine equations. The issue of solvability of this equations taking into account model non-minimality is also considered.

3.1. Emulation of Output Derivatives (α)

Let $(A(s), B(s))$ be coprime. In order to emulate the k-th 'derivatives' of the plant output $Y_k(s) = s^k Y(s)$, $k \geq 0$, the following two Diophantine equations are taken into account (Demircioglu and Gawthrop, 1991; Gawthrop, 1987):

$$\text{(D2)}: \qquad A(s)E_k(s) + F_k(s) = s^k C(s)$$

$$\text{(D3)}: \qquad C(s)H_k(s) + G_k(s) = B(s)E_k(s)$$

with constituent polynomials described by the following two lemmas (see also Lemma 1 and Appendix C). Different interviews of conditions for solvability of the Diophantine equations were done, for instance, by Grimble (1994), Ježek (1993), Kučera (1993) and Ogata (1995).

Lemma 5. (Properties of the design polynomials resulting from the second (D2) and third (D3) Diophantine equations)

$$\deg E_k(s) = \begin{cases} 0 & \textit{if} \;\; k = 0 \\ k-1 & \textit{if} \;\; k \geq 1 \end{cases} \qquad \textit{with} \;\; E_k(s) = 0 \qquad \textit{if} \;\; k = 0$$

$$\deg F_k(s) = \begin{cases} = N_A - 1 & \textit{if} \;\; k = 0 \\ \leq N_A - 1 & \textit{if} \;\; k \geq 1 \end{cases} \qquad \textit{with} \;\; F_k(s) = C(s) \quad \textit{if} \;\; k = 0$$

$$\deg G_k(s) = \begin{cases} = N_B + k - 1 & \textit{if} \;\; 1 \leq k < \rho \\ \leq N_A - 2 & \textit{if} \;\; k \geq \rho \end{cases} \qquad \textit{with} \;\; G_k(s) = 0 \qquad \textit{if} \;\; k = 0$$

$$G_k(s) = B(s)E_k(s) \qquad \textit{if} \;\; 1 \leq k < \rho$$

Lemma 6. (Appearance of the third Diophantine equation (D3) for $\rho \geq 1$)

A. *Zero solution:* $G_k(s) = H_k(s) = 0$ *if* $k = 0$.

B. *Strictly proper rational solutions (only for* $\rho \geq 2$*):* $\deg G_k(s) = N_B + k - 1 < N_A - 1$ *and* $H_k(s) = 0$ *if* $1 \leq k < \rho$.

C. *Proper rational solution:* $\deg G_k(s) \leq N_A - 2$ *and* $\deg H_k(s) = 0$ *if* $k = \rho$.

D. *Improper rational solutions:* $\deg G_k(s) \leq N_A - 2$ *and* $\deg H_k(s) = k - \rho$ *if* $k > \rho$.

By (D1)–(D3) the following complementary (fourth) Diophantine equation can easily be derived:

$$\text{(D4)}: \qquad A(s)G_k(s) + B(s)F_k(s) = C(s)L_k(s), \quad k \geq 0$$

The operator form of the predictable part $Y_k^*(s)$ of $Y_k(s) = s^k Y(s)$ becomes $Y_k^*(s) = Y_k^-(s) + Y_k^+(s)$, $k \geq 0$, where $Y_k^-(s)$ denotes the 'observer' part

$$Y_k^-(s) = \frac{G_k(s)}{C(s)}U(s) + \frac{F_k(s)}{C(s)}Y(s)$$

with the control signal filtered by a strictly proper transfer function $G_k(s)/C(s)$ and the plant output filtered by a proper transfer function $F_k(s)/C(s)$, while $Y_k^+(s) = H_k(s)U(s)$ stands for the 'predictor' part that is completely determined by the quotient polynomials $H_k(s)$.

Thus, in order to obtain $Y_k^*(s)$, the pair $(F_k(s), G_k(s))$ should be determined, $k \geq 0$. This can be achieved in two design paths: by solving the coupled Diophantine equations (D2) and (D3) (the emulator path) or by solving the other coupled Diophantine equations (D1) and (D4) (the observer path). The equations (D1), (D2) and (D3) can easily be solved by utilising the recursive algorithm given in Appendix C. Equation (D4) is equivalent to the following set of linear equations with a non-singular Sylvester matrix (see also Appendix C):

$$\begin{matrix} \\ \\ \rho - 1 \left\{ \right. \end{matrix} \left[\begin{array}{c:c} \boldsymbol{T}_{N_A}^{N_B} & \\ \cdots\cdots & \boldsymbol{T}_{N_C}^{N_A} \\ \mathbf{0} & \end{array} \right] \left[\begin{array}{c} \boldsymbol{f}_k \\ \cdots \\ \boldsymbol{g}_k \end{array} \right] = \boldsymbol{T}_{N_A}^{N_C} \boldsymbol{l}_k \tag{4}$$

where the vectors

$$\boldsymbol{f}_k = \begin{bmatrix} f_{k,0} & \cdots & f_{k,N_A-1} \end{bmatrix}^T, \quad \boldsymbol{f}_k \in \mathbb{R}^{N_A}$$

$$\boldsymbol{g}_k = \begin{bmatrix} g_{k,0} & \cdots & g_{k,N_C-1} \end{bmatrix}^T, \quad \boldsymbol{g}_k \in \mathbb{R}^{N_C}$$

are composed of the coefficients of the residual polynomials $F_k(s) = \sum_{i=0}^{N_A-1} f_{k,i}s^i$ and $G_k(s) = \sum_{i=0}^{N_C-1} g_{k,i}s^i$, $k \geq 0$, respectively.

3.2. Emulation of Filtered Output Derivatives ($\bar{\alpha}$)

Let us develop an emulator/observer for estimation of the derivatives of the filtered output $\bar{Y}(s) = Y(s)/B(s)$, instead of the previously considered output $Y(s)$. It is important that (for the limited horizon procedure developed below) potential instability of the filter $1/B(s)$ has no significance, and that, in the non-trivial case of $N_B \geq 1$ considered beneath, the model order $N_A \geq 2$. Let $(A(s), B(s))$ be coprime. The following four Diophantine equations analogous to (D1)–(D4) make a suitable

Diophantine basis (for $k \geq 0$) for the CGPC design development:

$$(\bar{\text{D}}1): \qquad A(s)\bar{H}_k(s) + \bar{L}_k(s) = s^k$$

$$(\bar{\text{D}}2): \qquad A(s)\bar{E}_k(s) + B(s)\bar{F}_k(s) = s^k C(s)$$

$$(\bar{\text{D}}3): \qquad C(s)\bar{H}_k(s) + \bar{G}_k(s) = \bar{E}_k(s)$$

$$(\bar{\text{D}}4): \qquad A(s)\bar{G}_k(s) + B(s)\bar{F}_k(s) = C(s)\bar{L}_k(s)$$

The operator form of the k-th 'derivative' of $\bar{Y}(s)$ gains the form $\bar{Y}_k(s) = s^k\bar{Y}(s) = \bar{Y}_k^*(s) + \bar{V}_k^+(s)$ with the predictable part of $\bar{Y}_k(s)$

$$\bar{Y}_k^*(s) = \frac{\bar{E}_k(s)}{C(s)}U(s) + \frac{\bar{F}_k(s)}{C(s)}Y(s)$$

and the error part of $\bar{Y}_k(s)$

$$\bar{V}_k^+(s) = \frac{\bar{E}_k(s)}{B(s)}V(s)$$

where $\deg \bar{E}_k(s) = \max\{N_B - 1, N_C - N_A + k\}$, $\deg \bar{F}_k(s) \leq N_A - 1$. The transfer function $\bar{E}_k(s)/C(s)$ can be represented by a strictly proper rational part $\bar{G}_k(s)/C(s)$ and a polynomial part $\bar{H}_k(s)$, where (see ($\bar{\text{D}}$3))

- if $\deg \bar{E}_k(s) < N_C$: $\bar{G}_k(s) = \bar{E}_k(s)$, $\deg \bar{G}_k(s) = \deg \bar{E}_k(s)$, and $\bar{H}_k(s) = 0$;
- if $\deg \bar{E}_k(s) = N_C$: $\deg \bar{G}_k(s) \leq N_C - 1$, and $\deg \bar{H}_k(s) = 0$;
- if $\deg \bar{E}_k(s) > N_C$: $\deg \bar{G}_k(s) \leq N_C - 1$, and $\deg \bar{H}_k(s) = \deg \bar{E}_k(s) - N_C$.

Presuming $N_C = N_A - 1$ one obtains $\deg \bar{E}_k(s) = \max\{N_B - 1, k - 1\}$. Moreover, the following lemma holds:

Lemma 7. (Appearance of the third Diophantine decomposition ($\bar{D}$3))

A. *Zero solution:* $\bar{G}_k(s) = \bar{H}_k(s) = 0$ *if* $B(s)$ *is a factor of* $C(s)$ *and* $k \leq N_B$.

B. *Strictly proper rational solutions:* $\bar{G}_k(s) = \bar{E}_k(s)$, $\deg \bar{G}_k(s) = \max\{N_B - 1, k - 1\}$ *and* $\bar{H}_k(s) = 0$ *if* $k < N_A$.

C. *Proper rational solutions:* $\deg \bar{G}_k(s) \leq N_A - 2$ *and* $\deg \bar{H}_k(s) = 0$ *if* $k = N_A$.

D. *Improper rational solutions:* $\deg \bar{G}_k(s) \leq N_A - 2$ *and* $\deg \bar{H}_k(s) = k - N_A$ *if* $k > N_A$.

The emulator equation for $\bar{Y}_k^*(s)$ thus becomes $\bar{Y}_k^*(s) = \bar{Y}_k^-(s) + \bar{Y}_k^+(s)$, $k \geq 0$, in which $\bar{Y}_k^-(s)$ denotes the corresponding 'observer' part

$$\bar{Y}_k^-(s) = \frac{\bar{G}_k(s)}{C(s)}U(s) + \frac{\bar{F}_k(s)}{C(s)}Y(s)$$

with the control signal filtered via the strictly proper transfer function $\bar{G}_k(s)/C(s)$ and the plant output filtered by the proper transfer function $\bar{F}_k(s)/C(s)$, and $\bar{Y}_k^+(s) = \bar{H}_k(s)U(s)$ stands for the corresponding 'predictor' part, which is based on the polynomials $\bar{H}_k(s) = \sum_{i=0}^{k-N_A} \bar{h}_{k-i}s^i$, $k \geq N_A$, composed of the Markov parameters $\bar{h}_i$'s, $i \geq 0$, associated with the all-pole plant-allied system $1/A(s) = \sum_{i=0}^{\infty} \bar{h}_i s^{-i}$, $\bar{h}_0 = \cdots = \bar{h}_{N_A-1} = 0$. Moreover, since $B(s)$ does not appear in ($\bar{\text{D}}$1), the coefficients of the residual polynomials $\bar{L}_k(s) = \sum_{i=0}^{N_A-1} \bar{l}_{k,i}s^{-i}$, $k \geq 0$, characterised by $\bar{L}_k(s) = s^k$ ($\deg \bar{L}_k(s) = k$) if $0 \leq k \leq N_A - 1$ and $\deg \bar{L}_k(s) \leq N_A - 1$ for $k \geq N_A$, cannot be employed in order to detect the cancellation order N_Λ. This effect can only be achieved by means of (D1).

From the above development, it follows that, in order to obtain $Y_k^*(s)$, the pair $(\bar{F}_k(s), \bar{G}_k(s))$ has to be resolved. Similarly to the previously considered case, the two CGPC design paths are now available: the emulator path in which the coupled Diophantine equations ($\bar{\text{D}}$2) and ($\bar{\text{D}}$3) are utilised, and the observer path dealing with the other pair of the coupled Diophantine equations ($\bar{\text{D}}$1) and ($\bar{\text{D}}$4). Equations ($\bar{\text{D}}$1) and ($\bar{\text{D}}$3) can easily be solved in a standard recursive manner. Equation ($\bar{\text{D}}$3) is equivalent to the following set of linear equations with a non-singular Sylvester matrix (see Appendix D):

$$N_{\bar{E}_k} - N_B + 1 \left\{ \begin{bmatrix} \boldsymbol{T}_{N_A}^{N_B} & \vdots & \\ \cdots\cdots & \vdots & \boldsymbol{T}_{N_{\bar{E}_k}+1}^{N_A} \\ \boldsymbol{0} & \vdots & \end{bmatrix} \right. \begin{bmatrix} \bar{\boldsymbol{f}}_k \\ \cdots \\ \bar{\boldsymbol{e}}_k \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \cdots \\ \boldsymbol{c} \\ \cdots \\ \boldsymbol{0} \end{bmatrix} \begin{matrix} \}k \\ \\ \\ \\ \}N_{\bar{E}_k} - k + 1 \end{matrix}$$

where $N_{\bar{E}_k} = \max\{N_B - 1, k - 1\}$,

$$\boldsymbol{c} = [c_0 \ \cdots \ c_{N_A-1}]^T, \quad \boldsymbol{c} \in \mathbb{R}^{N_A}$$

and the vectors

$$\bar{\boldsymbol{f}}_k = [\bar{f}_{k,0} \ \cdots \ \bar{f}_{k,N_A-1}]^T, \quad \bar{\boldsymbol{f}}_k \in \mathbb{R}^{N_A}$$

$$\bar{\boldsymbol{e}}_k = [\bar{e}_{k,0} \ \cdots \bar{e}_{k,N_{\bar{E}_k}}]^T, \quad \bar{\boldsymbol{e}}_k \in \mathbb{R}^{N_{\bar{E}_k}+1}$$

are composed of the coefficients of the polynomials $\bar{F}_k(s) = \sum_{i=0}^{N_A-1} \bar{f}_{k,i}s^i$ and $\bar{E}_k(s) = \sum_{i=0}^{N_{\bar{E}_k}} \bar{e}_{k,i}s^i$ for $k \geq 0$, respectively. Consequently, eqn. ($\bar{\text{D}}$4) leads to the following set of linear equations with a non-singular Sylvester matrix (see Appendix D):

$$\rho - 1 \left\{ \begin{bmatrix} \boldsymbol{T}_{N_A}^{N_B} & \vdots & \\ \cdots\cdots & \vdots & \boldsymbol{T}_{N_C}^{N_A} \\ \boldsymbol{0} & \vdots & \end{bmatrix} \right. \begin{bmatrix} \bar{\boldsymbol{f}}_k \\ \cdots \\ \bar{\boldsymbol{g}}_k \end{bmatrix} = \boldsymbol{T}_{N_A}^{N_C} \bar{\boldsymbol{l}}_k$$

where

$$\bar{\boldsymbol{l}}_k = [\bar{l}_{k,0} \ \cdots \ \bar{l}_{k,N_A-1}]^T, \quad \bar{\boldsymbol{l}}_k \in \mathbb{R}^{N_A}$$

$$\bar{\boldsymbol{g}}_k = [\bar{g}_{k,0} \quad \cdots \quad \bar{g}_{k,N_C-1}]^T, \quad \bar{\boldsymbol{g}}_k \in \mathbb{R}^{N_C}$$

are composed of the coefficients of the polynomials $\bar{L}_k(s)$ and $\bar{G}_k(s) = \sum_{i=0}^{N_C-1} \bar{g}_{k,i}s^i$, $k \geq 0$, respectively.

3.3. Estimation of the Future Output and Future Filtered Output of the Plant (α and $\bar{\alpha}$)

The time-domain estimate of the k-th derivative of the plant output $Y(s)$ takes the form

$$(\alpha): \qquad y_k^*(t) = L^{-1}\left[Y_k^*(s)\right] = \begin{cases} y_k^-(t) & \text{if} \quad k < \rho \\ y_k^-(t) + y_k^+(t) & \text{if} \quad k \geq \rho \end{cases} \tag{5}$$

where $y_k^-(t) = L^{-1}[Y_k^-(s)]$ for $k \geq 0$, $y_k^+(t) = \sum_{i=0}^{k-\rho} h_{k-i}u_i(t)$ for $k \geq \rho$, and $u_i(t) = \mathrm{d}^i u(t)/\mathrm{d}\,t^i$, $i \geq 0$. Similarly, the time-domain estimate of the k-th derivative of the filtered output $\bar{Y}(s) = Y(s)/B(s)$ is represented via

$$(\bar{\alpha}): \qquad \bar{y}_k^*(t) = L^{-1}\left[\bar{Y}_k^*(s)\right] = \begin{cases} \bar{y}_k^-(t) & \text{if} \quad k < N_A \\ \bar{y}_k^-(t) + \bar{y}_k^+(t) & \text{if} \quad k \geq N_A \end{cases} \tag{6}$$

where $\bar{y}_k^-(t) = L^{-1}[\bar{Y}_k^-(s)]$ for $k \geq 0$ and $\bar{y}_k^+(t) = \sum_{i=0}^{k-N_A} \bar{h}_{k-i}u_i(t)$ for $k \geq N_A$.

Let us introduce $\hat{t}$, as the variable of future time, and $\tau \in [0,T]$, $T \in \mathbb{R}_+$, standing for the relative variable of future time: $\tau = \hat{t} - t$. Moreover, let

$$\mathcal{B}_{k_1,k_2}(t;t_1,t_2) = \left\{\{t^k/k!\}_{k=k_1}^{k_2}, \quad t \in [t_1,t_2], \quad 0 \leq k_1 \leq k_2\right\} \tag{7}$$

denote the weighted natural functional basis.

Case α. The future output $y(\hat{t})$ can be approximated in $\mathcal{B}_{0,N_y}(\tau;0,T)$ as $y(\hat{t}) \cong \tilde{y}(\hat{t})|_{\hat{t}=t+\tau} = \sum_{k=0}^{N_y} \tau^k y_k(t)/k!$, where $y_i(t) = \mathrm{d}^i y(t)/\mathrm{d}\,t^i$, $i \geq 0$, and N_y denotes the plant output prediction order. Seeking for a realisable form $\hat{y}(\hat{t})$ of the above predictor $\tilde{y}(\hat{t})$, one can replace the derivatives of the output by their estimates (5): $\tilde{y}(\hat{t}) \cong \hat{y}(\hat{t})|_{\hat{t}=t+\tau} = \sum_{k=0}^{N_y} \tau^k y_k^*(t)/k!$. From Lemma 6, Parts C and D, it follows that the designed control sequence can be found for $N_y \geq \rho$. The following matrix representation of $\hat{y}(\hat{t})$ can then be obtained: $\hat{y}(\hat{t})|_{\hat{t}=t+\tau} = \boldsymbol{t}_{0,N_y}^T(\tau)\boldsymbol{H}_{N_y,N_u}\boldsymbol{u}_{N_u}(t) +$

$t^T_{0,N_y}(\tau)y^-_{N_y}(t)$, where

$$H_{N_y,M} = \begin{bmatrix} h_0 & 0 & \cdots & 0 \\ h_1 & h_0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ h_{N_y} & h_{N_y-1} & \cdots & h_{N_y-M} \end{bmatrix}, \quad H_{N_y,M} \in \mathbb{R}^{(N_y+1)\times(M+1)}, \quad N_y \geq 0$$

$$t_{m,n}(\tau) = [\tau^m/m! \ \cdots \ \tau^n/n!]^T, \quad t_{m,n}(\tau) \in \mathbb{R}^{n-m+1}, \quad 0 \leq m \leq n$$

$$u_i(t) = [u(t) \ u_1(t) \ \cdots \ u_i(t)]^T, \quad u_i(t) \in \mathbb{R}^{i+1}, \quad i \geq 0$$

$$y^-_i(t) = \left[y(t) \ y^-_1(t) \ \cdots \ y^-_i(t)\right]^T, \quad y^-_i(t) \in \mathbb{R}^{i+1}, \quad i \geq 0 \tag{8}$$

while the control prediction order N_u satisfies the design constraint $N_u \leq N_y - \rho$. An important feature of $H_{N_y,M}$ (with $N_y \geq \rho$ and $0 \leq M \leq N_y$), being composed of the coefficients of the quotient polynomials from (D1) and (D3), is the fact that it has a zero ρ-row upper submatrix and a lower triangular Toeplitz submatrix.

Case $\bar{\alpha}$. When considering the case of estimation of the future filtered output, by virtue of Lemma 7, Parts C and D, one concludes that the control sequence can be designed for $N_y \geq N_A$. By assuming that $N_u \leq N_y - N_A$, the formula $\hat{\bar{y}}(\hat{t}) = |_{\hat{t}=t+\tau} = t^T_{0,N_y}(\tau)\bar{H}_{N_y,N_u}u_{N_u}(t) + t^T_{0,N_y}(\tau)\bar{y}^-_{N_y}(t)$ can easily be derived, where $\bar{y}^-_i(t) = [\bar{y}^-_0(t) \ \cdots \ \bar{y}^-_i(t)]^T$, $\bar{y}^-_i(t) \in \mathbb{R}^{i+1}$, $i \geq 0$ (cf. also (6)) and

$$\bar{H}_{N_y,M} = \begin{bmatrix} \bar{h}_0 & 0 & \cdots & 0 \\ \bar{h}_1 & \bar{h}_0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \bar{h}_{N_y} & \bar{h}_{N_y-1} & \cdots & \bar{h}_{N_y-M} \end{bmatrix}, \quad \bar{H}_{N_y,M} \in \mathbb{R}^{(N_y+1)\times(M+1)}, \quad N_y \geq 0$$

In this case $\bar{H}_{N_y,M}$ for $N_y \geq N_A$, $0 \leq M \leq N_y$, being composed of the coefficients of the quotient polynomials of ($\bar{\text{D}}$1) and ($\bar{\text{D}}$3), has a zero N_A-row upper submatrix and a lower triangular Toeplitz submatrix.

4. CGPC Design

In this section, consequent CGPC control design algorithms are described. Taking the original CGPC design as a basis for a further development, two analytical design methods for predictive control are introduced: the first one (α) that utilises the emulated future output of the plant and is recommended for minimum-phase plant models, and the other one ($\bar{\alpha}$) that applies emulation of the future filtered output and allows for handling both minimum-phase and non-minimum-phase plant models. As will be shown, by examining the properties of the resulting closed-loop control systems, a set of prototype characteristic polynomials can be established. With this a simple design procedure can be formulated, in which the cardinal 'tuning knobs' of the CGPC methodology, i.e. the orders of output and control prediction, as well as the horizon of observation, are directly related to certain basic time-domain design specifications.

4.1. Basic CGPC Design

Supposing that $w(t)$ is the reference signal sampled at a time instant t, one can denote the future reference by

$$(\alpha): \qquad \hat{w}(\hat{t})|_{\hat{t}=t+\tau} = w(t)$$

and the filtered future reference by

$$(\bar{\alpha}): \qquad \hat{\bar{w}}(\hat{t})|_{\hat{t}=t+\tau} = w(t)\boldsymbol{t}^T_{0,N_y}(\tau)\bar{\boldsymbol{b}}_{N_y} \cong L^{-1}[W(s)/B(s)]$$

where $\bar{\boldsymbol{b}}_i = [\bar{b}_0 \quad \cdots \quad \bar{b}_i]^T$, $\bar{\boldsymbol{b}}_i \in \mathbb{R}^{i+1}$ is composed of the Markov parameters $\bar{b}_i$, for $i \geq 0$, associated with the all-pole inverse-plant-allied system $1/B(s) = \sum_{i=0}^{\infty} \bar{b}_i s^{-i}$, $\bar{b}_0 = \cdots = \bar{b}_{N_B-1} = 0$. Let us define two future control error signals:

$$(\alpha): \qquad e(\hat{t}) = r\hat{w}(\hat{t}) - \hat{y}(\hat{t})$$

and

$$(\bar{\alpha}): \qquad \bar{e}(\hat{t}) = \bar{r}\hat{\bar{w}}(\hat{t}) - \hat{\bar{y}}(\hat{t})$$

where $r, \bar{r} \in \mathbb{R}$ are some pre-scaling coefficients. Moreover, let the following quadratic indices be introduced for an observation horizon $T > 0$: $J(\boldsymbol{u}_{N_u}(t)) = \int_0^T e^2(t+\tau)\mathrm{d}\tau$ and $\bar{J}(\boldsymbol{u}_{N_u}(t)) = \int_0^T \bar{e}^2(t+\tau)\mathrm{d}\tau$. Minimisation of these indices with respect to the future input $\boldsymbol{u}_{N_u}(t)$ yields, respectively,

$$(\alpha): \qquad \boldsymbol{u}^*_{N_u}(t) = -\boldsymbol{K}_{N_u,N_y}\boldsymbol{y}_{w,N_y}(t), \quad N_y \geq \rho, \quad N_u \leq N_y - \rho$$

and

$$(\bar{\alpha}): \qquad \bar{\boldsymbol{u}}^*_{N_u}(t) = -\bar{\boldsymbol{K}}_{N_u,N_y}\bar{\boldsymbol{y}}^{\sim}_{w,N_y}(t), \quad N_y \geq N_A, \quad N_u \leq N_y - N_A$$

where

$$\boldsymbol{K}_{N_u,N_y} = \boldsymbol{T}^{-1}_{N_u,N_y}\boldsymbol{H}^T_{N_y,N_u}\boldsymbol{T}_{N_y}, \quad \boldsymbol{K}_{N_u,N_y} \in \mathbb{R}^{(N_u+1)\times(N_y+1)}$$

$$\boldsymbol{y}_{w,N_y}(t) = L^{-1}\left[\boldsymbol{Y}_{w,N_y}(s)\right], \qquad \boldsymbol{y}_{w,N_y}(t) \in \mathbb{R}^{N_y+1}$$

$$\bar{\boldsymbol{K}}_{N_u,N_y} = \bar{\boldsymbol{T}}^{-1}_{N_u,N_y}\bar{\boldsymbol{H}}^T_{N_y,N_u}\boldsymbol{T}_{N_y}, \quad \bar{\boldsymbol{K}}_{N_u,N_y} \in \mathbb{R}^{(N_u+1)\times(N_y+1)}$$

$$\begin{aligned}
\bar{\boldsymbol{y}}^{\sim}_{w,N_y}(t) &= -\bar{r}w(t)\bar{\boldsymbol{b}}_{N_y} + \bar{\boldsymbol{y}}^{-}_{N_y}(t), \quad \bar{\boldsymbol{y}}^{\sim}_{w,N_y}(t) \in \mathbb{R}^{N_y+1} \\
\boldsymbol{T}_{N_u,N_y} &= \boldsymbol{H}^T_{N_y,N_u}\boldsymbol{T}_{N_y}\boldsymbol{H}_{N_y,N_u}, \quad \boldsymbol{T}_{N_u,N_y} \in \mathbb{R}^{(N_u+1)\times(N_u+1)} \\
\bar{\boldsymbol{T}}_{N_u,N_y} &= \bar{\boldsymbol{H}}^T_{N_y,N_u}\boldsymbol{T}_{N_y}\bar{\boldsymbol{H}}_{N_y,N_u}, \quad \bar{\boldsymbol{T}}_{N_u,N_y} \in \mathbb{R}^{(N_u+1)\times(N_u+1)} \\
\boldsymbol{T}_{N_y} &= \boldsymbol{T}^{0,N_y}_{0,N_y}(0,T), \qquad \boldsymbol{T}_{N_y} \in \mathbb{R}^{(N_y+1)\times(N_y+1)} \\
\boldsymbol{T}^{m,n}_{k,l}(\tau_1,\tau_2) &= \int_{\tau_1}^{\tau_2} \boldsymbol{t}_{k,l}(\tau)\boldsymbol{t}^T_{m,n}(\tau)\,\mathrm{d}\tau, \quad \boldsymbol{T}^{m,n}_{k,l}(\tau_1,\tau_2) \in \mathbb{R}^{(l-k+1)\times(n-m+1)} \\
& \qquad 0 \le k \le l, \quad 0 \le m \le n \\
\boldsymbol{Y}_{w,N_y}(s) &= \Big[-rW(s) + Y(s) \;\; Y^{-}_1(s) \;\; \cdots \;\; Y^{-}_{N_y}(s)\Big]^T
\end{aligned} \tag{9}$$

The first co-ordinate of $\boldsymbol{u}^*_{N_u}(t)$ or $\bar{\boldsymbol{u}}^*_{N_u}(t)$ determines the optimal control input $u(t)$ at time t for Cases α and $\bar{\alpha}$

$$(\alpha): \qquad u(t) = -\boldsymbol{k}^T_{N_y}\boldsymbol{y}_{w,N_y}(t) \tag{10}$$

$$(\bar{\alpha}): \qquad \bar{u}(t) = -\bar{\boldsymbol{k}}^T_{N_y}\bar{\boldsymbol{y}}^{\sim}_{w,N_y}(t) \tag{11}$$

where $\boldsymbol{k}^T_{N_y} = [k_0 \;\; \cdots \;\; k_{N_y}]$, $\boldsymbol{k}_{N_y} \in \mathbb{R}^{N_y+1}$, is the first row of $\boldsymbol{K}_{N_u,N_y}$ and $\bar{\boldsymbol{k}}^T_{N_y} = [\bar{k}_0 \;\; \cdots \;\; \bar{k}_{N_y}]$, $\bar{\boldsymbol{k}}_{N_y} \in \mathbb{R}^{N_y+1}$, is the first row of $\bar{\boldsymbol{K}}_{N_u,N_y}$. Control (11) can be rewritten in a unified form:

$$(\bar{\alpha}): \qquad \bar{u}(t) = -\bar{\boldsymbol{k}}^T_{N_y}\bar{\boldsymbol{y}}_{w,N_y}(t) \tag{12}$$

where $\bar{\boldsymbol{y}}_{w,N_y}(t) = L^{-1}[\bar{\boldsymbol{Y}}_{w,N_y}(s)]$, $\bar{\boldsymbol{y}}_{w,N_y}(t) \in \mathbb{R}^{N_y+1}$, and

$$\bar{\boldsymbol{Y}}_{w,N_y}(s) = \Big[-rW(s) + \bar{Y}^{-}_0(s) \;\; \bar{Y}^{-}_1(s) \;\; \cdots \;\; \bar{Y}^{-}_{N_y}(s)\Big]^T$$

while the pre-scaling factors r and $\bar{r}$ are related by

$$\bar{r} = r\frac{\bar{k}_0}{\bar{\boldsymbol{k}}^T_{N_y}\bar{\boldsymbol{b}}_{N_y}} \tag{13}$$

The above strategy can also be viewed as partial-state feedback control.

4.2. Control Law Based on Prediction of the Future Output (α)

The linear (partial) feedback of (10) results in the following explicit closed-loop control law:

$$U(s) = grW(s) - M(s)U(s) - N(s)Y(s) \tag{14}$$

where $g = k_0$ is the effective scalar controller gain, with $M(s) = G^0_{N_y}(s)/C(s)$ being a strictly proper transfer function and $N(s) = F^0_{N_y}(s)/C(s)$ being a proper transfer function (Fig. 1(a)). The numerator polynomials can then be defined as $F^0_{N_y}(s) = \boldsymbol{k}^T_{N_y}\boldsymbol{F}_{N_y}\boldsymbol{s}_{N_A-1}$, $\deg F^0_{N_y}(s) \le N_A - 1$, and $G^0_{N_y}(s) = \boldsymbol{k}^T_{N_y}\boldsymbol{G}_{N_y}\boldsymbol{s}_{N_C-1}$, $\deg G^0_{N_y}(s) \le N_C - 1 = N_A - 2$, where $\boldsymbol{s}_i = [s^0 \ s^1 \ \cdots \ s^i]^T$, $i \ge 0$, while the entries of the matrices

$$\boldsymbol{F}_{N_y} = \begin{bmatrix}\boldsymbol{f}_0 & \cdots & \boldsymbol{f}_{N_y}\end{bmatrix}^T, \quad \boldsymbol{F}_{N_y} \in \mathbb{R}^{(N_y+1)\times N_A}$$

$$\boldsymbol{G}_{N_y} = \begin{bmatrix}\boldsymbol{g}_0 & \cdots & \boldsymbol{g}_{N_y}\end{bmatrix}^T, \quad \boldsymbol{G}_{N_y} \in \mathbb{R}^{(N_y+1)\times N_C}$$

are composed of the coefficients of the residual polynomials of (D2) and (D3), respectively.

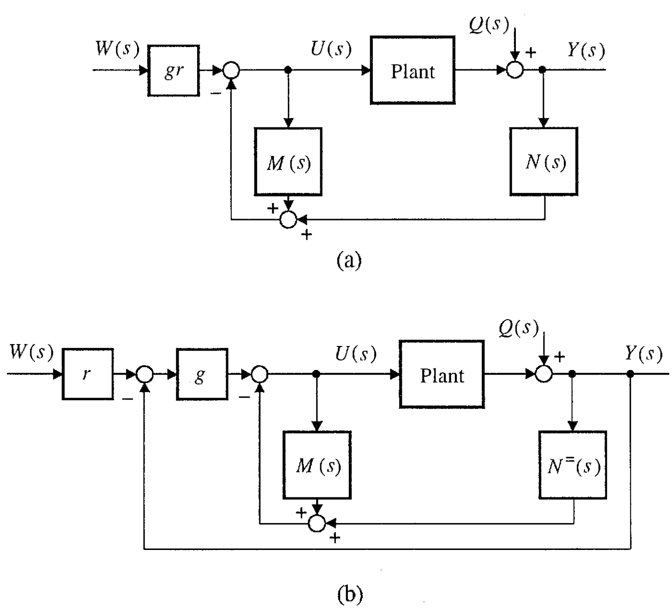

Fig. 1. Two CGPC closed-loop control system configurations.

With a unity positional feedback distinguished as in Fig. 1(b), one has the closed-loop control law $U(s) = g(rW(s) - Y(s)) - M(s)U(s) - N^{=}(s)Y(s)$, where

$N^{=}(s) = F^{0=}_{N_y}(s)/C(s)$ is a proper transfer function with $F^{0=}_{N_y}(s) = F^0_{N_y}(s) - gC(s) = \boldsymbol{k}^T_{N_y} \boldsymbol{F}^{=}_{N_y} \boldsymbol{s}_{N_A-1}$ and

$$\boldsymbol{F}^{=}_{N_y} = \begin{bmatrix} \boldsymbol{0}_{N_A} & \boldsymbol{f}_1 & \cdots & \boldsymbol{f}_{N_y} \end{bmatrix}^T, \quad \boldsymbol{F}^{=}_{N_y} \in \mathbb{R}^{(N_y+1)\times N_A}$$

Consider now the instrumental polynomials $E^0_{N_y}(s) = \boldsymbol{k}^T_{N_y} \boldsymbol{E}_{N_y} \boldsymbol{s}_{N_y-1}$, $\deg E^0_{N_y}(s) \leq N_y - 1$, and $L^0_{N_y}(s) = \boldsymbol{k}^T_{N_y} \boldsymbol{L}_{N_y} \boldsymbol{s}_{N_A-1}$, $\deg L^0_{N_y}(s) \leq N_A - 1$, with the matrix

$$\boldsymbol{E}_{N_y} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ e_1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ e_{N_y} & e_{N_y-1} & \cdots & e_1 \end{bmatrix}, \quad \boldsymbol{E}_{N_y} \in \mathbb{R}^{(N_y+1)\times N_y}$$

composed of the coefficients of the quotient polynomials of (D2): $E_k(s) = e_0 = 0$ if $k = 0$ and $E_k(s) = \sum_{i=0}^{k-1} e_{k-i} s^i$ if $k \geq 1$. Note that the e_i's, for $i \geq 0$, are Markov parameters of the modelled disturbance channel: $C(s)/A(s) = \sum_{i=0}^{\infty} e_i s^{-i}$. From the Diophantine equations (D1)–(D3) one obtains the following representations of the considered polynomials:

$$\begin{aligned} F^0_{N_y}(s) &= C(s)\boldsymbol{k}^T_{N_y}\boldsymbol{s}_{N_y} - A(s)E^0_{N_y}(s) \\ G^0_{N_y}(s) &= \begin{cases} B(s)E^0_{N_y}(s) & \text{if } N_y < \rho \\ B(s)E^0_{N_y}(s) - C(s)H^0_{N_y,N_y-\rho}(s) & \text{if } N_y \geq \rho \end{cases} \\ L^0_{N_y}(s) &= \begin{cases} B(s)\boldsymbol{k}^T_{N_y}\boldsymbol{s}_{N_y} & \text{if } N_y < \rho \\ B(s)\boldsymbol{k}^T_{N_y}\boldsymbol{s}_{N_y} - A(s)H^0_{N_y,N_y-\rho}(s) & \text{if } N_y \geq \rho \end{cases} \end{aligned} \tag{15}$$

where $H^0_{N_y,M}(s) = \boldsymbol{k}^T_{N_y}\boldsymbol{H}_{N_y,M}\boldsymbol{s}_M$, $\deg H^0_{N_y,M}(s) \leq M$, $M \geq 0$. By virtue of (D4), one gets

$$A(s)G^0_{N_y}(s) + B(s)F^0_{N_y}(s) = C(s)L^0_{N_y}(s)$$

Thus the characteristic polynomial of the resulting closed-loop system takes the form

$$P(s) = P_0(s)C(s) \quad \text{with} \quad P_0(s) = A(s) + L^0_{N_y}(s) \tag{16}$$

4.3. Control Law Based on Prediction of Future Filtered Output ($\bar{\alpha}$)

Also the linear (partial) feedback given by (12) yields the explicit closed-loop control law (14), where $g = \bar{k}_0$ is the effective scalar controller gain, $M(s) = \bar{G}^0_{N_y}(s)/C(s)$ denotes a strictly proper transfer function and $N(s) = \bar{F}^0_{N_y}(s)/C(s)$ is a proper transfer function. The numerator polynomials can now be defined as

$\bar{F}^0_{N_y}(s) = \bar{\boldsymbol{k}}^T_{N_y}\bar{\boldsymbol{F}}_{N_y}\boldsymbol{s}_{N_A-1}$, $\deg \bar{F}^0_{N_y}(s) \le N_A - 1$, and $\bar{G}^0_{N_y}(s) = \bar{\boldsymbol{k}}^T_{N_y}\bar{\boldsymbol{G}}_{N_y}\boldsymbol{s}_{N_C-1}$, $\deg \bar{G}^0_{N_y}(s) \le N_C - 1$, where the entries of the matrices

$$\bar{\boldsymbol{F}}_{N_y} = \begin{bmatrix} \bar{\boldsymbol{f}}_0 & \cdots & \bar{\boldsymbol{f}}_{N_y} \end{bmatrix}^T, \quad \bar{\boldsymbol{F}}_{N_y} \in \mathbb{R}^{(N_y+1)\times N_A}$$

$$\bar{\boldsymbol{G}}_{N_y} = \begin{bmatrix} \bar{\boldsymbol{g}}_0 & \cdots & \bar{\boldsymbol{g}}_{N_y} \end{bmatrix}^T, \quad \bar{\boldsymbol{G}}_{N_y} \in \mathbb{R}^{(N_y+1)\times N_C}$$

are created with the coefficients of the polynomials given by ($\bar{\text{D}}$1)–($\bar{\text{D}}$4). Let us now introduce the instrumental polynomials $\bar{E}^0_{N_y}(s) = \bar{\boldsymbol{k}}^T_{N_y}\bar{\boldsymbol{E}}_{N_y}\boldsymbol{s}_{N_y-1}$ of $\deg \bar{E}^0_{N_y}(s) \le N_y - 1$, $\bar{L}^0_{N_y}(s) = \bar{\boldsymbol{k}}^T_{N_y}\bar{\boldsymbol{L}}_{N_y}\boldsymbol{s}_{N_A-1}$ of $\deg \bar{L}^0_{N_y}(s) \le N_A - 1$, and $\bar{H}^0_{N_y,M}(s) = \bar{\boldsymbol{k}}^T_{N_y}\bar{\boldsymbol{H}}_{N_y,M}\boldsymbol{s}_M$ of $\deg \bar{H}^0_{N_y,M}(s) \le M$, $M \ge 0$, where the matrix

$$\bar{\boldsymbol{E}}_{N_y} = \left[\begin{array}{ccc:ccc} \bar{e}_{0,0} & \cdots & \bar{e}_{0,N_B-1} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \bar{e}_{N_B,0} & \cdots & \bar{e}_{N_B,N_B-1} & 0 & \cdots & 0 \\ \hdashline \bar{e}_{N_B+1,0} & \cdots & \bar{e}_{N_B+1,N_B-1} & \bar{e}_{N_B+1,N_B} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ \bar{e}_{N_y,0} & \cdots & \bar{e}_{N_y,N_B-1} & \bar{e}_{N_y,N_B} & \cdots & \bar{e}_{N_y,N_y-1} \end{array}\right]$$

$\bar{\boldsymbol{E}}_{N_y} \in \mathbb{R}^{(N_y+1)\times N_y}$, with $\bar{e}_{k,l} = \bar{e}_{k-1,l-1}$ for $k > N_B + 1$ and $N_B + 1 \le l \le k - 1$, is composed of the coefficients of the polynomials $\bar{E}_k(s)$, $k \le N_y$, of the ($\bar{\text{D}}$2) Diophantine equation, and the matrix

$$\bar{\boldsymbol{L}}_{N_y} = \begin{bmatrix} \bar{\boldsymbol{l}}_0 & \cdots & \bar{\boldsymbol{l}}_{N_y} \end{bmatrix}^T, \quad \bar{\boldsymbol{L}}_{N_y} \in \mathbb{R}^{(N_y+1)\times N_A}$$

is made up with the aid of the coefficients of the residual polynomials of the ($\bar{\text{D}}$1) Diophantine equation.

From Diophantine equations ($\bar{\text{D}}$1)–($\bar{\text{D}}$3) it follows that

$$C(s)\bar{\boldsymbol{k}}^T_{N_y}\boldsymbol{s}_{N_y} = A(s)\bar{E}^0_{N_y}(s) + B(s)\bar{F}^0_{N_y}(s)$$

$$\bar{G}^0_{N_y}(s) = \begin{cases} \bar{E}^0_{N_y}(s) & \text{if } \; N_y < N_A \\ \bar{E}^0_{N_y}(s) - C'(s)\bar{H}^0_{N_y,N_y-N_A}(s) & \text{if } \; N_y \ge N_A \end{cases}$$

$$\bar{L}^0_{N_y}(s) = \begin{cases} \bar{\boldsymbol{k}}^T_{N_y}\boldsymbol{s}_{N_y} & \text{if } \; N_y < N_A \\ \bar{\boldsymbol{k}}^T_{N_y}\boldsymbol{s}_{N_y} - A(s)\bar{H}^0_{N_y,N_y-N_A}(s) & \text{if } \; N_y \ge N_A \end{cases}$$

On account of ($\bar{\text{D}}$4) it can immediately be deduced that

$$A(s)\bar{G}^0_{N_y}(s) + B(s)\bar{F}^0_{N_y}(s) = C(s)\bar{L}^0_{N_y}(s)$$

Thus the characteristic polynomial of the resulting closed-loop system takes the form

$$P(s) = P_0(s)C(s) \quad \text{with} \quad P_0(s) = A(s) + \bar{L}^0_{N_y}(s) \tag{17}$$

For the unity positional-feedback control law (12) and Fig. 1(b) we have $N^{=}(s) = \bar{F}^{0=}_{N_y}(s)/C(s)$, where $\bar{F}^{0=}_{N_y}(s) = \bar{F}^0_{N_y}(s) - gC(s)$.

4.4. Analytical ACGPC Design for Minimum-Phase Plants (α)

Let $(A(s), B(s))$ be coprime, as previously. Let $N_u \geq 0$ be a free design parameter and, having in mind the general design restriction that $N_u \leq N_y - \rho$, assume that for a given relative order ρ the output prediction order is established at its minimal value: $N_y = \rho + N_u$. In such a case the gain matrix $\boldsymbol{K}_{N_u,\rho+N_u} \in \mathbb{R}^{(N_u+1)\times(\rho+N_u+1)}$ of Sec. 4.1 becomes

$$\boldsymbol{K}_{N_u,\rho+N_u} = \left(\boldsymbol{H}^T_{\rho+N_u,N_u}\boldsymbol{T}_{\rho+N_u}\boldsymbol{H}_{\rho+N_u,N_u}\right)^{-1}\boldsymbol{H}^T_{\rho+N_u,N_u}\boldsymbol{T}_{\rho+N_u} \tag{18}$$

Rewriting $\boldsymbol{H}_{\rho+N_u,N_u}$ as

$$\boldsymbol{H}_{\rho+N_u,N_u} = \begin{bmatrix} \boldsymbol{0}_{\rho\times(N_u+1)} \\ \cdots\cdots\cdots\cdots \\ \boldsymbol{H}^\rho_{N_u} \end{bmatrix}$$

where $\boldsymbol{H}^\rho_{N_u}$ denotes the following non-singular submatrix of a lower triangular Toeplitz structure:

$$\boldsymbol{H}^\rho_{N_y} = \begin{bmatrix} h_\rho & 0 & \cdots & 0 \\ h_{\rho+1} & h_\rho & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ h_{N_y} & h_{N_y-1} & \cdots & h_\rho \end{bmatrix}, \quad \boldsymbol{H}^\rho_{N_u} \in \mathbb{R}^{(N_u+1)\times(N_u+1)}$$

yields

$$\boldsymbol{K}_{N_u,\rho+N_u} = \left(\boldsymbol{H}^\rho_{N_u}\right)^{-1}\left[\left(\boldsymbol{T}^{\rho,\rho+N_u}_{\rho,\rho+N_u}(0,T)\right)^{-1}\boldsymbol{T}^{0,\rho-1}_{\rho,\rho+N_u}(0,T) \;\vdots\; \boldsymbol{I}_{N_u+1}\right] \tag{19}$$

Consequently, the gain vector $\boldsymbol{k}^T_{\rho+N_u}$ of (10) takes the form

$$\boldsymbol{k}^T_{\rho+N_u} = h_\rho^{-1}\left[\boldsymbol{\nu}^T_{\rho,N_u}(0,T) \;\vdots\; 1 \;\; 0 \;\; \cdots \;\; 0\right], \quad \boldsymbol{k}_{\rho+N_u} \in \mathbb{R}^{\rho+N_u+1} \tag{20}$$

where $h_\rho = b_{N_B}/a_{N_A} = b_{N_B}$ and

$$\boldsymbol{\nu}^T_{\rho,N_u}(0,T) = [1 \;\; 0 \;\; \cdots \;\; 0]\left(\boldsymbol{T}^{\rho,\rho+N_u}_{\rho,\rho+N_u}(0,T)\right)^{-1}\boldsymbol{T}^{0,\rho-1}_{\rho,\rho+N_u}(0,T),$$

$$\boldsymbol{\nu}_{\rho,N_u}(0,T) \in \mathbb{R}^\rho \tag{21}$$

Since the non-singularity of $\boldsymbol{T}_{\rho,\rho+N_u}^{\rho,\rho+N_u}(0,T)$ is guaranteed by the linear independence of the function basis (7), the solution to (21) always exists. From (18) it follows that $\boldsymbol{K}_{N_u,\rho+N_u}\boldsymbol{H}_{\rho+N_u,N_u} = \boldsymbol{I}_{N_u+1}$ and $H^0_{\rho+N_u,N_u}(s) = 1$. Hence the following form of the polynomial $P_0(s)$, being a factor of the resulting closed-loop characteristic polynomial $P(s) = C(s)P_0(s)$, can easily be established (cf. (15)):

$$P_0(s) = B(s)K_{\rho,N_u}(s)$$

where

$$K_{\rho,N_u}(s) = \boldsymbol{k}^T_{\rho+N_u}\boldsymbol{s}_{\rho+N_u}$$

By examining the above polynomial, the following corollary can be drawn (Kowalczuk and Suchomski, 1998b).

Corollary 1. *The CGPC design based on emulation of the plant-output derivatives and setting* $N_y = \rho + N_u$ *is restricted to minimum-phase models of the plant.*

Another important outcome of the above representation of the characteristic polynomial is that successive coefficients of its 'consciously' designed polynomial factor are determined by the coordinates of the gain vector $\boldsymbol{k}_{\rho+N_u}$. What is more, from (8) and (9) it follows that the component matrices of (19) can be written down as

$$\begin{aligned}&\boldsymbol{T}^{0,\rho-1}_{\rho,\rho+N_u}(0,T)\\ &\quad= T^{\rho}\boldsymbol{P}_{\rho,\rho+N_u}\begin{bmatrix} T/(\rho+1) & \cdots & T^{\rho}/(2\rho) \\ \vdots & \ddots & \vdots \\ T^{N_u+1}/(\rho+N_u+1) & \cdots & T^{\rho+N_u}/(2\rho+N_u)\end{bmatrix}\boldsymbol{P}_{0,\rho-1}\end{aligned} \tag{22}$$

$$\begin{aligned}&\boldsymbol{T}^{\rho,\rho+N_u}_{\rho,\rho+N_u}(0,T)\\ &\quad= T^{2\rho}\boldsymbol{P}_{\rho,\rho+N_u}\begin{bmatrix} T/(2\rho+1) & \cdots & T^{N_u+1}/(2\rho+N_u+1) \\ \vdots & \ddots & \vdots \\ T^{N_u+1}/(2\rho+N_u+1) & \cdots & T^{2N_u+1}/(2\rho+2N_u+1)\end{bmatrix}\boldsymbol{P}_{\rho,\rho+N_u}\end{aligned} \tag{23}$$

where $\boldsymbol{P}_{m,n} = \mathrm{diag}\{1/k!\}^n_{k=m}$, $m \le n$, $\boldsymbol{P}_{m,n} \in \mathbb{R}^{(n-m+1)\times(n-m+1)}$. By considering the above matrices in (21), the following form of the CGPC gain vector of (20) can be derived:

$$\boldsymbol{k}^T_{\rho+N_u} = h_\rho^{-1}\left[\widetilde{k}^0_{\rho,N_u}/T^{\rho}\ \ \widetilde{k}^1_{\rho,N_u}/T^{\rho-1}\ \ \cdots\ \ \widetilde{k}^{\rho}_{\rho,N_u}/T^0 \ \vdots\ 0\ \cdots\ 0\right]$$

where the composite vector, defined for the normalised observation horizon $T = 1$,

$$\begin{aligned}&\left[\widetilde{k}^0_{\rho,N_u}/T^{\rho}\ \ \widetilde{k}^1_{\rho,N_u}/T^{\rho-1}\ \ \cdots\ \ \widetilde{k}^{\rho}_{\rho,N_u}/T^0\right]\Big|_{T=1}\\ &\quad= \left[\boldsymbol{\nu}^T_{\rho,N_u}(0,T)\ \vdots\ 1\right]\Big|_{T=1} = \left[\boldsymbol{\nu}^T_{\rho,N_u}(0,1)\ \vdots\ 1\right]\end{aligned} \tag{24}$$

depends solely on the relative order ρ of the plant and the control prediction order N_u. Furthermore, by defining a normalised complex operator variable as $p = Ts$, the following set of two-parameter prototype closed-loop polynomials can be acquired:

$$\widetilde{K}_{\rho,N_u}(p) = \sum_{i=0}^{\rho} \widetilde{k}^i_{\rho,N_u} p^i, \quad \widetilde{k}^\rho_{\rho,N_u} = 1, \quad \deg \widetilde{K}_{\rho,N_u}(p) = \rho$$

Each of these monic polynomials constitutes (for chosen parameters ρ and N_u) the design polynomial factor of the characteristic polynomial

$$K_{\rho,N_u}(s) = h_\rho^{-1} T^{-\rho} \widetilde{K}_{\rho,N_u}(p)|_{p=Ts} \tag{25}$$

As is shown in Appendix E, the coefficients $\widetilde{k}^i_{\rho,N_u}$, $i = 0, \ldots, \rho$, of these polynomials take the form

$$\widetilde{k}^i_{\rho,N_u} = \begin{cases} \dfrac{\rho!}{i!} \dfrac{(2\rho+1)}{(\rho+i+1)} & \text{if} \quad N_u = 0 \\[2ex] \dfrac{\rho!}{N_u! i!} \displaystyle\prod_{j=1}^{N_u+1} \dfrac{(2\rho+j)}{(\rho+i+j)} \prod_{j=1}^{N_u} (\rho - i + j) & \text{if} \quad N_u \geq 1 \end{cases} \tag{26}$$

It can easily be seen that for the starting point $k = 0$ and a given value of ρ, the design computations can be performed in the following recursive manner:

$$\widetilde{k}^i_{\rho,k} = \frac{(2\rho+k+1)(\rho-i+k)}{k(\rho+i+k+1)} \widetilde{k}^i_{\rho,k-1} \quad \text{for} \quad k = 1, \ldots, N_u \quad \text{and} \quad i = 0, \ldots, \rho$$

Now, the controller gain takes the form

$$g = k_0 = \frac{\widetilde{k}^0_{\rho,N_u}}{h_\rho T^\rho} = \frac{\rho!}{N_u! h_\rho T^\rho (\rho + N_u + 1)} \prod_{j=1}^{N_u+1} (2\rho + j)$$

An important feature of this approach is that the coefficients of $\widetilde{K}_{\rho,N_u}(p)$, as described by (26), can be derived explicitly, without the necessity (expressed in (21)) of inverting the Hilbert-type matrices $\boldsymbol{T}^{\rho,\rho+N_u}_{\rho,\rho+N_u}(0,1)$, which are extremely ill-conditioned even for a relatively small N_u (Kowalczuk and Suchomski, 1997a). By considering the following matrix transfer function defined for two input signals $(W(s), Q(s))$ and two output signals $(Y(s), U(s))$ indicated in Fig. 1:

$$\begin{bmatrix} Y(s) \\ U(s) \end{bmatrix} = \begin{bmatrix} T_{wy}(s) & T_{qy}(s) \\ T_{wu}(s) & T_{qu}(s) \end{bmatrix} \begin{bmatrix} W(s) \\ Q(s) \end{bmatrix} \tag{27}$$

one obtains

$$T_{wy}(s) = \frac{k_0 r}{K_{\rho,N_u}(s)}, \qquad T_{wu}(s) = \frac{k_0 r A(s)}{B(s) K_{\rho,N_u}(s)} \tag{28}$$

$$T_{qy}(s) = \frac{A(s) E^0_{N_y}(s)}{C(s) K_{\rho,N_u}(s)}, \qquad T_{qu}(s) = \frac{-A(s) F^0_{N_y}(s)}{B(s) C(s) K_{\rho,N_u}(s)} \tag{29}$$

Thus the ACGPC closed-loop control system under consideration (Case α) is nominally internally stable if and only if $B(s), C(s)$ and $K_{\rho,N_u}(s)$ are Hurwitz polynomials. By examining the coefficients of $\widetilde{K}_{\rho,N_u}(p)$, the following corollary of practical importance can be formulated:

Corollary 2. *For a practically pertinent range of the relative order* ρ, *the polynomials* $\widetilde{K}_{\rho,N_u}(p)$ *are of Hurwitz property if:*

$$\begin{array}{llll} \rho \leq 4 \ \textit{and} \ N_u \geq 0, & \rho = 5 \ \textit{and} \ N_u \geq 1, & \rho = 6,7 \ \textit{and} \ N_u \geq 2 \\ \rho = 8 \ \textit{and} \ N_u \geq 3, & \rho = 9 \ \textit{and} \ N_u \geq 4, & \rho = 10 \ \textit{and} \ N_u \geq 5 \end{array}$$

With the nominal performance of the closed-loop system in mind, in order to assure the unity DC gain of $T_{wy}(s)$ the unity set-point pre-scaling factor, $r = 1$, should be assumed. What is more, the proposed CGPC method results in robust zeroing of positional error if $T_{qy}(0) \cong 0$, which is guaranteed by the presence of open-loop integral action, i.e. when $A(0) = 0$. On the other hand, taking into account both the stability robustness and performance robustness requirements one can consider the modulus of the output sensitivity function $T_{qy}(s)$ as a convenient tool of robustness analysis (cf. Green and Limebeer, 1995; Grimble, 1994; Weinmann, 1991). In many cases, simple robust stability conditions can be formulated in terms of upper bounds for common robustness indices. A good example makes the modulus margin Δ_m of the closed-loop system defined as $\Delta_m = 1/||T_{qy}(s)||_\infty$ (Landau *et al.*, 1998; Morari and Zafiriou, 1989; Zhou *et al.*, 1996).

From (29) it is clear that the modulus margin can, to some extent, be shaped by properly choosing the control prediction order N_u, the time scaling factor T, and the observer polynomial $C(s)$. The prototype transfer functions $\widetilde{k}^0_{\rho,N_u}/\widetilde{K}_{\rho,N_u}(p)$ can easily be scanned to yield their time-domain prototype specifications. A sample of results concerning overshoot $\widetilde{\kappa}$ and its instant $\widetilde{T}_\kappa$, as well as a couple of settling times, achieved for $\rho = 3$ and different N_u is given in Appendix F. The necessary scaling factor T of the time axis (and the observation horizon) can easily be obtained as a ratio of two corresponding time parameters, e.g. $T = T_{s2\%}/\widetilde{T}_{s2\%}$. Moreover, from (28) it follows that the considered CGPC system has rather poor properties of tracking velocity signals: the system is of the 1-type independently of the number of open-loop integrators. The nominal steady-state tracking error for the reference signal of unity velocity can be determined as

$$e_\nu = T\widetilde{e}_\nu(\rho, N_u) = \frac{T(\rho + N_u + 1)}{\rho + N_u} \prod_{j=1}^{N_u+1} \frac{(\rho + j - 1)}{(\rho + j + 1)}$$

4.5. ACGPC Design for Minimum and Non-minimum Phase Plants ($\bar{\alpha}$)

The development given in the previous section combined with the methodology given in Section 4.3 can result in a general design method for minimum and non-minimum phase plants. Let $N_u \geq 0$ be the only free design parameter. The output prediction order can then be established as $N_y = N_A + N_u$. Consequently, by considering the gain matrix $\bar{\boldsymbol{K}}_{N_u,N_A+N_u} \in \mathbb{R}^{(N_u+1)\times(N_A+N_u+1)}$, the following gain vector can be found:

$$\bar{\boldsymbol{k}}^T_{N_A+N_u} = \left[\widetilde{k}^0_{N_A,N_u}/T^{N_A} \;\; \widetilde{k}^1_{N_A,N_u}/T^{N_A-1} \;\; \cdots \;\; \widetilde{k}^{N_A}_{N_A,N_u}/T^0 \;\vdots\; 0 \;\cdots\; 0\right]$$

where for $i = 0, \dots, N_A$

$$\widetilde{k}^i_{N_A,N_u} = \begin{cases} \dfrac{N_A!}{i!} \dfrac{(2N_A+1)}{(N_A+i+1)} & \text{if} \quad N_u = 0 \\[2ex] \dfrac{N_A!}{N_u! i!} \displaystyle\prod_{j=1}^{N_u+1} \dfrac{(2N_A+j)}{(N_A+i+j)} \prod_{j=1}^{N_u} (N_A - i + j) & \text{if} \quad N_u \geq 1 \end{cases}$$

Hence a set of monic two-parameter prototype polynomials: $\widetilde{K}_{N_A,N_u}(p) = \sum_{i=0}^{N_A} \widetilde{k}^i_{N_A,N_u} p^i$, $\widetilde{k}^{N_A}_{N_A,N_u} = 1$, $\deg \widetilde{K}_{N_A,N_u}(p) = N_A$, results. It can easily be established that with the initial iteration $k = 0$, computation of $\widetilde{k}^i_{N_A,N_u}$ can be performed in the following recursive way: for $k = 1, \dots, N_u$ and $i = 0, \dots, N_A$ one has

$$\widetilde{k}^i_{N_A,k} = \frac{(2N_A+k+1)(N_A-i+k)}{k(N_A+i+k+1)} \widetilde{k}^i_{N_A,k-1}$$

In this case, the controller scalar gain is given by

$$g = \bar{k}_0 = \frac{\widetilde{k}^0_{N_A,N_u}}{T^{N_A}} = \frac{N_A!}{N_u! T^{N_A}(N_A+N_u+1)} \prod_{j=1}^{N_u+1} (2N_A+j)$$

As for the all-pole plant-allied system $1/A(s)$: $\bar{h}_{N_A} = 1/a_{N_A} = 1$ and $\bar{H}^0_{N_y,N_y-N_A}(s) = \bar{H}^0_{N_A+N_u,N_u}(s) = 1$. Consequently, the factor $P_0(s)$ of the resulting closed-loop characteristic polynomial $P(s)$ of (17) becomes

$$P_0(s) = \bar{K}_{N_A,N_u}(s)$$

with

$$\bar{K}_{N_A,N_u}(s) = \bar{\boldsymbol{k}}^T_{N_A+N_u} \boldsymbol{s}_{N_A+N_u} = \bar{h}^{-1}_{N_A} T^{-N_A} \widetilde{K}_{N_A,N_u}(p)\big|_{p=Ts}$$

where $\widetilde{K}_{N_A,N_u}(p) = \widetilde{K}_{\rho,N_u}(p)|_{\rho=N_A}$ of (25).

The transfer functions (27) take the form

$$T_{wy}(s) = \frac{\bar{k}_0 r B(s)}{\bar{K}_{N_A,N_u}(s)}, \qquad T_{wu}(s) = \frac{\bar{k}_0 r A(s)}{\bar{K}_{N_A,N_u}(s)}$$

$$T_{qy}(s) = \frac{A(s)\bar{E}^0_{N_y}(s)}{C(s)\bar{K}_{N_A,N_u}(s)}, \quad T_{qu}(s) = \frac{-A(s)\bar{F}^0_{N_y}(s)}{C(s)\bar{K}_{N_A,N_u}(s)}$$

The pre-scaling factor $r = 1/b_0$ assures that $T_{wy}(s)$ is nominally of unity DC gain.

As in the previously considered CGPC design, the zero steady-state error property for positional references can be robustified by assuming $A(0) = 0$. In such a case, the nominal steady-state tracking error for the unit-velocity reference-signal is determined by

$$e_\nu = \frac{T(N_A + N_u + 1)}{N_A + N_u} \prod_{j=1}^{N_u+1} \frac{(N_A + j - 1)}{(N_A + j + 1)} - \frac{b_1}{b_0} = T\widetilde{e}_\nu(N_A, N_u) - \frac{b_1}{b_0}$$

that, in certain limited cases, can be eliminated by properly tuning the design parameters N_u and T. This is, however, only a nominal, not robust, property.

5. ACGPC Design for Non-Minimal Models (α and $\bar{\alpha}$)

The generic solutions given in the previous section for minimal models of the plant can equally serve as a basis for designing the CGPC controllers for non-minimal models. Let $(A(s), B(s))$ be reducible with the greatest common divisor $\Lambda(s)$ of $\deg \Lambda(s) = N_\Lambda > 0$. According to the type of model cancellations, the following cases can be considered here:

- 'Design Unilateral' (DU): a Hurwitz $C(s)$ of $\deg C(s) = N_C = N_A - 1$ is arbitrarily chosen and a high-order controller is designed based on the triple $(A(s), B(s), C(s))$,
- 'Design Bilateral' (DB): a Hurwitz $C'(s)$ of $\deg C'(s) = N_{C'} = N_A - N_\Lambda - 1$ is arbitrarily chosen and a low-order controller is designed based on the triple $(A(s), B(s), C'(s))$,
- 'Controller Bilateral' (CB): a Hurwitz $C'(s)$ of $\deg C'(s) = N_{C'} = N_A - N_\Lambda - 1$ is arbitrarily chosen and a high-order controller is designed based on the triple $(A(s), B(s), C'(s)\Lambda(s))$ with the required Hurwitz $\Lambda(s)$ (the non-minimality of the controller gives a possibility of additional parameterisation of the control loop),
- 'Object Bilateral' (OB): the design is based on the triple $(A'(s), B'(s), C'(s))$, obtained by performing an appropriate reduction both in the control channel and in the disturbance channel of the original non-minimal model, and a low-order controller is designed assuming that the resulting polynomial $C'(s)$ of $\deg C'(s) = N_{C'} = N_A - N_\Lambda - 1$ is Hurwitz.

A summary of the design principles is given in Tables 1 (Case α) and 2 (Case $\bar{\alpha}$). For simplicity of presentation, the dependence on s is omitted and the following notational conventions are introduced: 'E' and 'O' correspond to the emulator and observer design paths, respectively; 'G' and 'F' denote the corresponding numerators of the input and output observer filters $M(s)$ and $N(s)$, respectively; 'R' refers to the recursive way of solving the Diophantine equations; 'L' reflects the fact that a set of linear equations is to be solved; 'LS' indicates a least-squares problem; and, finally, 'LS+p' should be read as a parameterised least-squares problem.

Table 1. Design principles for non-minimal models: Case α.

Case	Path	Design Equations	Method of solving	Solutions	Key commentary
DU	E	$AE_k + F_k = s^k C$ $CH_k + G_k = BE_k$	R R	G/C F/C	
	O	$AH_k + L_k = s^k B$ $AG_k + BF_k = CL_k$	R LS+p	G/C F/C	
DB	E	$AE'_k + \Lambda F'_k = s^k C'\Lambda$ $C'\Lambda H'_k + \Lambda G'_k = BE'_k$	LS LS	G'/C' F'/C'	$\Lambda(s)$ should be evaluated independently.
	O	$AH'_k + L_k = s^k B$ $AG'_k + BF'_k = C'L_k$	R LS	G'/C' F'/C'	
CB	E	$AE_k + F_k = s^k C'\Lambda$ $C'\Lambda H_k + G_k = BE_k$	R R	$G/(C'\Lambda)$ $F/(C'\Lambda)$	$\Lambda(s)$ should be Hurwitz and evaluated independently.
	O	$AH_k + L_k = s^k B$ $AG_k + BF_k = C'\Lambda L_k$	R LS+p	$G/(C'\Lambda)$ $F/(C'\Lambda)$	$\Lambda(s)$ should be Hurwitz and evaluated independently.
OB	E	$A'E'_k + F'_k = s^k C'$ $C'H'_k + G'_k = B'E'_k$	R R	G'/C' F'/C'	
	O	$A'H'_k + L'_k = s^k B'$ $A'G'_k + B'F'_k = C'L'_k$	R L	G'/C' F'/C'	

Additional points and comments are specified below, where only cases of practical importance are given in detail.

1. $(\alpha + \mathrm{DU} + \mathrm{E})$:

- a non-minimal controller always exists,
- neither N_Λ nor $\Lambda(s)$ need to be evaluated,
- there is no possibility of controller parameterisation.

Table 2. Design principles for non-minimal models: Case $\bar{\alpha}$.

Case	Path	Design Equations	Method of solving	Solutions	Key commentary
DU	E	—	—	—	No solution!
	O	—	—	—	No solution!
DB	E	$A\bar{E}'_k + B\bar{F}'_k - s^k C'\Lambda = 0$ $C'\bar{H}'_k + \bar{G}'_k = \bar{E}'_k$	LS R	$\bar{G}'/C'$ $\bar{F}'/C'$	$\Lambda(s)$ is determined as a by-product of the method, effective when $(C',\ \Lambda)$ is coprime
	O	$A\bar{H}'_k + \Lambda\bar{L}'_k = s^k\Lambda$ $A\bar{G}'_k + B\bar{F}'_k = C'\Lambda\bar{L}'_k$	R LS	G'/C' F'/C'	$\Lambda(s)$ should be evaluated independently
CB	E	$A\bar{E}_k + B\bar{F}_k - s^k C'\Lambda = 0$ $C'\Lambda\bar{H}_k + \bar{G}_k = \bar{E}_k$	LS R	$G/(C'\Lambda)$ $F/(C'\Lambda)$	$\Lambda(s)$ is determined as a by-product of the method, effective when $(C',\ \Lambda)$ is coprime and $\Lambda(s)$ is Hurwitz.
	O	$A\bar{H}_k + \bar{L}_k = s^k$ $A\bar{G}_k + B\bar{F}_k - C'\bar{L}_k\Lambda = 0$	R LS+p	$G/(C'\Lambda)$ $F/(C'\Lambda)$	$\Lambda(s)$ is determined as a by-product of the method, effective when $(C',\ \Lambda)$ is coprime and $\Lambda(s)$ is Hurwitz.
OB	E	$A'\bar{E}'_k + B'\bar{F}'_k = s^k C'$ $C'\bar{H}'_k + \bar{G}'_k = \bar{E}'_k$	L R	$\bar{G}'/C'$ $\bar{F}'/C'$	
	O	$A'\bar{H}'_k + \bar{L}'_k = s^k$ $A'\bar{G}'_k + B'\bar{G}'_k = C'\bar{L}'_k$	R L	$\bar{G}'/C'$ $\bar{F}'/C'$	

2. $(\alpha + \mathrm{DU} + \mathrm{O})$:

- a practically convenient parameterisation of the solutions to (D4) can be obtained by the following splitting of the vector $\boldsymbol{f}_k,\ k \geq 0$:

$$\boldsymbol{f}_k = \begin{bmatrix} \boldsymbol{f}_k^0 \\ \cdots \\ \boldsymbol{f}_k^\Lambda \end{bmatrix}, \quad \boldsymbol{f}_k^0 \in \mathbb{R}^{N_A - N_\Lambda}, \quad \boldsymbol{f}_k^\Lambda \in \mathbb{R}^{N_\Lambda},$$

where $\boldsymbol{f}_k^\Lambda$ denotes its free part,

- the modified Diophantine equation (D4) is equivalent to the following set of linear equations:

$$\begin{bmatrix} \boldsymbol{T}^{N_B}_{N_A-N_\Lambda} & \vdots & \boldsymbol{T}^{N_A}_{N_A-1} \\ \cdots\cdots\cdots\cdots & \vdots & \\ \boldsymbol{0}_{\rho+N_\Lambda-1,N_A-N_\Lambda} & \vdots & \end{bmatrix} \begin{bmatrix} \boldsymbol{f}^0_k \\ \cdots \\ \boldsymbol{g}_k \end{bmatrix} = \boldsymbol{T}^{N_C}_{N_A}\boldsymbol{l}_k - \begin{bmatrix} \boldsymbol{0}_{N_A-N_\Lambda} \\ \cdots\cdots \\ \boldsymbol{T}^{N_B}_{N_\Lambda}\boldsymbol{f}^\Lambda_k \\ \cdots\cdots \\ \boldsymbol{0} \end{bmatrix} \begin{matrix} \\ \\ \\ \\ \}\rho-1 \end{matrix} \tag{30}$$

3. $(\alpha + \mathrm{DB} + \mathrm{O})$:

- the vectors

$$\boldsymbol{f}'_k = \left[f'_{k,0} \ \cdots \ f'_{k,N_A-N_\Lambda-1}\right]^T, \quad \boldsymbol{f}'_k \in \mathbb{R}^{N_A-N_\Lambda}$$

$$\boldsymbol{g}'_k = \left[g'_{k,0} \ \cdots \ g'_{k,N_C-N_\Lambda-1}\right]^T, \quad \boldsymbol{g}'_k \in \mathbb{R}^{N_C-N_\Lambda}$$

of the coefficients of polynomials $F'_k(s) = \sum_{i=0}^{N_A-N_\Lambda-1} f'_{k,i}s^i$ and $G'_k(s) = \sum_{i=0}^{N_C-N_\Lambda-1} g'_{k,i}s^i$, $k \geq 0$, are obtained via solving the following set of linear equations (note that there is no need for $\Lambda(s)$ to be explicitly evaluated, see Appendix C):

$$\rho-1\left\{\begin{bmatrix} \boldsymbol{T}^{N_B}_{N_A-N_\Lambda} & \vdots & \boldsymbol{T}^{N_A}_{N_A-N_\Lambda-1} \\ \cdots\cdots & \vdots & \\ \boldsymbol{0} & \vdots & \end{bmatrix}\right. \begin{bmatrix} \boldsymbol{f}'_k \\ \cdots \\ \boldsymbol{g}'_k \end{bmatrix} = \boldsymbol{T}^{N_{C'}}_{N_A}\boldsymbol{l}_k \tag{31}$$

4. $(\bar{\alpha}+\mathrm{DB}+\mathrm{E})$: If the minimal controller is designed based on the pair $(A(s), B(s))$, an appropriate reduced-order emulator of the derivatives of the system output, filtered via the minimal all-pole inverse-plant-allied filter $1/B'(s) = \sum_{i=0}^{\infty} \bar{b}'_i s^{-i}$, i.e. $\bar{Y}'(s) = Y(s)/B'(s)$, should be utilised instead of the previously used emulator matched to $\bar{Y}(s) = Y(s)/B(s)$. This implies that now the filtered future reference should be defined as $\hat{\bar{w}}(\hat{t})|_{\hat{t}=t+\tau} = \hat{\bar{w}}'(\hat{t})|_{\hat{t}=t+\tau} = w(t)\boldsymbol{t}^T_{0,N_y}(\tau)\bar{\boldsymbol{b}}'_{N_y} \cong L^{-1}[W(s)/B'(s)]$, where the vector $\bar{\boldsymbol{b}}'_i = [\bar{b}'_0 \ \cdots \ \bar{b}'_i]^T$, $\bar{\boldsymbol{b}}'_i \in \mathbb{R}^{i+1}$, is composed of the Markov parameters $\bar{b}'_i$, for $i \geq 0$ with $\bar{b}'_0 = \cdots = \bar{b}'_{N_B-N_\Lambda-1} = 0$, associated with the filter $1/B'(s)$. Consequently, for the output prediction order established as $N_y = N_A - N_\Lambda + N_u$ and with a free N_u one obtains the gain vector $\bar{\boldsymbol{k}}'_{N_A-N_\Lambda+N_u} \in \mathbb{R}^{N_A-N_\Lambda+N_u+1}$, the scalar controller gain $g = \bar{k}'_0$ and the pre-scaling factor $r = 1/b'_0 = \lambda_0/b_0$ assuring that $T_{wy}(s)$ is nominally of unit DC gain (cf. Sec. 4.5). The characteristic polynomial of the resulting closed-loop control system takes the form $P(s) = C'(s)P_0(s)$, where $P_0(s) = \bar{K}'_{N_A-N_\Lambda,N_u}(s)$ with $\bar{K}'_{N_A-N_\Lambda,N_u}(s) = \bar{\boldsymbol{k}}'^T_{N_A-N_\Lambda+N_u}\boldsymbol{s}_{N_A-N_\Lambda+N_u}$. If solely the cancellation order N_Λ is known, i.e. if $\Lambda(s)$ is unavailable (no explicit model reduction is performed), the procedure of seeking for the vectors

$$\begin{aligned} \boldsymbol{\lambda}_\Lambda &= \left[\lambda_0 \;\; \dots \;\; \lambda_{N_\Lambda-1}\right]^T, & \boldsymbol{\lambda}_\Lambda &\in \mathbb{R}^{N_\Lambda} \\ \bar{\boldsymbol{f}}'_k &= \left[\bar{f}'_{k,0} \;\; \dots \;\; \bar{f}'_{k,N_A-N_\Lambda-1}\right]^T, & \bar{\boldsymbol{f}}'_k &\in \mathbb{R}^{N_A-N_\Lambda} \\ \bar{\boldsymbol{e}}'_k &= \left[\bar{e}'_{k,0} \;\; \dots \;\; \bar{e}'_{k,N_{\bar{E}'_k}}\right]^T, & \bar{\boldsymbol{e}}'_k &\in \mathbb{R}^{N_{\bar{E}'_k}+1} \end{aligned}$$

requires solving the following set of linear equations:

$$N_{\bar{E}'_k} - N_B + N_\Lambda + 1 \left\{ \begin{bmatrix} \boldsymbol{T}^{N_B}_{N_A-N_\Lambda} & \vdots & & \vdots & \boldsymbol{T}^{N_{C'}}_{k,N_\Lambda} \\ \dots\dots\dots & \vdots & \boldsymbol{T}^{N_A}_{N_{\bar{E}'_k}+1} & \vdots & \dots\dots\dots\dots\dots \\ \mathbf{0} & \vdots & & \vdots & \mathbf{0}_{N_{\bar{E}'_k}-k+2,N_\Lambda} \end{bmatrix} \right. \begin{bmatrix} \bar{\boldsymbol{f}}'_k \\ \dots \\ \bar{\boldsymbol{e}}'_k \\ \dots \\ \boldsymbol{\lambda}_\Lambda \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0}_{N_\Lambda+k} \\ \dots\dots \\ \boldsymbol{c}' \\ \dots\dots \\ \mathbf{0} \end{bmatrix} \}N_{\bar{E}'_k} - k + 1$$

where

$$\boldsymbol{T}^{N_{C'}}_{k,N_\Lambda} = \begin{bmatrix} \mathbf{0} \\ \dots\dots \\ -\boldsymbol{T}^{C'}_{N_\Lambda} \end{bmatrix}^{\}k}, \quad \boldsymbol{T}^{N_{C'}}_{k,N_\Lambda} \in \mathbb{R}^{(N_A+k-1)\times N_\Lambda}$$

$$\boldsymbol{c}' = \left[c'_0 \;\; \dots \;\; c'_{N_A-N_\Lambda-1}\right]^T, \quad \boldsymbol{c}' \in \mathbb{R}^{N_A-N_\Lambda}$$

for the unknowns $\Lambda(s) = \sum_{i=0}^{N_\Lambda} \lambda_i s^i$ of $\deg \Lambda(s) = N_\Lambda$ and $\lambda_{N_\Lambda} = 1$, $\bar{F}'_k(s) = \sum_{i=0}^{N_A-N_\Lambda-1} \bar{f}'_{k,i} s^i$ of $\deg \bar{F}'_k(s) = N_A - N_\Lambda - 1$, and $\bar{E}'_k(s) = \sum_{i=0}^{N_{\bar{E}'_k}} \bar{e}'_{k,i}$ of $\deg \bar{E}'_k(s) = N_{\bar{E}'_k} = \max\{N_B - N_\Lambda - 1, k-1\}$, respectively. For solvability of the design problem the pair $(C'(s), \Lambda(s))$ is to be coprime. If this condition is not satisfied, the common divisor $\Lambda(s)$ should be independently estimated (as is required in the case of $(\bar{\alpha}$+DB+O)).

5. ($\bar{\alpha}$+CB+E): In order to obtain the emulator, the following sets of linear equations should be solved:

$$\begin{bmatrix} \boldsymbol{T}^{N_B}_{N_A-N_\Lambda} & \vdots & & \vdots & \boldsymbol{T}^{N_{C'}}_{k,N_\Lambda} \\ \cdots\cdots & \vdots & \boldsymbol{T}^{N_A}_{N_{\bar{E}_k}+1} & \vdots & \cdots\cdots \\ \boldsymbol{0}_{N_{\bar{E}_k}-N_B+N_\Lambda+1,N_A-N_\Lambda} & \vdots & & \vdots & \boldsymbol{0}_{N_{\bar{E}_k}-k+2,N_\Lambda} \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{f}}^0_k \\ \cdots \\ \bar{\boldsymbol{e}}_k \\ \cdots \\ \boldsymbol{\lambda}_\Lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{0}_{N_\Lambda+k} \\ \cdots \\ \boldsymbol{c}' \\ \cdots \\ \boldsymbol{0} \end{bmatrix} \Big\} N_{\bar{E}_k}-k+1 \; - \begin{bmatrix} \boldsymbol{0}_{N_A-N_\Lambda} \\ \cdots \\ \boldsymbol{T}^{N_B}_{N_\Lambda}\bar{\boldsymbol{f}}^\Lambda_k \\ \cdots \\ \boldsymbol{0} \end{bmatrix} \Big\} N_{\bar{E}_k}-N_B+1$$

$$\bar{\boldsymbol{f}}_k = \begin{bmatrix} \bar{\boldsymbol{f}}^0_k \\ \cdots \\ \bar{\boldsymbol{f}}^\Lambda_k \end{bmatrix}, \quad \bar{\boldsymbol{f}}^0_k \in \mathbb{R}^{N_A-N_\Lambda}, \quad \bar{\boldsymbol{f}}^\Lambda_k \in \mathbb{R}^{N_\Lambda}$$

where $\bar{\boldsymbol{f}}^\Lambda_k$ denotes the vector of free parameters, $k \geq 0$. The simplest choice $\bar{\boldsymbol{f}}^\Lambda_k = \boldsymbol{0}_{N_\Lambda}$ results in $\deg \bar{F}_k(s) \leq N_A - N_\Lambda$. Irreducibility of $(C'(s), \Lambda(s))$ guarantees the problem solvability.

6. ($\bar{\alpha}$+CB+O): The solutions $(\bar{H}_k(s), \bar{L}_k(s))$ to the corresponding Diophantine equation ($\bar{\text{D}}1$) obtained for $A(s) = A'(s)\Lambda(s)$ and the solutions $(\bar{H}'_k(s), \bar{L}'_k(s))$ obtained for $A'(s)$ are generally characterised by $\bar{H}_k(s) \neq \bar{H}'_k(s)$ and $\bar{L}_k(s) \neq \bar{L}'_k(s)\Lambda(s)$ for $k \geq 0$. Moreover, the following set of linear equations is involved in the controller synthesis:

$$\begin{bmatrix} \boldsymbol{T}^{N_B}_{N_A-N_\Lambda} & \vdots & & \vdots & -\boldsymbol{T}^{N_{C'}}_{N_A+N_\Lambda-1}\boldsymbol{T}^{N_{\bar{L}_k}}_{N_\Lambda} \\ \cdots\cdots & \vdots & \boldsymbol{T}^{N_A}_{N_A-1} & \vdots & \cdots\cdots \\ \boldsymbol{0}_{\rho+N_\Lambda-1,N_A-N_\Lambda} & \vdots & & \vdots & \boldsymbol{0}^T_{N_\Lambda} \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{f}}^0_k \\ \cdots \\ \bar{\boldsymbol{g}}_k \\ \cdots \\ \boldsymbol{\lambda}_\Lambda \end{bmatrix}$$

$$= \boldsymbol{T}^{N_{C'}}_{N_A+N_\Lambda} \begin{bmatrix} \boldsymbol{0}_{N_\Lambda} \\ \cdots \\ \bar{\boldsymbol{l}}_k \end{bmatrix} - \begin{bmatrix} \boldsymbol{0}_{N_A-N_\Lambda} \\ \cdots \\ \boldsymbol{T}^{N_B}_{N_\Lambda}\bar{\boldsymbol{f}}^\Lambda_k \\ \cdots \\ \boldsymbol{0} \end{bmatrix} \Big\} \rho - 1$$

where $\bar{\boldsymbol{f}}^\Lambda_k \in \mathbb{R}^{N_\Lambda}$ stands for a free parameter, $k \geq 0$. The coprimeness of $(C'(s)\bar{L}_k(s), \Lambda(s))$ or $(C'(s), \Lambda(s))$ for a Hurwitz $\Lambda(s)$ guarantees solvability of the design problem.

6. Pole Placement Perspectives

Let us consider some connections between the developed analytical ACGPC design and an emulation-based pole-placement methodology reconstructed below (references corresponding to the classical observer approach to the pole assignment can easily be found in any modern text-book by Brogan (1991), Ogata (1995), or Wellstead and Zarrop (1991), for instance). Assume that the control system is of the previously considered observer-like structure given in Fig. 1(a) (cf. Gawthrop, 1987; Middleton and Goodwin, 1990), where $M(s) = G(s)/C(s)$ and $N(s) = F(s)/C(s)$, with $G(s) = \sum_{i=0}^{N_C-1} g_i s^i$ of $\deg G(s) = N_C - 1 = N_A - 2$ and $F(s) = \sum_{i=0}^{N_A-1} f_i s^i$ of $\deg F(s) = N_A - 1$, being now certain free design polynomials. In the case of unity positional feedback, the control system can be represented as in Fig. 1(b), where $F^{=}(s) = F(s) - gC(s)$.

6.1. Basic Pole Placement Design

Let $A(s)$ and $B(s)$ be coprime. The characteristic polynomial of the closed-loop system under consideration takes the form $P(s) = A(s)C(s) + A(s)G(s) + B(s)F(s)$, $\deg P(s) = 2N_A - 1$. Assume that $D(s) = \sum_{i=0}^{N_A} d_i s^i = \prod_{i=0}^{N_A}(s - p_i)$, $d_{N_A} = 1$, $\deg D(s) = N_A$, denote a monic polynomial chosen accordingly to desired closed-loop poles p_i, $i = 1, \dots, N_A$. Hence, taking $P(s) = C(s)D(s)$ leads to the Diophantine equation

$$A(s)G(s) + B(s)F(s) = C(s)\big(D(s) - A(s)\big)$$

which is equivalent to the following set of linear equations with a non-singular Sylvester matrix (Liu and Patton, 1998):

$$\begin{array}{c} \\ \\ \rho - 1 \{ \end{array} \begin{bmatrix} \boldsymbol{T}_{N_A}^{N_B} & \vdots & \\ \cdots\cdots & \vdots & \boldsymbol{T}_{N_A-1}^{N_A} \\ \mathbf{0} & \vdots & \end{bmatrix} \begin{bmatrix} \boldsymbol{f} \\ \cdots \\ \boldsymbol{g} \end{bmatrix} = \boldsymbol{q}$$

where

$$\boldsymbol{f} = \begin{bmatrix} f_0 & \dots & f_{N_A-1} \end{bmatrix}^T, \quad \boldsymbol{f} \in \mathbb{R}^{N_A}$$

$$\boldsymbol{g} = \begin{bmatrix} g_0 & \dots & g_{N_A-2} \end{bmatrix}^T, \quad \boldsymbol{g} \in \mathbb{R}^{N_A-1}$$

$$\boldsymbol{q} = \boldsymbol{T}_{N_A}^{N_C}(\boldsymbol{d} - \boldsymbol{a}), \quad \boldsymbol{q} \in \mathbb{R}^{2N_A-1}$$

$$\boldsymbol{d} = \begin{bmatrix} d_0 & \dots & d_{N_A-1} \end{bmatrix}^T, \quad \boldsymbol{d} \in \mathbb{R}^{N_A}$$

In order to assure the unity DC gain of the tracking transfer function $T_{wy}(s)$, one has to set $gr = d_0/b_0$. In the case of integral action existing in the plant (i.e. if $A(0) = 0$) it occurs that $b_0 f_0 = c_0 d_0$. Thus the simplest tuning rule with $g = d_0/b_0 = f_0/c_0$ and $r = 1$ results in the unity DC gain of the closed-loop control system and differentiation in the internal loop ($F^{=}(0) = 0$). If the plant is minimum-phase (i.e. $B(s)$ is Hurwitz) a special setting for $D(s)$ can be recommended:

$$D(s) = b_{N_B}^{-1} D_\rho(s) B(s) \tag{32}$$

with $D_\rho(s) = \sum_{i=0}^{\rho} d_i^\rho s^i = \prod_{i=1}^{\rho}(s - p_i)$, $d_\rho^\rho = 1$, $\deg D_\rho(s) = \rho$, being a monic polynomial associated with the desired set of closed-loop poles locations p_i, $i = 1,\ldots,\rho$. The solution to the above leads to $T_{wy}(s) = (grb_{N_B})/D_\rho(s)$. This means that for $g = d_0^\rho / b_{N_B}$ and $r = 1$ one gets a possibility of explicitly shaping the closed-loop transients by employing an appropriately prepared prototype polynomial $D_\rho(s)$.

6.2. Pole Placement Design for Non-Minimal Models

In the case of non-minimal models one has four ways of handling this issue, which is shown in Table 3. Symbols C', D', F' and G' used in this table are respectively related to a simplified observer polynomial $C'(s)$ of $\deg C'(s) = N_A - N_\Lambda - 1$, a reduced-in-degree polynomial $D'(s) = \sum_{i=0}^{N_A - N_\Lambda} d_i' s^i = \prod_{i=1}^{N_A - N_\Lambda}(s - p_i)$ with $d'_{N_A - N_\Lambda} = 1$ and $\deg D'(s) = N_A - N_\Lambda$, chosen accordingly to the desired set of closed-loop poles p_i, $i = 1,\ldots,N_A - N_\Lambda$, a design polynomial $G'(s) = \sum_{i=0}^{N_A - N_\Lambda - 2} g_i' s^i$ with $\deg G'(s) = N_A - N_\Lambda - 2$, and $F'(s) = \sum_{i=0}^{N_A - N_\Lambda - 1} f_i' s^i$ having $\deg F'(s) = N_A - N_\Lambda - 1$. Each of the above Diophantine equations can easily be transformed into an equivalent set of linear equations similarly to the methodology described in the previous section (details are omitted). The only interesting case (DB) with a Hurwitz $B(s)$ is considered below. By assuming that $D(s)$ is constructed as in (32), the following set of linear equations can be obtained:

$$\rho - 1 \left\{ \begin{bmatrix} \boldsymbol{T}_{N_A - N_\Lambda}^{N_B} & \vdots & \\ \cdots\cdots & \vdots & \boldsymbol{T}_{N_A - N_\Lambda - 1}^{N_A} \\ 0 & \vdots & \end{bmatrix} \right. \begin{bmatrix} \boldsymbol{f}' \\ \cdots \\ \boldsymbol{g}' \end{bmatrix} = \boldsymbol{q}_\rho$$

$$\boldsymbol{f}' = [f_0 \ \cdots \ f_{N_A - N_\Lambda - 1}]^T, \quad \boldsymbol{f}' \in \mathbb{R}^{N_A - N_\Lambda}$$

$$\boldsymbol{g}' = [g_0 \ \cdots \ g_{N_A - N_\Lambda - 2}]^T, \quad \boldsymbol{g}' \in \mathbb{R}^{N_A - N_\Lambda - 1}$$

$$\boldsymbol{q}_\rho = \boldsymbol{T}_{N_A}^{N_{C'}} \left(b_{N_B}^{-1} \boldsymbol{T}_{N_B}^{N_{D_\rho}} \boldsymbol{b} + \boldsymbol{d}_\rho - \boldsymbol{a} \right), \quad \boldsymbol{q}_\rho \in \mathbb{R}^{2N_A - N_\Lambda - 1}$$

$$\boldsymbol{d}_\rho = \left[0_{N_B}^T \ \vdots \ d_0^\rho \ \cdots \ d_{\rho-1}^\rho \right]^T, \quad \boldsymbol{d}_\rho \in \mathbb{R}^{N_A}$$

Since $\Lambda(s)$ divides $D_\rho(s)B(s) - A(s)$, the exact solution to the design problem exists and can easily be obtained by applying any standard least-squares technique. At the same time there is no requirement for $(C'(s), \Lambda(s))$ to be coprime.

Table 3. Design principles for non-minimal models.

Case	Design Equations	Method of solving	Solutions	Key commentary
DU	$AG + BF - CD'\Lambda = -AC$	LS+p	G/C F/C	$\Lambda(s)$ is determined as a by-product of the method, effective when (CD', Λ) is coprime.
DB	$AG' + BF' - C'D'\Lambda = -AC'$	L	G'/C' F'/C'	$\Lambda(s)$ is determined as a by-product of the method, effective when $(C'D', \Lambda)$ is coprime.
CB	$AG + BF - C'(D - A)\Lambda = 0$	LS+p	$G/(C'\Lambda)$ $F/(C'\Lambda)$	$\Lambda(s)$ is a by-product of the method, effective when $(C'D, \Lambda)$ is coprime and $\Lambda(s)$ Hurwitz.
OB	$A'G' + B'F' = C'(D' - A')$	L	G'/C' F'/C'	

6.3. ACGPC from the Pole Placement Viewpoint

Considering the control strategies given by the linear combinations described by (10) or (12), we can deduce that once we have obtained the appropriate output-derivative-allied estimates, i.e. $y_{w,N_y}(t)$ and $\bar{y}_{w,N_y}(t)$, the only question that needs to be answered is how to choose the gain vector, $k_{N_y}^T$ or $\bar{k}_{N_y}^T$, respectively. The previously proposed analytical rules for ACGPC controller design can just be identified as a particular choice of these vectors expressed in common terms of predictive control lexis. In fact, such a choice may be made completely arbitrary. What is more, by employing easy-to-find and clear relationships between the gain vectors and the closed-loop characteristic polynomials, a 'natural' tuning rule for these vectors can be established. Now, the problem of the existence and computability of a controller satisfying certain design requirements can be formulated in terms of the existence and computability of the appropriate sets of polynomials, $\{F_k(s), G_k(s)\}$ or $\{\bar{F}_k(s), \bar{G}_k(s)\}$, $k = 0, \ldots, N_y$. The requisite answers can be found in Tables 1 and 2.

There are two examples considered below that additionally illustrate the simplicity of this approach:

(i) Let us consider a minimum-phase plant described by a non-minimal model $(A'(s)\Lambda(s), B'(s)\Lambda(s), C(s))$. Assume that the $(\alpha + \text{DU} + \text{E})$ design means that the control law has the form of (10) in which the emulated output derivatives are utilised. Considering the factor of the characteristic polynomial defined in (16) and (15), we obtain $P_0(s) = A'(s)(1 - H^0_{N_y,N_y-\rho}(s)) + B'(s)\boldsymbol{k}^T_{N_y}\boldsymbol{s}_{N_y}$. Taking the feedback gain of $\mathbb{R}^{N_y+1}$, $N_y \geq \rho$, in the form $\boldsymbol{k}_{N_y} = h_\rho^{-1}[\widetilde{k}_0 \ \ \dots \ \ \widetilde{k}_{\rho-1} \ \ 1 \ \ 0 \ \ \dots \ \ 0]^T$, we get $H^0_{N_y,N_y-\rho}(s) = 1$. Thus, simply letting $N_y = \rho$ yields the factorial form of the closed-loop characteristic polynomial $P(s) = h_\rho^{-1}B'(s)C(s)K_\rho(s)$, in which $K_\rho(s) = \sum_{i=0}^{\rho}\widetilde{k}_i s^i$, $\widetilde{k}_\rho = 1$. Thus the corresponding tracking transfer function $T_{wy}(s) = (\widetilde{k}_0 r)/K_\rho(s)$ can directly be shaped in order to satisfy the design specifications. The free parameters $\widetilde{k}_i$, $i = 0, \dots, \rho-1$, can instantly be utilised so as to guarantee the desired set of closed-loop poles p_i, $i = 1, \dots, \rho$: $K_\rho(s) = \prod_{i=1}^{\rho}(s - p_i)$. Then the scalar gain is $g = k_0 = h_\rho^{-1}\widetilde{k}_0 = a_{N_A}/b_{N_B}\prod_{i=1}^{\rho}(-p_i)$. Thus, with $r = 1$ the nominal transfer function $T_{wy}(s)$ has unity DC gain.

(ii) Let the controlled plant be described by a non-minimal model $(A'(s)\Lambda(s), B'(s)\Lambda(s), C'(s))$. By assuming the $(\bar{\alpha} + \mathrm{DB} + \mathrm{E})$ design with both the control strategy of (12) and the simplest setting $N_y = N_A - N_\Lambda$ one can directly establish the controller parameters $\bar{\boldsymbol{k}}'_{N_y}$, specifically, by considering the monic factor $\bar{P}'_0(s) = \sum_{i=0}^{N_A-N_\Lambda}\bar{k}'_i s^i = \prod_{i=1}^{N_A-N_\Lambda}(s - p_i)$, $\bar{k}'_{N_A-N_\Lambda} = 1$, of the characteristic polynomial $\bar{P}'(s) = C'(s)\bar{P}'_0(s)$, corresponding to the desired set of closed-loop poles p_i, $i = 1, \dots, N_A - N_\Lambda$. In order to achieve the unity DC gain of the nominal transfer function $T_{wy}(s)$, the scalar gain $g = \bar{k}'_0$ and the pre-scaling factor $r = \lambda_0/b_0$ should be implemented.

The only matter left for consideration is the problem of how to generalise the non-filtered (α) methodology in order to obtain a universal design rule for both minimum-phase and non-minimum-phase plants. This is possible if the model of (1) is minimal. Then, from the design polynomial $P_0(s) = D(s)$ generated by the desired closed-loop poles set, a suitable set of linear equations for the controller parameters $\boldsymbol{k}_{N_y}$ can instantly be obtained: $\boldsymbol{L}^T_{N_y}\boldsymbol{k}_{N_y} = \boldsymbol{d} - \boldsymbol{a}$. With $N_y = N_A - 1$ the matrix $\boldsymbol{L}^T_{N_A-1}$ is non-singular. It is also possible in the case of non-minimal models characterised by a known (formerly determined) $N_\Lambda > 0$. Then one finds the parameters $\boldsymbol{k}_{N_y}|_{N_y=N_A-N_\Lambda-1}$ associated with the polynomial $D'(s)$ generated by the limited set of desired closed-loop poles p_i, $i = 1, \dots, N_A - N_\Lambda$. This can surely be achieved via employing the reduced minimal model $(A'(s), B'(s), C'(s))$. If merely the cancellation order N_Λ is detected (and no effective model reduction has been performed), the parameters of the controller of a non-redundant structure (Case α +DB+O) as well as of a redundant structure (Case α+DU+O) can be derived by solving

$$\begin{bmatrix} \boldsymbol{L}^T_{N_A-N_\Lambda-1} & \vdots & -\boldsymbol{T}^{N_{D'}}_{N_\Lambda} \end{bmatrix} \begin{bmatrix} \boldsymbol{k}_{N_A-N_\Lambda-1} \\ \cdots\cdots\cdots \\ \boldsymbol{\lambda}_\Lambda \end{bmatrix} = \boldsymbol{d}' - \boldsymbol{a}$$

where $\boldsymbol{d}' = [\boldsymbol{0}^T_{N_\Lambda} \ \vdots \ d'_0 \ \ \dots \ \ d'_{N_A-N_\Lambda-1}]^T$, $\boldsymbol{d}' \in \mathbb{R}^{N_A}$.

It is worth noticing that $\boldsymbol{L}^T_{N_A-N_\Lambda-1}$ corresponding to a non-coprime pair $(A(s), B(s))$ can always be obtained. Clearly, a solution to the above equation exists if and only if the polynomials $D'(s)$ and $\Lambda(s)$ are relatively prime. As, in general, one has no additional prior knowledge about the common factor $\Lambda(s)$, using the rank deficiency detection algorithm, described in Appendix B, is recommended. In particular, if $\operatorname{rank}[\boldsymbol{L}^T_{N_A-N_\Lambda-1} \;\vdots\; -\boldsymbol{T}^{N_{D'}}_{N_\Lambda}] < N_A$, then $D'(s)$ has to be properly redesigned. In order to enforce the unity DC gain of the nominal transfer function $T_{wy}(s)$, the pre-scaling coefficient r should be taken as $r = d_0/(k_0 b_0)$ if $N_\Lambda = 0$, or $r = d_0\lambda_0/(k_0 b_0)$ if $N_\Lambda > 0$. If $A(0) = 0$, i.e. if there is integral action in the control channel, one always observes that $r = 1$.

7. Illustrative Example

The ACGPC design method proposed in this paper is analytical in its roots. Nevertheless, for the reader's convenience, let us consider a simulated example of our ACGPC implementation, which can also be viewed as another evidence of the design correctness, applicability and simplicity.

Assume that a non-minimum phase and non-stable plant is characterised by the non-minimal model $A(s) = -1.5s + s^2 - 1.5s^3 + s^4 = s(s-1.5)(s^2+1)$, $B(s) = -1.5 + 1.3s - 0.2s^2 = -0.2(s-5)(s-1.5)$, $N_A = 4$, $N_B = 2$ and $\rho = 2$. Let the desired control specifications, defined for the closed-loop step-response, be described by the indices $\kappa \cong 0.05$, $T_{s2\%} \leq 2\,\mathrm{sec}$ and $u(0) \leq 75$. The algorithm given in Appendix B applied to determine the rank deficiency of the matrix

$$\boldsymbol{L}^T_3 = \begin{bmatrix} -1.5 & 0 & 0 & 0 \\ 1.3 & -1.5 & -0.3 & 1.5 \\ -0.2 & 1.3 & -1.3 & -1.3 \\ 0 & -0.2 & 1.0 & 0.2 \end{bmatrix}$$

yields $N_\Lambda = 1$.

Let the reduced-degree observer polynomial be of the form $C'(s) = 1 + s + 0.2s^2$. Now, the filters $M(s) = G'^0_{N_y}(s)/C'(s)$ with $\deg G'^0_{N_y}(s) = 1$ and $N(s) = F'^0_{N_y}(s)/C'(s)$ with $\deg F'^0_{N_y}(s) = 2$ are to be designed. By performing the computations described in Appendix D one ascertains that $\Lambda(s) = -1.5 + s$ and

$$\bar{\boldsymbol{F}}'_3 = \begin{bmatrix} 1 & 1.1154 & 0.4231 \\ 0 & 0.5769 & 1.1154 \\ 0 & -1.1154 & 0.5769 \\ 0 & -0.5769 & -1.1154 \end{bmatrix}$$

$$\bar{E}'_3 = \left[\begin{array}{c:cc} 0.0846 & 0 & 0 \\ 0.4231 & 0 & 0 \\ \hdashline 1.1154 & 0.2 & 0 \\ 0.5769 & 0 & 0.2 \end{array}\right], \quad \bar{G}'_3 = \begin{bmatrix} 0.0846 & 0 \\ 0.4231 & 0 \\ 1.1154 & 0.2 \\ -0.4231 & 0 \end{bmatrix}$$

The set-point pre-scaling factor is $r = \lambda_0/b_0 = 1$. From the data listed in Table 4 of Appendix F it follows that the value of the control prediction order can suitably be chosen as $N_u = 2$. Since $u(0) = rg = r\bar{k}'_0 = r\widetilde{k}'^0_{3,2}/T^3$, the time scaling factor is constrained by $T \geq (\widetilde{k}'^0_{3,2}/u(0))^{1/3} = 1.4978\,\mathrm{sec}$. Thus let the observation horizon be $T = 1.5\,\mathrm{sec}$. This leads to the gain $g = 74.667$, and the observer numerators $G'^0_{N_y}(s) = 32.1795 + 1.8s$ and $F'^0_{N_y}(s) = 74.6667 + 94.8205s + 78.4974s^2$. Simulation of the resulting closed-loop system shows that $\kappa = 0.054$ and $T_{s2\%} = 1.74\,\mathrm{sec}$ (see also Figs. 2 and 3).

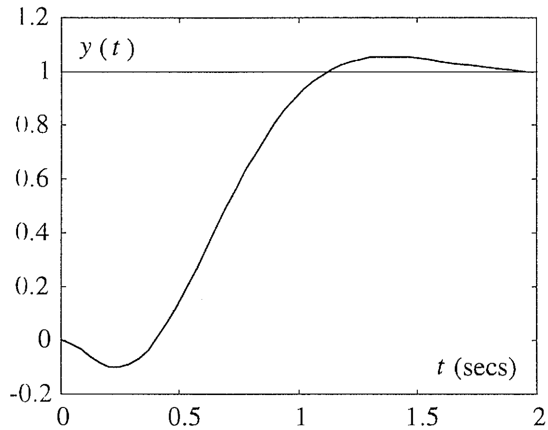

Fig. 2. Step response.

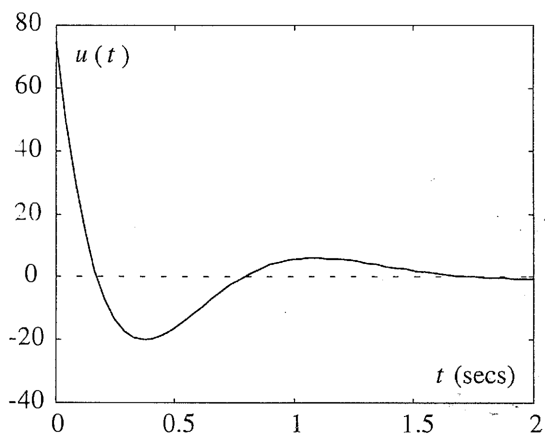

Fig. 3. Controlling signal.

8. Concluding Remarks

In our considerations, scalar linear continuous-time plants are represented with the aid of two conjoined rational transfer functions describing the controlled part and the disturbed part of the plant, respectively. Two cases of model-based prediction concerning the future output (Case α) and the future filtered output of the plant (Case $\bar{\alpha}$) are developed. Suitable emulation of the output derivatives serves as a basis for the signal prediction. The emulation can be performed by using two design paths, referred to as the Emulator (E) and the Observer (O) paths, respectively, resulting from resolution of a set of coupled Diophantine equations, taken from a suitable Diophantine basis (different for each case, α and $\bar{\alpha}$). The prediction of type α is useful in the CGPC design for minimum-phase plant models, while the prediction of type $\bar{\alpha}$ (with the output signal filtered by the numerator polynomial of the controlled plant transfer function) is appropriate for both minimum-phase and non-minimum-phase plant models.

All the necessary emulation procedures, the CGPC design algorithms, including the two (α and $\bar{\alpha}$) introduced predictive control laws are provided, and an analysis of the resulting closed-loop systems performed. Moreover, explicit formulae for closed-loop characteristic polynomials are given that serve as a basis for the analytical ACGPC design (both ways, α and $\bar{\alpha}$), which is the main objective of this presentation. In this approach, the principal CGPC 'tuning knobs', i.e. the output and control prediction orders as well as the horizon of observation, are directly related to common time-domain design specifications.

By observing the above-mentioned specific design parameterisation, simplicity and explicitness of the proposed design solution is obtained. The nominal performance and nominal stability of the closed-loop ACGPC systems for both minimum-phase and non-minimum-phase plants is guaranteed via a suitable choice of the control order that can be derived from the (relative) order of the plant model and arbitrary time-domain design specifications. These features, which are in contrast to the generic CGPC design, lay practical foundations for applying of the method in adaptive control systems.

Taking into account various aspects of the CGPC design (the design cases α and $\bar{\alpha}$, paths E and O, and related Diophantine equations, the methods of solving them, and the forms of solution), syncopating forethoughts are provided that indicate meaningful consequences of miscellaneous types of potential non-minimality of plant models, including design unilateral (DU), design bilateral (DB), controller bilateral (CB), and object bilateral (OB) types of non-minimality (reducibility).

What is more, a method of detecting the cancellation order of non-minimal plant models along with a way of reconstructing their minimal models are proposed that are established by examining the rank deficiency of a testing matrix composed of the coefficients of residual polynomials yielded by a properly defined set of Diophantine equations, resulting, in turn, from the transfer function of the controlled part of the plant model. There is also a collection of specific studies provided in the Appendices given below, where the prototype design characteristic polynomials are catalogued and certain computational aspects of the developed procedures are explained in detail.

A self-imposing analysis of the proposed analytical emulation-based ACGPC design methodology from the viewpoint of pole placement (PP), employing a design-identified characteristic polynomial, is also performed. It is shown that the ACGPC method can easily act toward a special case of the PP technique. A specific property of ACGPC is that the characteristic polynomial of the ACGPC closed-loop system is a resultant polynomial, conforming to the postulate of minimisation of the quadratic index, which is composed with the aid of the predicted system-output signal and governed with the use of the traditional CGPC gears: the prediction orders and observation horizon. Consequently, the family of implemented prototype polynomials is obtained by a suitable parameterisation of a "fundamental" factor of the closed-loop characteristic polynomial resulting from the ACGPC design.

Settling the deliberated ACGPC project on the catalogued properties of the polynomial family allows for rationalisation (minimisation) of the degrees of design freedom. For a given system relative order (in Case α), or the true/minimal system order (in Case $\bar{\alpha}$), the only key knob for the designer is the control prediction order which affects the closed-loop stability margin, the overshoot of the step response and the control signal magnitude. At the same time, one has to keep in mind that the observation horizon resulting from the design specifications exerts influence on the speed of transient processes and the control signal, as well. This type of synthesis, characterised by a limited design freedom, appears to be beneficial as compared to both the generic CGPC and PP methodologies. In particular, in most practical cases of the PP scheme, where all the poles of the closed-loop system transfer function have to be discriminated, the designer allocates only one pair of dominant complex poles and puts the remaining poles somewhere "deep in LHP." This procedure applied to higher-order systems can, however, easily result in unduly large control signals.

Another important outcome of the proposed approach is the fact that the common subordination of the control signal both by the control weighting factor λ of the cost function and the anticipative filtration of the control error (Kowalczuk *et al.*, 1996), is now abandoned. Instead, the control effort is practically restricted by using a constrained control signal (in terms of its instant value), implied by the scalar gain coefficient g (entailing the observation horizon T). Consequently, the designer is able to trade off effectively between the control energy and output transients (e.g. the settling time).

In this paper, two practically important issues of nominal stability and nominal performance of the CGPC control systems are deliberated at length. On the other hand, the only modelling uncertainty included in the design is the existence of cancellations in the plant model. This issue is completely solved by effective detection of the corresponding plant model cancellation order. Thus, the under consideration plant models are otherwise nominal. Nevertheless, it is equally important that the proposed analytical control design approach provides the means for discussion of the two other complementary issues of robust stability and robust performance in cases of modelling uncertainty. For instance, the control structure comprising a freely designed polynomial $C(s)$ allows for shaping robust and noise properties of the closed-loop CGPC systems (with an interplay included). The presented here analytical considerations can also be used in synthesis of predictive control of delay systems (Kowalczuk and Suchomski, 1999).

Appendices

A. Proofs of Lemmas 2, 3 and 4

Proof of Lemma 2. (Only if) Suppose that $A(s)$ and $B(s)$ have a common factor $\Lambda(s) = \lambda_0 + s$: $A(s) = A'(s)\Lambda(s)$, $\deg A'(s) = N_A - 1$, and $B(s) = B'(s)\Lambda(s)$, $\deg B'(s) = N_B - 1$. By virtue of (D1) one can conclude that

$$L_k(s) = L_k'(s)\Lambda(s), \quad \deg L_k'(s) = \deg L_k(s) - 1 \tag{33}$$

with L_k' satisfying the reduced degree Diophantine equation

$$\text{(D1}'\text{)} \qquad A'(s)H_k(s) + L_k'(s) = s^k B'(s), \quad k \geq 0$$

From (33) it follows that $\boldsymbol{L}_{N_A-1}^T = \boldsymbol{T}_{N_A-1}^{N_\Lambda}\boldsymbol{L}_{N_A-1}'^T$, where $\boldsymbol{L}_{N_A-1}'^T = [\boldsymbol{l}_0' \ \cdots \ \boldsymbol{l}_{N_A-1}']$, $\boldsymbol{L}_{N_A-1}' \in \mathbb{R}^{N_A \times (N_A-1)}$, has columns $\boldsymbol{l}_k' = [l_{k,0}' \ \cdots \ l_{k,N_A-2}']^T$, $\boldsymbol{l}_k' \in \mathbb{R}^{N_A-1}$, established by polynomials $L_k'(s)$, $0 \leq k \leq N_A - 1$, and $\boldsymbol{T}_{N_A-1}^{N_\Lambda} \in \mathbb{R}^{N_A \times (N_A-1)}$. Clearly, $\operatorname{rank} \boldsymbol{L}_{N_A-1}^T = \operatorname{rank} \boldsymbol{L}_{N_A-1}'^T \leq N_A - 1$, a contradiction.

(If) From Lemma 1 it follows that the first ρ columns of $\boldsymbol{L}_N^T$ are composed of the coefficients of the numerator polynomial $B(s)$, thus one has $\rho \leq \operatorname{rank} \boldsymbol{L}_{N_A-1}^T \leq N_A$. Therefore, in the sequel only the non-trivial case of $\rho < N_A$ will be analysed. Suppose that $\boldsymbol{L}_{N_A-1}^T$ is singular. Then the set of homogeneous linear equations $\boldsymbol{L}_{N_A-1}^T \boldsymbol{x} = \boldsymbol{0}_{N_A}$ has a non-zero solution for $\boldsymbol{x} \in \mathbb{R}^{N_A}$. By splitting $\boldsymbol{x} = [\alpha_0 \ \cdots \ \alpha_{\rho-1} \ \vdots \ \beta_0 \ \cdots \ \beta_{N_B-1}]^T$ and a simple matrix algebra one obtains the relation $B(s)\sum_{k=0}^{\rho-1}\alpha_k s^k + \sum_{k=0}^{N_B-1}\beta_k L_{\rho+k}(s) = 0$. With the use of the above and the first Diophantine equation (D1) one gains $A(s)W_B(s) = B(s)W_A(s)$, where $W_A(s) = \sum_{k=0}^{\rho-1}\alpha_k s^k + \sum_{k=0}^{N_B-1}\beta_k s^{\rho+k}$ and $W_B(s) = \sum_{k=0}^{N_B-1}\beta_k H_{k+\rho}(s)$. Since $\deg W_A(s) \leq N_A - 1$, the previous assumption (that $\deg A(s) = N_A$) is thus contradicted.

Proof of Lemma 3. Since $\forall k \geq 0 : \ L_k(s) = L_k'(s)\Lambda(s)$, with $L_k'(s) = \sum_{i=0}^{N_A-N_\Lambda-1} l_{k,i}' s^i$ satisfying the corresponding (first) Diophantine equation appropriately reduced-in-degree, one has $\boldsymbol{L}_{N_A-1}^T = \boldsymbol{T}_{N_A-N_\Lambda}^{N_\Lambda}\boldsymbol{L}_{N_A-1}'^T$, where $\boldsymbol{L}_{N_A-1}' = [\boldsymbol{l}_0' \ \cdots \ \boldsymbol{l}_{N_A-1}']^T$, $\boldsymbol{L}_{N_A-1}' \in \mathbb{R}^{N_A \times (N_A-N_\Lambda)}$, and columns $\boldsymbol{l}_k' = [l_{k,0}' \ \cdots \ l_{k,N_A-N_\Lambda-1}']^T$, $\boldsymbol{l}_k' \in \mathbb{R}^{N_A-N_\Lambda}$, are composed of the coefficients of polynomials $L_k'(s)$, $k = 0, \ldots, N_A - 1$, while $\boldsymbol{T}_{N_A-N_\Lambda}^{N_\Lambda} \in \mathbb{R}^{N_A \times (N_A-N_\Lambda)}$. From Lemma 2 it follows that $\operatorname{rank} \boldsymbol{L}_{N_A-1}'^T = \operatorname{rank} \boldsymbol{L}_{N_A-N_\Lambda-1}'^T = N_A - N_\Lambda$. Since $\boldsymbol{T}_{N_A-N_\Lambda}^{N_\Lambda}$ is of full column rank, i.e. $\operatorname{rank} \boldsymbol{T}_{N_A-N_\Lambda}^{N_\Lambda} = N_A - N_\Lambda$, the claim is proved.

Proof of Lemma 4. The proof is straightforward, and follows directly from the equality $A(s)B'(s) = B(s)A'(s)$. Since

$$\left[\boldsymbol{T}_{N_A-N_\Lambda}^{N_B} \ \vdots \ -\boldsymbol{T}_{N_B-N_\Lambda}^{N_A}\right] = \boldsymbol{T}_{N_A+N_B-2N_\Lambda}^{N_\Lambda}\left[\boldsymbol{T}_{N_A-N_\Lambda}^{N_{B'}} \ \vdots \ -\boldsymbol{T}_{N_B-N_\Lambda}^{N_{A'}}\right]$$

where $\operatorname{rank} \boldsymbol{T}^{N_\Lambda}_{N_A+N_B-2N_\Lambda} = N_A + N_B - 2N_\Lambda$ as well as $\operatorname{rank}[\boldsymbol{T}^{N_{B'}}_{N_A-N_\Lambda} \vdots - \boldsymbol{T}^{N_{A'}}_{N_B-N_\Lambda}] = N_A + N_B - N_\Lambda$, one concludes that the matrices of (2) and (3) are of full column rank. Since $\boldsymbol{t}_{ab} \in \mathcal{R}[\boldsymbol{T}^{N_B}_{N_A-N_\Lambda} \vdots -\boldsymbol{T}^{N_A}_{N_B-N_\Lambda}]$ and $\bar{\boldsymbol{t}}_{ab} \in \mathcal{R}[\boldsymbol{T}^{N_B}_{N_A-N_\Lambda}]$, the exact solution to (2) can be obtained by applying any least-squares method (Björck, 1996; Golub and Van Loan, 1996). Once the coprime pair $(A'(s), B'(s))$ is derived, the coefficients $\boldsymbol{\lambda}_\Lambda = [\lambda_0 \ \ldots \ \lambda_{N_\Lambda-1}]^T$, $\boldsymbol{\lambda}_\Lambda \in \mathbb{R}^{N_\Lambda}$, of the common factor $\Lambda(s)$ can be estimated by solving another simple least-squares problem (given here for $N_\Lambda < N_B$)

$$\begin{bmatrix} \boldsymbol{T}^{N_{A'}}_{N_\Lambda} \\ \cdots\cdots \\ \boldsymbol{T}^{N_{B'}}_{N_\Lambda} \end{bmatrix} \boldsymbol{\lambda}_\Lambda = \begin{bmatrix} \boldsymbol{a} \\ \cdots \\ \boldsymbol{b} \end{bmatrix} \begin{bmatrix} \mathbf{0}_{N_\Lambda} \\ \cdots\cdot \\ \boldsymbol{a}' \\ \cdots\cdot \\ \mathbf{0}_{N_\Lambda} \\ \cdots\cdot \\ \boldsymbol{b}' \end{bmatrix}$$

B. Determination of Cancellation Order N_Λ

Let $N_\Lambda > 0$. From Lemma 3 it follows that $\boldsymbol{l}_{N_A-N_\Lambda} \in \mathcal{R}[\boldsymbol{L}^T_{N_A-N_\Lambda-1}]$, which is equivalent to $\boldsymbol{l}_{N_A-N_\Lambda} \in \mathcal{N}[\boldsymbol{P}_{N_A-N_\Lambda-1}]$, where $\mathcal{N}[\boldsymbol{P}_{N_A-N_\Lambda-1}]$ denotes the null space of the projector $\boldsymbol{P}_{N_A-N_\Lambda-1} = \boldsymbol{I}_{N_A} - \boldsymbol{L}^T_{N_A-N_\Lambda-1}(\boldsymbol{L}^T_{N_A-N_\Lambda-1})^+$, $\boldsymbol{P}_{N_A-N_\Lambda-1} \in \mathbb{R}^{N_A \times N_A}$, onto the orthogonal complement $\mathcal{R}^\perp[\boldsymbol{L}^T_{N_A-N_\Lambda-1}]$ of the range space of $\boldsymbol{L}^T_{N_A-N_\Lambda-1}$. Therefore N_Λ can be computed by using the following recursive algorithm for Moore-Penrose generalised inversion (Boullion and Odell, 1971) of subsequent left submatrices of $\boldsymbol{L}^T_{N_A-1}$.

The algorithm (Kowalczuk *et al.*, 1996; Kowalczuk and Suchomski, 1996)

Initialisation $(i = 0)$:

$$\begin{aligned} \boldsymbol{n}_0 &= \boldsymbol{l}_0 \\ \boldsymbol{n}_0^+ &= \|\boldsymbol{n}_0\|_2^{-2}\boldsymbol{n}_0^T, \quad (\boldsymbol{L}_0^T)^+ = \boldsymbol{n}_0^+ \\ \boldsymbol{P}_0 &= \boldsymbol{I}_{N_A} - \boldsymbol{n}_0\boldsymbol{n}_0^+ \end{aligned}$$

Iteration $(i \leftarrow i + 1$; End: if $i = N_A)$:

$$\begin{aligned} \boldsymbol{n}_i &= \boldsymbol{P}_{i-1}\boldsymbol{l}_i \\ \|\boldsymbol{n}_i\|_2^2 &= \boldsymbol{n}_i^T\boldsymbol{n}_i \quad (\Rightarrow \text{End : if } \|\boldsymbol{n}_i\|_2^2 < \varepsilon) \\ \boldsymbol{n}_i^+ &= \|\boldsymbol{n}_i\|_2^{-2}\boldsymbol{n}_i^T, \quad \boldsymbol{p}_i = (\boldsymbol{L}^T_{i-1})^+\boldsymbol{l}_i \end{aligned}$$

$$(L_i^T)^+ = \begin{bmatrix} (L_{i-1}^T)^+ - p_i n_i^+ \\ \cdots\cdots\cdots\cdots \\ n_i^+ \end{bmatrix}$$

$$P_i = P_{i-1} - n_i n_i^+$$

Termination:

$$N_\Lambda = N_A - i$$

where ε denotes a small real number determining the accuracy of computations.

A geometric characterisation of the iterative mechanism of the algorithm is given in Fig. 4, where $\|n_i\|_2/\|l_i\|_2 = \sin\alpha_i \leq \|P_{i-1}\|_s$, $i \geq 1$, with $\|\cdot\|_s$ denoting the spectral norm, can be interpreted as a measure of the angular distance between l_i and $\mathcal{R}[L_{i-1}^T]$. Another important outcome of the algorithm is the fact that a numerical estimate of the pseudoinverse $(L_i^T)^+$, $i = 1, \ldots, N_A - N_\Lambda$, is also available and can be utilised while considering the corresponding linear least-squares problem.

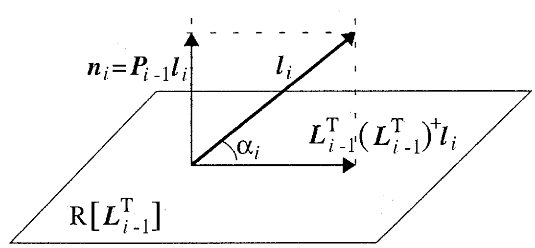

Fig. 4. Geometric interpretation of the algorithm for determination of the cancellation order.

C. Solution of Diophantine Equations (D1)–(D4)

C.1. Diophantine equations (D1)–(D3) can easily be solved be employing the following recursive algorithms that can immediately be verified by direct computations (see also Grimble, 1992; Ježek, 1993; Kučera, 1993):

$E_k(s)$: $\quad e_0 = 0$

$$e_1 = c_{N_C}$$

$$e_i = c_{N_C - i+1} - \sum_{j=\max\{1, i-N_A\}}^{i-1} e_j a_{N_A - i+j}, \quad i \geq 2$$

and $\quad c_{N_C - l} = 0 \quad$ if $\quad l > N_C$

$H_k(s)$:

$$h_0 = \cdots = h_{\rho-1} = 0$$

$$h_\rho = b_{N_B}$$

$$h_{\rho+i} = b_{N_B-i} - \sum_{j=\max\{0,i-N_A\}}^{i-1} h_{\rho+j} a_{N_A-i+j}, \quad i \geq 1$$

$$\text{and} \quad b_{N_B-i} = 0 \quad \text{if} \quad i > N_B$$

$L_k(s)$:

$$l_{k,i} = b_{i-k}, \quad 0 \leq k \leq \rho - 1, \quad 0 \leq i \leq N_A - 1$$

$$\text{and} \quad b_l = 0 \quad \text{if} \quad l < 0 \quad \text{or} \quad l > N_B$$

$$l_{k,i} = b_{i-k} - \sum_{j=\max\{\rho,k-i\}}^{k} h_j a_{i-k+j}, \quad k \geq \rho, \quad 0 \leq i \leq N_A - 1$$

$$\text{and} \quad b_l = 0 \quad \text{if} \quad l < 0$$

$F_k(s)$:

$$f_{0,i} = c_i, \quad 0 \leq i \leq N_A - 1$$

$$f_{k,i} = c_{i-k} - \sum_{j=\max\{1,k-i\}}^{k} e_j a_{i-k+j}, \quad k \geq 1, \quad 0 \leq i \leq N_A - 1$$

$$\text{and} \quad c_l = 0 \quad \text{if} \quad l < 0$$

$G_k(s)$:

$$g_{0,i} = 0, \quad 0 \leq i \leq N_A - 2$$

$$g_{k,i} = \sum_{j=\max\{1,k-i\}}^{\min\{k,k-N_B-i\}} e_j b_{i-k+j}, \quad 1 \leq k \leq \rho - 1, \quad 0 \leq i \leq N_A - 2$$

$$g_{k,i} = \sum_{j=\max\{1,k-i\}}^{\min\{k,k-N_B-i\}} e_j b_{i-k+j} - \sum_{j=\max\{\rho,k-i\}}^{k} h_j c_{i-k+j}, \quad k \geq \rho$$

$$0 \leq i \leq N_A - 2$$

C.2. Since the matrices of (4), (30) and (31) do not depend on k, the following block equation (D4) should be solved for any $k \geq 0$:

$$\begin{bmatrix} f_0 & \cdots & f_k \\ \cdots & \cdots & \cdots \\ g_0 & \cdots & g_k \end{bmatrix} = \begin{matrix} \\ \\ \rho-1\{ \end{matrix}\left[\begin{matrix} T_{N_A}^{N_B} \\ \cdots \\ 0 \end{matrix} \; \vdots \; T_{N_A-1}^{N_A}\right]^{-1} T_{N_A}^{N_C} L_k^T$$

$$\begin{bmatrix} f_0^0 & \cdots & f_k^0 \\ \cdots & \cdots & \cdots \\ g_0 & \cdots & g_k \end{bmatrix} = \left[\begin{matrix} T_{N_A-N_\Lambda}^{N_B} \\ \cdots \\ 0_{\rho+N_\Lambda-1,N_A-N_\Lambda} \end{matrix} \; \vdots \; T_{N_A-1}^{N_A}\right]^{+} T_{N_A}^{N_C} L_k^T,$$

$$\begin{bmatrix} f'_0 & \cdots & f'_k \\ \cdots & \cdots & \cdots \\ g'_0 & \cdots & g'_k \end{bmatrix} = \begin{matrix} \\ \\ \rho-1\{ \end{matrix}\left[\begin{matrix} T_{N_A-N_\Lambda}^{N_B} \\ \cdots \\ 0 \end{matrix} \; \vdots \; T_{N_A-N_\Lambda-1}^{N_A}\right]^{+} T_{N_A}^{N_{C'}} L_k^T$$

where the minimal model $(A(s), B(s), C(s))$, the non-minimal model $(A(s), B(s), C(s))$ with the zero parameterisation $\begin{bmatrix} f_0^\Lambda & \cdots & f_k^\Lambda \end{bmatrix} = 0_{N_\Lambda,k+1}$, and the non-minimal model $(A(s), B(s), C'(s))$ can be used, respectively.

D. Solution of Diophantine Equation ($\bar{D}4$)

The vectors $\bar{e}_k$, $k \geq 0$, have the following form:

$$\bar{e}_k = \begin{cases} \bar{e}_k^0 = \begin{bmatrix} \bar{e}_{k,0} & \cdots & \bar{e}_{k,N_B-1} \end{bmatrix}^T, \quad \bar{e}_k, \bar{e}_k^0 \in \mathbb{R}^{N_B} \quad \text{if } 0 \leq k \leq N_B \\ \begin{bmatrix} \bar{e}_k^0 \\ \cdots \\ \bar{e}_k^- \end{bmatrix} = \begin{bmatrix} \bar{e}_{k,0} & \cdots & \bar{e}_{k,N_B-1} & \vdots & \bar{e}_{k,N_B} & \cdots & \bar{e}_{k,k-1} \end{bmatrix}^T, \\ \qquad \bar{e}_k \in \mathbb{R}^k, \quad \bar{e}_k^- \in \mathbb{R}^{k-N_B} \quad \text{if } k > N_B \end{cases}$$

The co-ordinates of the vector

$$\bar{e}_k^- = \begin{cases} \bar{e}_{N_B+1,N_B} = c_{N_A-1} & \text{if } k = N_B + 1 \\ \begin{bmatrix} \bar{e}_{k,N_B} \\ \cdots \\ \bar{e}_{k-1}^- \end{bmatrix} & \text{if } k > N_B + 1 \end{cases}$$

can be computed be applying the following recursive formula:

$$\bar{e}_{k,N_B} = c_{N_A+N_B-k} - \sum_{l=N_B+1}^{\min\{N_A+N_B,k-1\}} \bar{e}_{k,l} a_{N_A+N_B-l}, \quad k > N_B + 1$$

where $c_{N_A+N_B-k}=0$ for $k>N_A+N_B$. Then, for $k \geq N_B+1$, the solution takes the form

$$\left[\begin{array}{ccc:ccc} \bar{e}_0^0 & \cdots & \bar{e}_{N_B}^0 & \bar{e}_{N_B+1}^0 & \cdots & \bar{e}_k^0 \\ \hdashline \bar{f}_0 & \cdots & \bar{f}_{N_B} & \bar{f}_{N_B+1} & \cdots & \bar{f}_k \end{array}\right] = \left[\boldsymbol{T}_{N_B}^{N_A} \,\vdots\, \boldsymbol{T}_{N_A}^{N_B}\right]^{-1} \left[\boldsymbol{T}_{N_B+1}^{N_C} \,\vdots\, \bar{e}_{N_B+1}^{=} \cdots \bar{e}_k^{=}\right]$$

where

$$\bar{e}_k^{=} = c_k^0 - \boldsymbol{T}_{k12}^{N_A}\bar{e}_k^{-}, \quad \bar{e}_k^{=} \in \mathbb{R}^{N_A+N_B}, \quad k>N_B$$

$$c_k^0 = \begin{cases} \left[0_k^T \,\vdots\, c_0 \cdots c_{N_A+N_B-k-1}\right]^T & \text{if } N_B<k<N_A+N_B, \\ 0_{N_A+N_B}^T & \text{if } k \geq N_A+N_B, \end{cases} \quad c_k^0 \in \mathbb{R}^{N_A+N_B}$$

$$\boldsymbol{T}_k^{N_A} = \left[\begin{array}{c:c} \boldsymbol{T}_{N_B}^{N_A} & \boldsymbol{T}_{k12}^{N_A} \\ \hdashline 0_{k-N_B,N_B} & \boldsymbol{T}_{k22}^{N_A} \end{array}\right], \quad \begin{array}{l} \boldsymbol{T}_{k12}^{N_A} \in \mathbb{R}^{(N_A+N_B)\times(k-N_B)}, \\ \boldsymbol{T}_{k22}^{N_A} \in \mathbb{R}^{(k-N_B)\times(k-N_B)}, \end{array} \quad k>N_B$$

The exact solution exists because the Sylvester matrix $[\boldsymbol{T}_{N_B}^{N_A} \,\vdots\, \boldsymbol{T}_{N_A}^{N_B}]$ is non-singular.

E. Gain of CGPC Controller

From (22) and (23) it follows that for $T=1$ we have

$$\left(\boldsymbol{T}_{\rho,\rho+N_u}^{\rho,\rho+N_u}(0,1)\right)^{-1}\boldsymbol{T}_{\rho,\rho+N_u}^{0,\rho-1}(0,1) = \boldsymbol{P}_{\rho,\rho+N_u}^{-1}\left(\widetilde{\boldsymbol{T}}_{\rho,\rho+N_u}^{\rho,\rho+N_u}\right)^{-1}\widetilde{\boldsymbol{T}}_{\rho,\rho+N_u}^{0,\rho-1}\boldsymbol{P}_{0,\rho-1} \quad (34)$$

where

$$\widetilde{\boldsymbol{T}}_{k,l}^{m,n} = \begin{bmatrix} 1/(k+m+1) & \cdots & 1/(k+n+1) \\ \vdots & \ddots & \vdots \\ 1/(l+m+1) & \cdots & 1/(l+n+1) \end{bmatrix}, \quad 0 \leq k \leq l, \quad 0 \leq m \leq n,$$

$$\widetilde{\boldsymbol{T}}_{k,l}^{m,n} \in \mathbb{R}^{(l-k+1)\times(n-m+1)}$$

The above given matrix can be found in the square Hilbert matrix $\widetilde{\boldsymbol{T}}_{0,\rho+N_u}^{0,\rho+N_u}$ with the following partitioning:

$$\widetilde{\boldsymbol{T}}_{0,\rho+N_u}^{0,\rho+N_u} = \boldsymbol{P}_{0,\rho+N_u}^{-1}\boldsymbol{T}_{0,\rho+N_u}^{0,\rho+N_u}(0,1)\boldsymbol{P}_{0,\rho+N_u}^{-1} = \left[\begin{array}{c:c} \widetilde{\boldsymbol{T}}_{0,\rho-1}^{0,\rho-1} & \widetilde{\boldsymbol{T}}_{0,\rho-1}^{\rho,\rho+N_u} \\ \hdashline \widetilde{\boldsymbol{T}}_{\rho,\rho+N_u}^{0,\rho-1} & \widetilde{\boldsymbol{T}}_{\rho,\rho+N_u}^{\rho,\rho+N_u} \end{array}\right]$$

It is a known fact that Hilbert matrices exhibit bad numerical conditioning (Demmel, 1997). On the other hand, the inversion of $\widetilde{\boldsymbol{T}}^{0,\rho+N_u}_{0,\rho+N_u}$ can be obtained explicitly. Hence the inverse of the Hilbert matrix $\widetilde{\boldsymbol{T}}^{0,\rho+N_u}_{0,\rho+N_u}$

$$\left(\widetilde{\boldsymbol{T}}^{0,\rho+N_u}_{0,\rho+N_u}\right)^{-1} = \bar{\boldsymbol{T}}^{\rho+N_u} = \left[\bar{t}^{\rho,N_u}_{i,j}\right]^{\rho+N_u}_{i,j=0}, \quad \bar{\boldsymbol{T}}^{\rho+N_u} \in \mathbb{R}^{(\rho+N_u+1)\times(\rho+N_u+1)}$$

can be shown to have the following entries:

$$\bar{t}^{\rho,N_u}_{i,j} = \frac{(-1)^{j-i}}{(i+j+1)}\frac{(\rho+N_u+i+1)!(\rho+N_u+j+1)!}{(i!)^2(j!)^2(\rho+N_u-i)!(\rho+N_u-j)!}$$

An interesting feature resulting from the common partitioned matrix inversion formula (Barnett, 1971; Weinmann, 1991) is that a component of

$$\bar{\boldsymbol{T}}^{\rho+N_u} = \left[\begin{array}{c:c} \bar{\boldsymbol{T}}^{\rho,N_u}_{11} & \bar{\boldsymbol{T}}^{\rho,N_u}_{12} \\ \hdashline \bar{\boldsymbol{T}}^{\rho,N_u}_{21} & \bar{\boldsymbol{T}}^{\rho,N_u}_{22} \end{array}\right]$$

namely, $\bar{\boldsymbol{T}}^{\rho,N_u}_{21} = -(\widetilde{\boldsymbol{T}}^{\rho,\rho+N_u}_{\rho,\rho+N_u})^{-1}\widetilde{\boldsymbol{T}}^{0,\rho-1}_{\rho,\rho+N_u}\bar{\boldsymbol{T}}^{\rho,N_u}_{11}$, contains the necessary factor of the right-hand side of (34) and the partitioning complies with the original decomposition of $\widetilde{\boldsymbol{T}}^{0,\rho+N_u}_{0,\rho+N_u}$: $\bar{\boldsymbol{T}}^{\rho,N_u}_{11} \in \mathbb{R}^{\rho\times\rho}$, $\bar{\boldsymbol{T}}^{\rho,N_u}_{12} \in \mathbb{R}^{\rho\times(N_u+1)}$, $\bar{\boldsymbol{T}}^{\rho,N_u}_{21} \in \mathbb{R}^{(N_u+1)\times\rho}$ and $\bar{\boldsymbol{T}}^{\rho,N_u}_{22} \in \mathbb{R}^{(N_u+1)\times(N_u+1)}$. Rewriting $\boldsymbol{\nu}^T_{\rho,N_u}(0,1)$ in the form of $\boldsymbol{\nu}^T_{\rho,N_u}(0,1) = [\rho! \ 0 \ \cdots \ 0](\widetilde{\boldsymbol{T}}^{\rho,\rho+N_u}_{\rho,\rho+N_u})^{-1}\widetilde{\boldsymbol{T}}^{0,\rho-1}_{\rho,\rho+N_u}\boldsymbol{P}_{0,\rho-1}$ and using $\bar{\boldsymbol{T}}^{\rho,N_u}_{21}$ leads to

$$\bar{\boldsymbol{T}}^{\rho,N_u}_{11}\boldsymbol{P}^{-1}_{0,\rho-1}\boldsymbol{\nu}_{\rho,N_u}(0,1) = -\rho!\,\bar{\boldsymbol{t}}^{\rho,N_u} \tag{35}$$

where $\bar{\boldsymbol{t}}^{\rho,N_u} = [\bar{t}^{\rho,N_u}_{\rho,0} \ \cdots \ \bar{t}^{\rho,N_u}_{\rho,\rho-1}]^T$, $\bar{\boldsymbol{t}}^{\rho,N_u} \in \mathbb{R}^{\rho}$, and $\boldsymbol{P}^{-1}_{0,\rho-1} = \operatorname{diag}\{k!\}^{\rho-1}_{k=0}$. By direct calculations it can easily be verified that $\boldsymbol{\nu}_{\rho,N_u}(0,1)$ defined by (24) and (26) satisfies the relation (35).

F. Properties of Design Polynomials $\widetilde{\boldsymbol{K}}_{\rho,\boldsymbol{N}_u}(\boldsymbol{p})$

Consider the case of $\rho = 3$. Let $\widetilde{g} = gh_\rho T^\rho = \widetilde{k}^0_{\rho,N_u}$ denote the normalised CGPC controller gain. Four parameters of the normalised step response, corresponding to the prototype transfer function $\widetilde{g}/\widetilde{K}_{\rho,N_u}(p)$, are considered. In particular, $\widetilde{\kappa}$ denotes the overshoot occurring at the normalised time instant $\widetilde{T}_\kappa$, whereas $\widetilde{T}_{s2\%}$ and $\widetilde{T}_{s5\%}$ denote the normalised settling times.

Table 4. ACGPC-model specifications for $\rho = 3$.

N_u	$\widetilde{g}$	$\widetilde{K}_{\rho,N_u}(p)$	Pooles of $\widetilde{K}_{\rho,N_u}(p)$	$\widetilde{\kappa}\vert\widetilde{T}_\kappa$	$\widetilde{T}_{s5\%}\vert\widetilde{T}_{s2\%}$
0	10.5	$10.5 + 8.4p + 3.5p^2 + p^3$	$-0.774 \pm j2.186$, -1.952	0.1572|1.972	3.623|3.992
1	67.2	$67.2 + 33.6p + 8p^2 + p^3$	$-2.095 \pm j3.640$, -3.810	0.0693|1.204	1.402|1.582
2	252	$252 + 86.4p + 13.5p^2 + p^3$	$-3.695 \pm j5.253$, -6.109	0.0441|0.837	0.606|1.048
3	720	$720 + 180p + 20p^2 + p^3$	$-5.573 \pm j7.089$, -8.855	0.0333|0.621	0.447|0.742
4	1732.5	$1732.5 + 330p + 27.5p^2 + p^3$	$-7.728 \pm j9.172$, -12.044	0.0277|0.477	0.343|0.551
5	3696	$3696 + 554.4p + 36p^2 + p^3$	$-10.163 \pm j11.512$, -15.673	0.0244|0.380	0.272|0.424
6	7207.2	$7207.2 + 873.6p + 45.5p^2 + p^3$	$-12.880 \pm j14.114$, -19.741	0.0222|0.309	0.221|0.335
7	13104	$13104 + 1310.4p + 56p^2 + p^3$	$-15.878 \pm j16.982$, -24.244	0.0208|0.257	0.183|0.269
8	22522.5	$22522.5 + 1890p + 67.5p^2 + p^3$	$-19.159 \pm j20.118$, -29.181	0.0197|0.217	0.154|0.166

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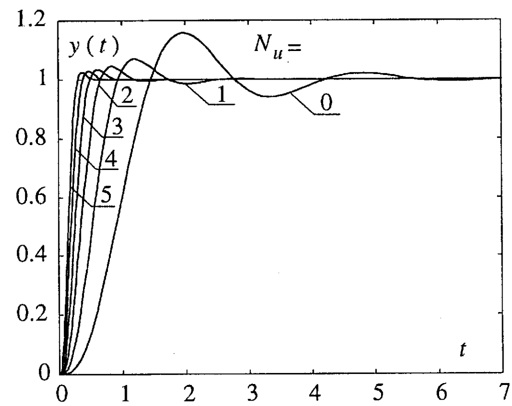

Fig. 5. CGPC-system response for $\rho = 3$.

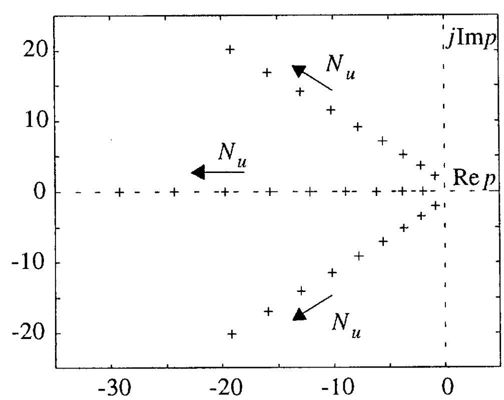

Fig. 6. Root loci of CGPC models for $\rho = 3$.

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