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# APPLICATION OF SENSITIVITY THEORY TO FUZZY LOGIC BASED FDI

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This paper describes an application of sensitivity theory to the analysis of a certain class of fuzzy systems which can be used for fault detection and isolation (FDI). The work is divided into three main tasks. The first is the mathematical representation of some class of fuzzy systems. This is followed by an application of sensitivity theory to fuzzy systems based on the approach detailed in the first part. Finally, this method is applied to a fuzzy fault diagnosis scheme for the two-tank system, and the results compared with those achieved by the application of sensitivity theory to a non-fuzzy diagnosis scheme for the same system. Simulation results for the fuzzy and non-fuzzy fault diagnosis schemes are presented, which verify the results obtained via the application of sensitivity theory.

**Keywords:** fuzzy systems, fault detection and isolation, fuzzy inference, sensitivity theory, residual analysis, two-tank system.

## 1. Introduction

In the last few years fuzzy systems and fuzzy models have been used intensively for numerous applications as a new method in control theory. Fuzzy systems are characterised by a high robustness, i.e. control loops are less susceptible to deviations of the plant parameters. With classical design methods, however, the analysis of the system sensitivity is a well-defined science. The problem arises with fuzzy systems where such analysis methods do not exist. Also, the inclusion of a fuzzy component produces a heterogeneous system (see e.g. Fig. 1): a plant described by a mathematical model and a knowledge-based fuzzy controller. Since the two methods used to describe the constituent parts of the overall system are completely different, it is difficult (if not impossible) to establish a method to analyse the entire loop. A solution to this problem is to represent the elements of the system in the same way, i.e. either by transforming the description of the plant into a fuzzy model, or by transforming the fuzzy component into a mathematical description to gain a homogeneous overall system. The second approach offers the possibility of applying well-known mathematical methods to the analysis of the system, and it is this approach which will be

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considered in more detail in this paper. Since a fuzzy system is not a mysterious black box, but simply a nonlinear function designed using the tool of fuzzy logic, it is (under certain assumptions) possible to derive this mathematical function.

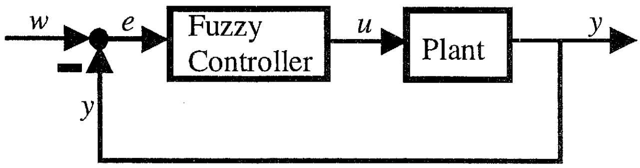


Fig. 1. Fuzzy control loop.

Robustness is often defined in terms of a system property (e.g. a steady-state value) and a class of disturbances (e.g. parameter changes) towards which the system property should be robust. Parameter sensitivity is in general the impact of parameter deviations on a variable which describes the system dynamics, e.g. the output $y$. In the literature, robustness and sensitivity are given various definitions. Some authors use robustness as a modern synonym for insensitivity, whilst others distinguish the two using e.g. the following criteria:

A. Robustness is concerned with finite parameter deviations (global property), whilst sensitivity applies to infinitesimal parameter deviations (local property).

B. The problem of sensitivity is the maintenance of nominal values whilst the problem of robustness does normally not consider nominal behaviour.

Although sensitivity theory does indeed consider very small parameter deviations, for finite but small parameter deviations, sensitivity functions can be expected to provide satisfying results for engineering practice. Such a definition is also dependent upon the system in question: for some systems even very small parameter deviations can cause undesired behaviour, and thus affect the robustness of the system in the sense of criterion (A) above.

The sensitivity function can be regarded as one tool to consider robustness, assuming the following four categories of parameter deviations $\Delta\alpha$ are to be taken into account:

1. $\Delta\alpha$ **infinitesimal**: The sensitivity function predicts nearly exactly the deviation $\Delta\xi$ of some system variable $\xi$.

2. $\Delta\alpha$ **small** (up to 30% (Frank, 1978)): The sensitivity function yields satisfying results for engineering practice.

3. $\Delta\alpha$ **large** (nominal parameter value multiplied several times): The sensitivity function indicates a tendency of how the control system will behave. This can be useful for the determination of the time interval in which $\Delta\alpha$ has the strongest influence and for the analysis of two different controllers for the same system.

4. $\Delta\alpha$ **very large**: The system reacts unpredictably.

The above approach to obtain a mathematical description of a fuzzy system, and then to use this mathematical form to determine the sensitivity of the system to changes in system parameters has successfully been applied to the analysis of a fuzzy control loop (Klotzek *et al.*, 1998; Klotzek and Frank, 1997).

Fuzzy models and fuzzy systems are of use not only in the field of control, but also within fault diagnosis, both as models of systems and as decision making elements (Isermann and Ballé, 1996). The properties of robustness and sensitivity play an important role in the effectiveness of a fault diagnosis system. A system should be sensitive to faults and robust to parameter changes (when such parameter changes are not themselves considered as faults). The ability to state in what way a system is robust to certain parameter changes can be used to aid the design of the system, and also to state minimum detectable fault sizes without the need for extensive simulation runs.

## 2. Sensitivity Theory

Parameter sensitivity provides information regarding the impact of deviations of a parameter $\alpha$ on a variable $\xi$ which describes the system dynamics. The deviation of the parameter $\alpha$ is given by (Frank, 1978):

$$\Delta\alpha = \alpha - \alpha_0 \tag{1}$$

where $\alpha_0$ is the nominal value of the parameter and $\alpha$ denotes the actual parameter value. Similarly, the resulting change in the system variable $\xi$ is given by

$$\Delta\xi = \xi - \xi_0 \tag{2}$$

where $\xi_0 = \xi(t, \alpha_0)$ is the nominal value of the system variable and $\xi$ stands for the actual system variable value. A sensitivity function $S(\alpha_0) = S(\alpha_0, \xi, t)$ can then be defined by the relation between $\Delta\alpha$ and $\Delta\xi$:

$$\Delta\xi \approx S(\alpha_0)\Delta\alpha \tag{3}$$

The sensitivity function can be considered as a mapping from the parameter space to the system variable space, as shown in Fig. 2.

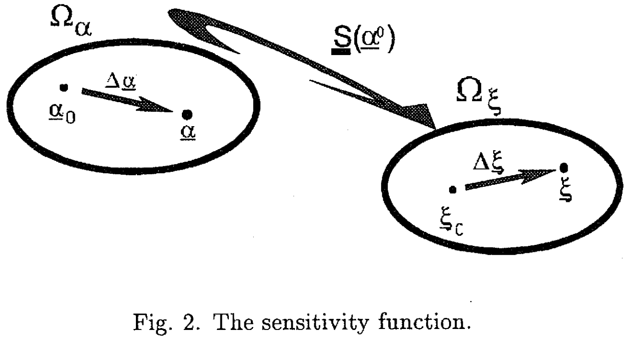


Fig. 2. The sensitivity function.

For $\xi(t) = y(t)$, the *output sensitivity function* for one parameter $\alpha$ is derived from the Taylor series at $\alpha_0$:

$$y(t,\alpha) = y(t,\alpha_0) + \left.\frac{\partial y}{\partial \alpha}\right|_{\alpha = \alpha_0} \Delta\alpha + \frac{1}{2}\left.\frac{\partial^2 y}{\partial \alpha^2}\right|_{\alpha = \alpha_0} (\Delta\alpha)^2 + \cdots \tag{4}$$

The linear approximation (terminating after the first-order term) is

$$y(t,\alpha) \approx y(t,\alpha_0) + \left.\frac{\partial y(t,\alpha)}{\partial \alpha}\right|_{\alpha = \alpha_0} \Delta\alpha \tag{5}$$

For infinitesimally small parameter deviations $\Delta\alpha$ we have

$$\Delta y(t,\alpha) = \underbrace{\left.\frac{\partial y(t,\alpha)}{\partial \alpha}\right|_{\alpha=\alpha_0}}_{S(\alpha_0)} \Delta\alpha \tag{6}$$

## 3. Mathematical Representation of a Fuzzy System

In order to apply sensitivity theory to a fuzzy system, the system has to be transformed into its mathematical representation. The study is limited to fuzzy systems with the following characteristics, since for fuzzy control and fuzzy fault models these type of fuzzy systems are applied most often:

- Sugeno-type system (Takagi and Sugeno, 1985);
- Two input variables fuzzified using triangular and trapezoid membership functions with complementary overlaps;
- One output variable fuzzified using singleton membership functions;
- AND operator: Algebraic product

$$\mu_{A\cap B}(x) = \mu_A(x)\mu_B(x)$$

- OR operator: Absolute sum

$$\mu_{A\cup B}(x) = \mu_A(x) + \mu_B(x)$$

- Defuzzification method: Centre of gravity for singletons.

For this class of fuzzy systems the absolute sum can be used (as opposed to the algebraic sum) because with complementary overlaps all truth values will be in the interval $[0, 1]$.

A general form of the input membership function is shown in Fig. 3. The capital letters denote linguistic variables, whilst the small letters describe the borders of the intervals where the same fuzzy sets are valid.

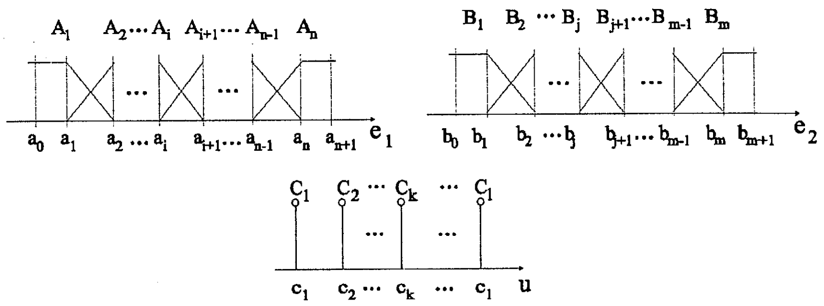


Fig. 3. A general form of input and output membership functions.

With such a system, it is possible to derive a mathematical function for the membership grade, *for each interval* along the input axis. For example, in the interval $e_1 \in [a_i, a_{i+1}] =: I_i$, the membership grades are

$$\mu_{A_i}(e_1) = \frac{e_1 - a_i}{a_{i+1} - a_i} \tag{7}$$

$$\mu_{A_{i+1}}(e_1) = \frac{a_{i+1} - e_1}{a_{i+1} - a_i} \tag{8}$$

Similar relationships can be derived for $e_2 \in [b_j, b_{j+1}] =: J_j$. This gives a mathematical function relating the input membership grades to the inputs. In order to determine a function relating the output $y$ to the inputs $e_1$ and $e_2$, i.e. to describe the fuzzy systems as a mathematical function, the contributing rules of the fuzzy system have to be combined. A tool has been implemented in Maple (a computer-algebra tool (Redfern, 1994)) which automatically derives the function *over all intervals*, for the class of systems described above. This tool processes different components of the fuzzy symbolically and so it generates a mathematical function representing the fuzzy system analytically, where the influence of all variables becomes visible (Klotzek *et al.*, 1999).

For each pair of intervals $(I_i, J_j)$ the mathematical function is found in the form

$$y_{ij}(e_1, e_2) = \tilde{k}_{1_{ij}} e_1 e_2 + \tilde{k}_{2_{ij}} e_1 + \tilde{k}_{3_{ij}} e_2 + \tilde{k}_{4_{ij}} \tag{9}$$

This can be written as

$$y_{ij}(e_1, e_2) = \begin{pmatrix} e_1 & 1 \end{pmatrix} M_{ij} \begin{pmatrix} e_2 \\ 1 \end{pmatrix} \tag{10}$$

were

$$M_{ij} = \begin{pmatrix} \tilde{k}_{1_{ij}} & \tilde{k}_{2_{ij}} \\ \tilde{k}_{3_{ij}} & \tilde{k}_{4_{ij}} \end{pmatrix} \tag{11}$$

The final result is thus a mathematical description of the fuzzy system in terms of the parameters of the input and output membership functions, i.e. the function for the fuzzy system over all intervals is

$$y_{\mathrm{FUZ}}(e_1,e_2)=\begin{cases}\tilde{k}_{1_{00}}e_1e_2+\tilde{k}_{2_{00}}e_1+\tilde{k}_{3_{00}}e_2+\tilde{k}_{4_{00}}, & \begin{matrix}e_1\in I_0\\ e_2\in J_0\end{matrix}\\ \vdots & \vdots\\ \tilde{k}_{1_{ij}}e_1e_2+\tilde{k}_{2_{ij}}e_1+\tilde{k}_{3_{ij}}e_2+\tilde{k}_{4_{ij}}, & \begin{matrix}e_1\in I_i\\ e_2\in J_j\end{matrix}\\ \vdots & \vdots\\ \tilde{k}_{1_{nm}}e_1e_2+\tilde{k}_{2_{nm}}e_1+\tilde{k}_{3_{nm}}e_2+\tilde{k}_{4_{nm}}, & \begin{matrix}e_1\in I_n\\ e_2\in J_m\end{matrix}\end{cases}\tag{12}$$

Each part $y_{ij}$ of the fuzzy controller is represented unequivocally by the coefficient matrix $M_{ij}$. So the analytical function of the fuzzy controller $y_{\mathrm{FUZ}}(e_1,e_2)$ is described completely by the following table of coefficient matrices:

| | $e_2\in J_0$ | $\cdots$ | $e_2\in J_j$ | $\cdots$ | $e_2\in J_m$ |
|---|---|---|---|---|---|
| $e_1\in I_0$ | $M_{00}$ | $\cdots$ | $M_{0j}$ | $\cdots$ | $M_{0m}$ |
| $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| $e_1\in I_i$ | $M_{i0}$ | $\cdots$ | $M_{ij}$ | $\cdots$ | $M_{im}$ |
| $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| $e_1\in I_n$ | $M_{n0}$ | $\cdots$ | $M_{nj}$ | $\cdots$ | $M_{nm}$ |

(13)

The matrices $M_{ij}$ can be determined by the following calculation:

For each pair of intervals $(I_i, J_j)$ the relevant section of the rule base ($A_i$, $A_{i+1}, B_j$ and $B_{j+i}$ are the only fuzzy sets that contribute with a truth value $>0$) is

| | $\cdots$ | $B_j$ | $B_{j+1}$ | $\cdots$ |
|---|---|---|---|---|
| $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ | $\cdots$ |
| $A_i$ | $\cdots$ | $c_{ij}$ | $c_{i,j+1}$ | $\cdots$ |
| $A_{i+1}$ | $\cdots$ | $c_{i+1,j}$ | $c_{i+1,j+1}$ | $\cdots$ |
| $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ | $\cdots$ |

(14)

where $c_{i,j}, c_{i,j+1}, c_{i+1,j}$ and $c_{i+1,j+1}$ are the corresponding values of the singletons $C_{i,j}, C_{i,j+1}, C_{i+1,j}$ and $C_{i+1,j+1}$.

The matrix of coefficients $M_{ij}$ can then be calculated by the following formula (dependent on the relevant entries of the rule base and the boundaries of the appropriate intervals in which $e_1$ and $e_2$ are located):

$$M_{ij} = \begin{pmatrix} \dfrac{c_{ij} - c_{i,j+1} - c_{i+1,j} + c_{i+1,j+1}}{(a_i - a_{i+1})(b_j - b_{j+1})} & \dfrac{-c_{ij} b_{j+1} + c_{i,j+1} b_j + c_{i+1,j} b_{j+1} - c_{i+1,j+1} b_j}{(a_i - a_{i+1})(b_j - b_{j+1})} \\ \dfrac{-c_{ij} a_{i+1} + c_{i,j+1} a_{i+1} + c_{i+1,j} a_i - c_{i+1,j+1} a_i}{(a_i - a_{i+1})(b_j - b_{j+1})} & \dfrac{c_{ij} a_{i+1} b_{j+1} - c_{i,j+1} a_{i+1} b_j - c_{i+1,j} a_i b_{j+1} + c_{i+1,j+1} a_i b_j}{(a_i - a_{i+1})(b_j - b_{j+1})} \end{pmatrix} \tag{15}$$

Note that by inserting for all variables the appropriate values, all matrices $M_{ij}$, $i = 0, \ldots, n$, $j = 0, \ldots, m$ are in $\mathbb{R}^{2\times 2}$, so a very simple mathematical function for the class of fuzzy controllers under consideration has been found.

## 4. Application of the Approach to Fuzzy Fault Diagnosis Systems

In this section, the previous investigations of the functional description of fuzzy systems and sensitivity theory will be applied to fuzzy fault models. Consider a system with output $y$ and inputs $e_1$ and $e_2$. In the case of a fault diagnosis scheme the system may represent a fault model of the plant. The sensitivity of the output $y$ of the fault model to a particular fault $f$ is given by

$$\frac{\partial y}{\partial f} = \frac{\partial y}{\partial e_1}\frac{\partial e_1}{\partial f} + \frac{\partial y}{\partial e_2}\frac{\partial e_2}{\partial f} \tag{16}$$

A similar sensitivity function can be written for disturbances $d$ to the plant:

$$\frac{\partial y}{\partial d} = \frac{\partial y}{\partial e_1}\frac{\partial e_1}{\partial d} + \frac{\partial y}{\partial e_2}\frac{\partial e_2}{\partial d} \tag{17}$$

If the sensitivity functions of eqns. (16) and (17) can be calculated, then it is possible to derive an expression for the relative sensitivities with respect to disturbance $d$ and fault $f$. This expression should then be maximised to ensure that the effect of faults on the output of the fault model is large when compared with the effects of disturbances,

$$\max \left[ \frac{\dfrac{\partial y}{\partial f}}{\dfrac{\partial y}{\partial d}} \right] \tag{18}$$

The results from the application of sensitivity theory to fuzzy systems can be used by the designer of a fuzzy FDI system in the following ways:

- To assess the relative robustness to system parameters of fuzzy and non-fuzzy models.
- To determine which parameters of the fuzzy system itself (i.e. which input membership parameters) affect the response of the system to particular faults and disturbances. This information can be used in two ways: firstly, as a guideline to the designer as to which parameters should be tuned when attempting to improve the system manually, and secondly, as the basis for a mathematical optimisation of the system parameters given certain constraints or specifications.

### 4.1. An Example System

In the following, a simple fuzzy system with two inputs and one output is considered. The theory detailed above is not limited to the number of inputs, however the representation becomes much more complex as the number of inputs increases.

The system under consideration is shown in Fig. 4, and consists of two tanks, connected via a valve. The input to the system is the pump outflow $Q_1$ and it is assumed that the height $h_2$ is available as a measurement.

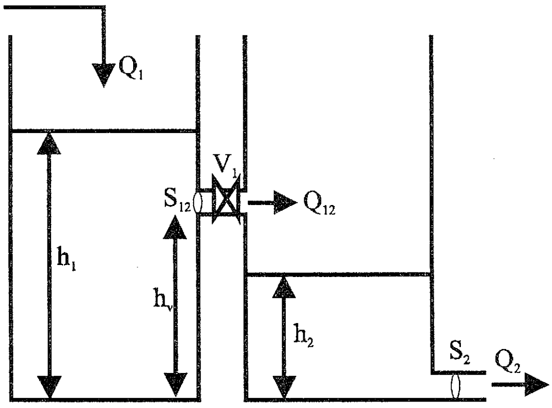


Fig. 4. The two-tank example.

The equations of the system are

$$\dot{h}_1 = \frac{1}{A_1}\left[Q_1 - a_z S_{12}\sqrt{2g(h_1 - h_v)}V_1\right] \tag{19}$$

$$\dot{h}_2 = \frac{1}{A_1}\left[a_z S_{12}\sqrt{2g(h_1 - h_v)}V_1 - a_z S_2\sqrt{2gh_2}\right] \tag{20}$$

$Q_1$ is a control input given by

$$Q_1 = 10^{-3}(w_1 - h_1) + 5 \times 10^{-6} \int (w_1 - h_1)\,\mathrm{d}t \tag{21}$$

where $w_1$ is the desired set-point of the level in Tank 1. Moreover, $A_1$, $h_v$, $a_z$, $S_{12}$ and $S_2$ are parameters of the system and $g$ is the acceleration due to gravity.

We consider the case where the valve $V_1$ between the two tanks is nominally 70% open (i.e. $V_1$ has a nominal value of 0.7). A fault can occur in the system if the position of this valve changes by ± 30%. A fuzzy fault model is constructed whose inputs are two measurements taken from the system, i.e. $e_1 = Q_1$ and $e_2 = h_2$. The desired output from this fuzzy model is the actual valve position. The input and output membership functions are given in Fig. 5.

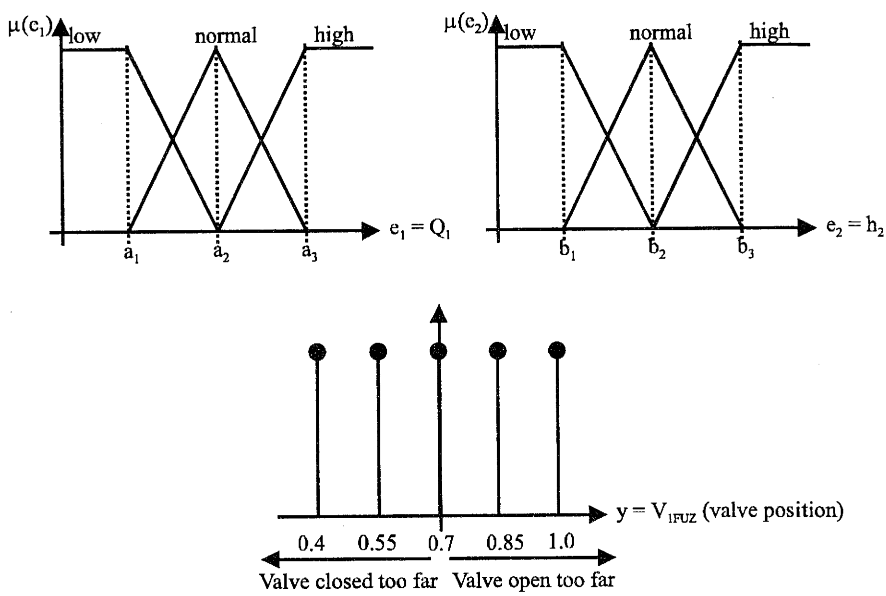


Fig. 5. Input and output membership functions for the two-tank example.

The rule base for the fuzzy fault model consists of a set of rules, each with two terms in the premise, e.g.

IF $Q_1$ is *LOW* AND $h_2$ is *LOW* THEN $V_1$ is *LOW*

where the low valve position is a closure of less than 70% ($V_1$ =0.7). The rules were derived from visual inspection of the outputs of a simulation in the presence of faults. The rule base can be summarised in Table 1 which gives the change in the value of $V_1$. A mathematical expression for this system was derived using the approach detailed above, which results in a table of coefficient matrices (Tables 2 and 3).

Table 1. The fuzzy rule base for the two tank system.

| | | $h_2$ low | $h_2$ normal | $h_2$ high |
|---|---|---|---|---|
| $Q_1$ | low | −0.3 | -1.5 | 0 |
| $Q_1$ | normal | −0.15 | 0 | 0.15 |
| $Q_1$ | high | 0 | 0.15 | 0.3 |

Table 2. Coefficient matrices for the fuzzy system.

| | $h_2 \in ]-\infty, b_1]$ | $h_2 \in [b_1, b_2]$ | $h_2 \in [b_2, b_3]$ | $h_2 \in [b_3, \infty[$ |
|---|---|---|---|---|
| $Q_1 \in ]-\infty, a_1]$ | $M_{11}$ | $M_{12}$ | $M_{13}$ | $M_{14}$ |
| $Q_1 \in [a_1, a_2]$ | $M_{21}$ | $M_{22}$ | $M_{23}$ | $M_{24}$ |
| $Q_1 \in [a_2, a_3]$ | $M_{31}$ | $M_{32}$ | $M_{33}$ | $M_{34}$ |
| $Q_1 \in [a_3, \infty[$ | $M_{41}$ | $M_{42}$ | $M_{43}$ | $M_{44}$ |

As an example of the procedure whereby the functions of Table 3 were derived, consider the case of $M_{23}$ which corresponds to $Q_1 \in [a_1, a_2]$, $h_2 \in [b_2, b_3]$. From Fig. 5 it can be seen that the corresponding linguistic values for these ranges are $Q_1$ is *low* or *normal* and $h_2$ is *normal* or *high*. The corresponding output singleton values for these ranges can then be obtained from Table 1. Thus, for this case, $a_1 = a_1$, $a_{i+1} = a_2$, $b_j = b_2$, $b_{j+1} = b_3$, $c_{i,j} = -0.15$, $c_{i,j+1} = 0$, $c_{i+1,j} = 0$, $c_{i+1,j+1} = 0.15$. Substituting these values into (15) yields the function for $M_{23}$ shown in Table 3.

The above can then be differentiated (for each interval) to yield the sensitivity of the output of the fuzzy system with respect to the inputs $Q_1$ and $h_2$. This gives

$$\frac{\partial V_{1\mathrm{FUZ}}}{\partial Q_1} = \begin{bmatrix} Q_1 & 1 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{matrix} f_1 \\ f_2 \end{matrix} & \begin{matrix} f_1 \\ f_2 \end{matrix} & \begin{matrix} f_1 \\ f_2 \end{matrix} & \begin{matrix} f_1 \\ f_2 \end{matrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} h_2 \\ 1 \end{bmatrix} \tag{22}$$

Table 3. Expressions for the coefficient matrices of the fuzzy system.

$$M_{11} = \begin{pmatrix} 0 & 0 \\ 0 & -0.3 \end{pmatrix}, \qquad M_{12} = \begin{pmatrix} 0 & 0 \\ \dfrac{-0.15}{b_1 - b_2} & \dfrac{-0.15b_1 + 0.3b_2}{b_2 - b_1} \end{pmatrix}$$

$$M_{13} = \begin{pmatrix} 0 & 0 \\ \dfrac{0.15}{b_3 - b_2} & \dfrac{-0.15b_3}{b_3 - b_2} \end{pmatrix}, \qquad M_{14} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$M_{21} = \begin{pmatrix} 0 & \dfrac{-0.15}{a_1 - a_2} \\ 0 & \dfrac{-0.15a_1 + 0.3a_2}{a_2 - a_1} \end{pmatrix}$$

$$M_{22} = \begin{pmatrix} 0 & \dfrac{-1}{6.67a_1 - 6.67a_2} \\ \dfrac{-1}{6.67b_1 - 6.67b_2} & \dfrac{0.15a_1b_2 + 0.15a_2b_1 - 0.3a_2b_2}{a_1b_1 - a_1b_2 - a_2b_1 + a_2b_2} \end{pmatrix}$$

$$M_{23} = \begin{pmatrix} 0 & \dfrac{-1}{6.67a_1 - 6.67a_2} \\ \dfrac{1}{6.67b_3 - 6.67b_2} & \dfrac{-0.15a_1b_2 + 0.15a_2b_3}{-a_1b_2 + a_1b_3 + a_2b_2 - a_2b_3} \end{pmatrix}$$

$$M_{24} = \begin{pmatrix} 0 & \dfrac{-0.15}{a_1 - a_2} \\ 0 & \dfrac{-0.15a_1}{a_1 - a_2} \end{pmatrix}, \qquad M_{31} = \begin{pmatrix} 0 & \dfrac{0.15}{a_3 - a_2} \\ 0 & \dfrac{-0.15a_3}{a_3 - a_2} \end{pmatrix}$$

$$M_{32} = \begin{pmatrix} 0 & \dfrac{1}{6.67a_3 - 6.67a_2} \\ \dfrac{-1}{6.67b_1 - 6.67b_2} & \dfrac{-1.5a_2b_1 + 1.5a_3b_2}{a_3b_1 - a_3b_2 - a_2b_1 + a_2b_2} \end{pmatrix}$$

$$M_{33} = \begin{pmatrix} 0 & \dfrac{1}{6.67a_3 - 6.67a_2} \\ \dfrac{1}{6.67b_3 - 6.67b_2} & \dfrac{3a_2b_2 - 0.15a_2b_3 - 0.15a_3b_2}{-a_3b_2 + a_3b_3 + a_2b_2 - a_2b_3} \end{pmatrix}$$

$$M_{34} = \begin{pmatrix} 0 & \dfrac{0.15}{a_3 - a_2} \\ 0 & \dfrac{-3a_2 + 0.15a_3}{a_3 - a_2} \end{pmatrix}, \quad M_{41} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_{42} = \begin{pmatrix} 0 & 0 \\ \dfrac{-0.15}{b_1 - b_2} & \dfrac{0.15b_1}{b_1 - b_2} \end{pmatrix}$$

$$M_{43} = \begin{pmatrix} 0 & 0 \\ \dfrac{0.15}{-b_2 + b_3} & \dfrac{-3b_2 + 0.15b_3}{-b_2 + b_3} \end{pmatrix}, \qquad M_{44} = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}$$

where

$$f_1 = \begin{bmatrix} 0 & 0 \\ 0 & -\dfrac{0.15}{a_1 - a_2} \end{bmatrix}, \quad f_2 = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{0.15}{a_3 - a_2} \end{bmatrix}$$

and

$$\frac{\partial V_{1\mathrm{FUZ}}}{\partial h_2} = \begin{bmatrix} Q_1 & 1 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & f_3 & f_4 & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & f_3 & f_4 & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & f_3 & f_4 & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & f_3 & f_4 & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} h_2 \\ 1 \end{bmatrix} \tag{23}$$

where

$$f_3 = \begin{bmatrix} 0 & 0 \\ 0 & -\dfrac{0.15}{b_1 - b_2} \end{bmatrix}, \quad f_4 = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{0.15}{b_3 - b_2} \end{bmatrix}$$

In order to demonstrate the use of the analysis method described in this paper, the fuzzy FDI system will be compared with a non-fuzzy alternative which is derived from the mathematical equations of the system. If we consider the steady state of the system, i.e. $\dot{h}_1 = 0$ and $\dot{h}_2 = 0$, then, from (19) and (20), assuming $S_{12} = S_2$, we have

$$Q_1 = a_z S_{12} \sqrt{2g(h_1 - h_v)} V_1 \tag{24}$$

$$h_2 = V_1 (h_1 - h_v) \tag{25}$$

Furthermore, from (24) it follows that

$$V_1 = \frac{Q_1}{a_z S_{12} \sqrt{2g(h_1 - h_v)}} \tag{26}$$

This equation can now be used to derive an estimate $V_{1\mathrm{EST}}$ of the valve position $V_1$ from the measured variables $Q_1$ and $h_1$:

$$V_{1\mathrm{EST}} = \frac{Q_1}{a_z S_{12} \sqrt{2g(h_1 - h_v)}} \tag{27}$$

Partial derivatives of $V_{1\mathrm{EST}}$ with respect to system parameters can then be found. For example, consider the derivative with respect to the parameter $a_z$:

$$\frac{\partial V_{1\mathrm{EST}}}{\partial a_z} = \frac{-Q_1}{a_z^2 S_{12} \sqrt{2g(h_1 - h_v)}} \tag{28}$$

Similarly, the effect of the parameter $a_z$ on the output $V_{1\mathrm{FUZ}}$ of the fuzzy system can then be found in the following way:

$$\frac{\partial V_{1\mathrm{FUZ}}}{\partial a_z} = \frac{\partial V_{1\mathrm{FUZ}}}{\partial Q_1}\frac{\partial Q_1}{\partial a_z} + \frac{\partial V_{1\mathrm{FUZ}}}{\partial h_2}\frac{\partial h_2}{\partial a_z}$$

$$\frac{\partial V_{1\mathrm{FUZ}}}{\partial Q_1} = \frac{-0.15}{a_2 - a_1}, \quad Q_1 \in [a_1, a_2]$$

$$\frac{\partial V_{1\mathrm{FUZ}}}{\partial Q_1} = \frac{0.15}{a_3 - a_2}, \quad Q_1 \in [a_2, a_3]$$

From (24) we have

$$\frac{\partial Q_1}{\partial a_z} = S_n\sqrt{2g(h_1 - h_v)}V_1 \tag{29}$$

and from (25) we get

$$\frac{\partial h_2}{\partial a_z} = 0 \tag{30}$$

Thus

$$\frac{\partial V_{1\mathrm{FUZ}}}{\partial a_z} = \begin{cases} \dfrac{-0.15 S_{12}\sqrt{2g(h_1 - h_v)}V_1}{a_2 - a_1}, & Q_1 \in [a_1, a_2] \\[2ex] \dfrac{0.15 S_{12}\sqrt{2g(h_1 - h_v)}V_1}{a_3 - a_2}, & Q_1 \in [a_2, a_3] \end{cases} \tag{31}$$

There are two sensitivity functions which are relevant in different ranges of the input variables.

The above sensitivity functions were calculated using the following nominal system parameter values:

$$\begin{cases} a_{z\mathrm{nom}} = 1 \\ A = 0.0154 \\ g = 9.81 \\ h_v = 0.3 \\ S_1 = S_{12} = 0.00002 \\ h_{1\mathrm{nom}} = 0.5 \\ Q_{1\mathrm{nom}} = 0.0000278 \end{cases} \tag{32}$$

This gives the fuzzy sensitivities

$$\frac{\partial V_{1\mathrm{FUZ}}}{\partial a_z} = \begin{cases} -0.594 \times 10^{-5}\dfrac{V_1}{a_2 - a_1}, & Q_1 \in [a_1, a_2] \\[2ex] 0.594 \times 10^{-5}\dfrac{V_1}{a_3 - a_2}, & Q_1 \in [a_2, a_3] \end{cases} \tag{33}$$

The sensitivity of the fuzzy system to the parameter $a_z$ is seen to be linearly dependent upon the valve position $V_1$ and the parameters of the fuzzy membership functions $a_1$, $a_2$ and $a_3$. The parameter $a_2$ corresponds to the nominal value of the input $Q_1$ when $V_1 = 0.7$. With this value inserted, the sensitivity functions can now be plotted (Fig. 6).

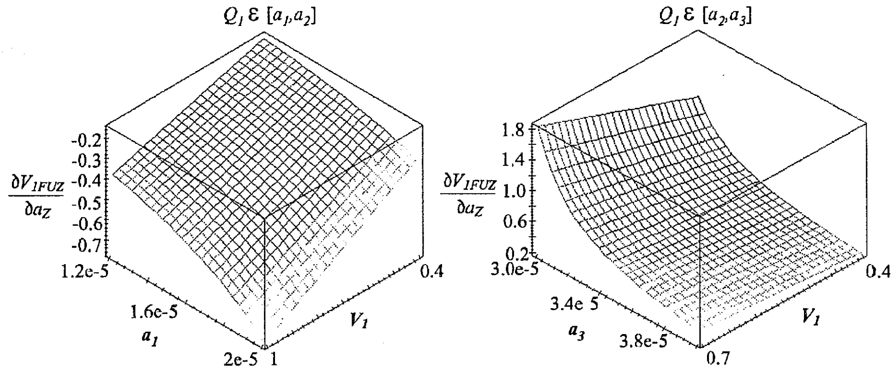


Fig. 6. Sensitivity functions for the fuzzy system.

The plots show clearly that the fuzzy system is increasingly less sensitive to the change in $a_z$ as the parameter $a_1$ decreases and $a_3$ increases. This is no surprise as it corresponds to a decrease in the gradient of the input membership functions, thus to less sensitivity overall to changes in the inputs to the fuzzy system, irrespective of the cause of those changes. The choice of the parameters $a_1$ and $a_3$ is, as usual, a compromise between the sensitivity to faults and insensitivity to disturbances. Of more interest is the fact that the sensitivity is proportionally dependent on the valve position $V_1$.

If the parameters $a_1$ and $a_3$ are fixed (e.g. at values determined from the values of $Q_1$ and $h_1$ for the maximum fault cases), the relationship between the sensitivity to $a_z$ and the valve position $V_1$ can clearly be seen (Fig. 7).

For the non-fuzzy sensitivity, we have

$$\frac{\partial V_{1\mathrm{EST}}}{\partial a_z} = -5 \times 10^5 \frac{Q_1}{\sqrt{19.62 h_1 - 5.886}} \tag{34}$$

The non-fuzzy sensitivity is shown as a function of $Q_1$ and $h_1$ in Fig. 8. The sensitivity is dependent upon $Q_1$ and $h_1$. However, the variable $h_1$ varies so little that the effect of changes in $h_1$ is negligible. The sensitivity of the function to $a_z$ can therefore be considered to be linearly dependent upon $Q_1$. When $Q_1$ varies from $2 \times 10^{-5}$ to $4 \times 10^{-5}$, the sensitivity function varies from $-0.5$ to $-1$.

The results of the analysis of the fuzzy and non-fuzzy FDI systems above show that for this fault (a change in position of $V_1$), the fuzzy system is less sensitive to a change in the parameter $a_z$. Note that it is not the result itself which is of importance

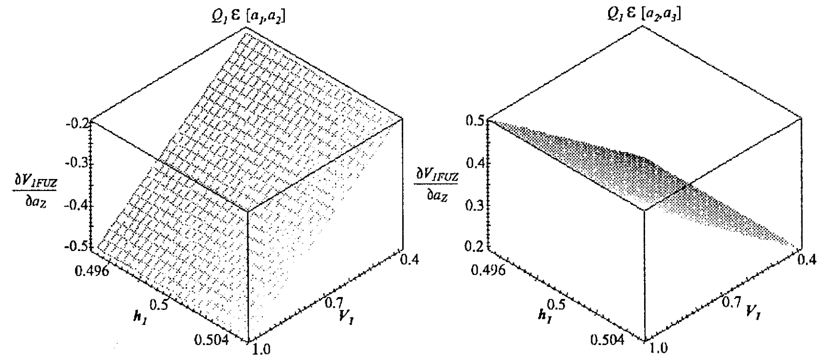


Fig. 7. The sensitivity of the fuzzy system when $a_1$ and $a_3$ are fixed.

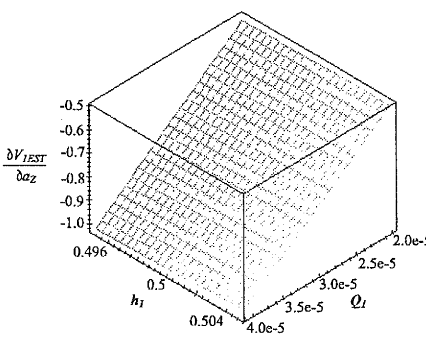


Fig. 8. The sensitivity of the non-fuzzy system.

here, but the fact that using the method described in this paper it was possible to obtain this result in the first place.

## 5. Simulation Results

Figure 9 shows simulation results for the fuzzy and non-fuzzy systems using nominal parameter values, for the fault-free case (the valve position 0.7), and where a fault is present in the valve position (valve positions of 0.4 and 1.0). The plots show the estimated value of the valve position $V_1$, derived by the two methods using simulated input and output data. As can be expected, for the case where the system parameters have their nominal values, the non-fuzzy (mathematical model-based) method performs better than the fuzzy system. This is no surprise — it is often the case that a model-based method will produce better results, *if a suitable and accurate model is available.* In the results of Fig. 9, it can be seen that the fuzzy system necessitates more time to reach the desired value of the valve position, and in the case of a valve position of 0.4, there is a small steady-state error.

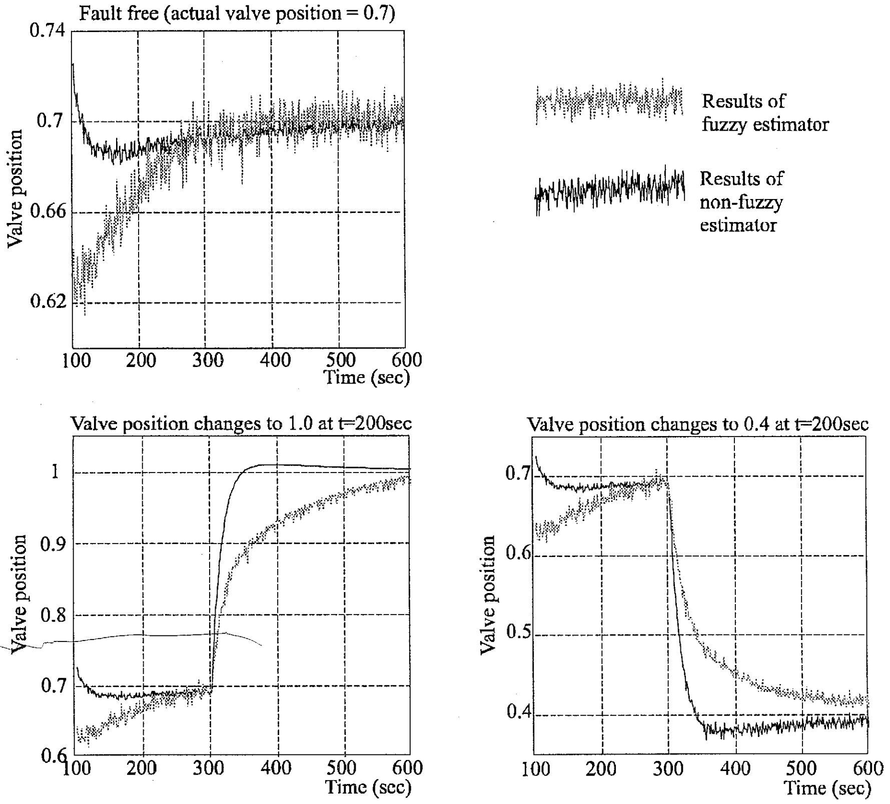


Fig. 9. Estimation of the valve position with nominal parameter values.

Figure 10 shows the same cases (fault-free, and valve positions of 0.4 and 1.0), this time with the parameter of interest ($a_z$) at 70% of its nominal value. Both of the systems, fuzzy and non-fuzzy, are affected by the parameter change. Even in the fault-free case, the estimated parameters reveal large errors. For the fuzzy estimator, the steady-state error is approximately 15% when compared with the non-fuzzy error of approximately 30%. Thus, the fuzzy system is only half as sensitive to the change in the system parameter as in the fault-free case. When the valve is open too far (the actual valve position 1.0), the fuzzy estimate gives an error of approximately 18%, when compared with 30% in the case of the non-fuzzy system. For the last case, where the valve is closed too far (the valve position of 0.4), the fuzzy system yields a steady-state error of 5% when compared with 32% for the non-fuzzy system.

These simulation results confirm the findings of the sensitivity analysis of the previous section. The application of sensitivity theory revealed that the sensitivity

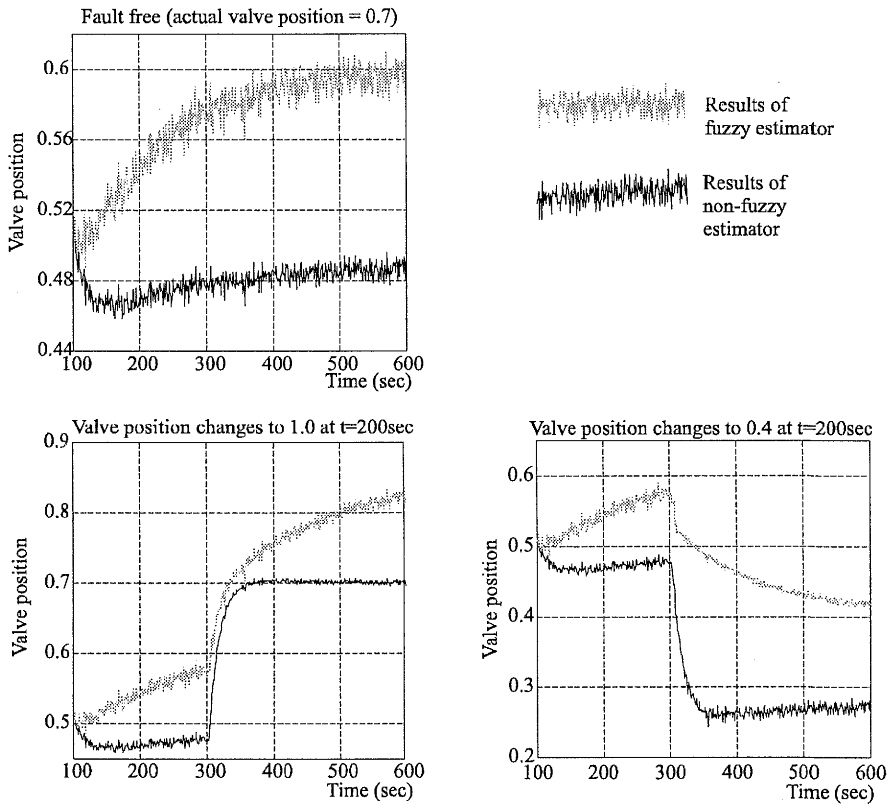


Fig. 10. Estimation of the valve position with $a_z$ at 70% of the nominal value.

of the fuzzy system is dependent on the position of the valve itself. This can clearly be seen in the results of Fig. 10: the smaller the valve position, the smaller the parameter-change induced error between the estimated and the actual valve position.

For the non-fuzzy system, the sensitivity analysis revealed that the sensitivity of the estimate of the valve position was linearly dependent on $Q_1$. The faults considered, however, did not cause enough change in the value of $Q_1$ for this effect to be seen.

## 6. Conclusions

The method of deriving a mathematical description of a fuzzy system (under certain constraints) described in this paper allows for an analysis of the sensitivity of fuzzy systems. An example of a simple fuzzy FDI system for the two-tank system is given, along with the method used to allow for an analysis of the sensitivity of the system to a parameter change. This analysis provides information about the way in which the sensitivity of the fuzzy system changes according to the conditions under which the system operates. A comparison with a non-fuzzy system, also analysed using sensitivity theory, makes the advantages of a fuzzy FDI system evident. Note that the purpose of this analysis was not to show that one system was better than the other, but to demonstrate the power of the analysis tool described in this paper.

The type and structure of the fuzzy models to which the method of deriving the mathematical function described here can be applied is restricted. This may cause some problems in applications where the fuzzy model is of a more complex nature, i.e. in extensive fuzzy models. For simple controllers and for FDI, however, the restrictions do not affect the validity of the approach.

One of the main applications of fuzzy systems in FDI is the decision making stage (i.e. the analysis of residuals or other symptoms). Such fuzzy reasoning systems are, or can be, of exactly of the form which is considered in this paper.

Further work is required to determine the extent to which the application of sensitivity theory to the robustness analysis of fuzzy systems can be of use to fuzzy FDI system designers. This paper represents a basis upon which such investigations can be based.

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