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REGIONAL GRADIENT CONTROLLABILITY OF PARABOLIC SYSTEMS

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The purpose of this paper is to show for parabolic systems how one can achieve a final gradient in a subregion ω of the system domain Ω. First, we give a definition and delineate some properties of this new concept, and then we introduce the concept of regionally gradient strategic actuators. The importance of the spatial structure and location of the actuators in achieving regional gradient controllability is emphasized. Consequently, we concentrate on the determination of a control which would realize a given final gradient on ω with minimum energy. The developed approach is original and leads to numerical algorithms for constructing optimal controls. This approach is also illustrated by an example.

Keywords: parabolic systems, regional gradient controllability, G-strategic actuator

1. Introduction

The analysis of distributed parameter systems is related to a set of concepts such as controllability, observability, stability, etc. that permit to better understand those systems and consequently enable us to manage them in a better way. Many works which deal with the problem of steering a system (S) to a prescribed state defined on a space domain Ω, were related in (Curtain and Zwart, 1995), also see the references therein. Later, the concept of regional controllability was developed in (Zerrik, 1993). It consists in steering the system to a prescribed state defined only in a subregion $\omega \subset \Omega$. It was extended by Zerrik *et al.* (1998) to regional boundary controllability, where ω is a part of the boundary $\partial\Omega$ of Ω. An extension that is very important in practical applications is that of regional gradient controllability, i.e. we are interested in steering the system gradient to a desired function given only on a part ω of Ω.

The principal reason behind introducing this concept is that it provides a means to deal with some problems from the real world, e.g. in thermic isolation problems it happens that the control is only required to cancel the temperature gradient before crossing the brick. The purpose of this paper is to give an account of some original results related to the regional gradient controllability problem.

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The paper is organized as follows. Section 2 is focused on the system under consideration and mathematical formulation of the regional gradient controllability problem. In the next section, we introduce and characterize the concept of regional gradient strategic actuators. Section 4 is devoted to the determination of an optimal control to achieve regional gradient controllability. In Section 5, we give some useful results regarding computation of optimal controls. In the last section, we concentrate on the relation between the considered subregion and actuator location.

2. Regional Gradient Controllability

2.1. Problem Statement

Let Ω be an open, bounded and regular subset of $\mathbb{R}^n$ with boundary $\partial\Omega$. We consider

- a parabolic system (S) defined in $\Omega\times]0,T[$ where $T>0$ is given,
- a given initial state y_0,
- a subregion ω of Ω which may be connected or not,
- controls which may be applied via various types of actuators (pointwise, zone, internal).

We set $Q=\Omega\times]0,T[\,,\ \Sigma=\partial\Omega\times]0,T[$, and consider

$$\begin{cases} \dfrac{\partial y}{\partial t}(x,t)=Ay(x,t)+Bu(t) & \text{in } Q, \\ y(\xi,t)=0 & \text{on } \Sigma, \\ y(x,0)=y_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where A is a second-order linear differential operator with compact resolvent, which generates a strongly continuous semi-group $(S(t))_{t\geq 0}$ on the Hilbert state space $L^2(\Omega)$. In what follows, A^* signifies the adjoint operator of A, $B\in\mathcal{L}(\mathbb{R}^p,L^2(\Omega))$, $u\in U=L^2(0,T;\ \mathbb{R}^p)$ and $y_0\in L^2(\Omega)$.

Let $y_u(\cdot)$ be the solution to (1) when it is excited by a control u and suppose that (1) has a unique solution such that $y_u(T)\in H^1(\Omega)$. If $\omega\subset\Omega$, we consider

$$\chi_\omega : \left(L^2(\Omega)\right)^n \longrightarrow \left(L^2(\omega)\right)^n$$
$$z \longrightarrow \chi_\omega z = z_{|\omega},$$

where χ_ω^* denotes the adjoint operator, given by

$$\chi_\omega^* y = \begin{cases} y & \text{in } \ \omega, \\ 0 & \text{in } \ \Omega\backslash\omega, \end{cases}$$

and

$$\begin{aligned}\widetilde{\chi}_\omega : L^2(\Omega) &\longrightarrow L^2(\omega)\\ z &\longrightarrow \widetilde{\chi}_\omega z = z_{|\omega}.\end{aligned}$$

Let ∇ be the operator given by

$$\begin{aligned}\nabla : H^1(\Omega) &\longrightarrow \left(L^2(\Omega)\right)^n\\ z &\longrightarrow \nabla z = \left(\frac{\partial z}{\partial x_1}, \dots, \frac{\partial z}{\partial x_n}\right)\end{aligned}$$

with adjoint ∇^*.

Let us recall that an actuator is conventionally defined by a couple (D, f), where:

i) $D \subset \Omega$ is the support of the actuator,

ii) f is the spatial distribution of the action on the support D. In the case of a pointwise actuator (internal or boundary), D is reduced to the location $\{b\}$ of the actuator and $f = \delta(\cdot - b)$ where δ is the Dirac mass concentated at zero.

For the definitions and properties of strategic and regional strategic actuators, we refer the reader to (El Jai and Pritchard, 1988; El Jai *et al.*, 1995).

2.2. Definition and Characterization

Definition 1. System (1) is said to be exactly (resp. weakly) regionally gradient controllable on ω if for all $g_d \in (L^2(\omega))^n$ and for all $\varepsilon > 0$, there exists a control $u \in U$ such that

$$\chi_\omega \nabla y_u(T) = g_d \quad \left(\text{resp. } \|\chi_\omega \nabla y_u(T) - g_d\|_{(L^2(\omega))^n} \le \varepsilon\right). \tag{2}$$

In what follows, we shall say that such a system is regionally G-controllable on ω (G stands for the gradient).

Without loss of generality, we may consider the case where $y_0 = 0$. Let H be the operator

$$H : U \longrightarrow H^1(\Omega)$$

defined by

$$\forall\, u \in U, \quad Hu = \int_0^T S(T - \tau) B u(\tau)\, \mathrm{d}\tau. \tag{3}$$

Then system (1) is exactly (resp. weakly) regionally G-controllable on ω iff $\mathrm{Im}\, \chi_\omega \nabla H = (L^2(\omega))^n$ (resp. $\overline{\mathrm{Im}\, \chi_\omega \nabla H} = (L^2(\omega))^n$).

It is clear that

1. The above definitions mean that we are only interested in the transfer of the system gradient to a desired function on the subregion $\omega \subset \Omega$, so the control u depends implicitly on ω.

2. The above definitions do not allow for pointwise or boundary controls since, for such systems $B \notin \mathcal{L}(\mathbb{R}^p, L^2(\Omega))$ and the solution $y_u(\cdot) \notin L^2(\Omega)$. However, the extension can be carried out in a similar manner if one takes regular controls such that $y_u(T) \in H^1(\Omega)$ (El Jai and Pritchard, 1988).

3. A system which is exactly (resp. weakly) regionally controllable (Zerrik, 1993) is exactly (resp. weakly) regionally G-controllable.

4. A system which is exactly (resp. weakly) regionally G-controllable on ω is exactly (resp. weakly) regionally G-controllable in ω_1 for any $\omega_1 \subset \omega$.

5. Let

$$J(u) = \int_0^T ||u(t)||^2_{\mathbb{R}^p} \, dt$$

be the transfer cost. Then for any $\omega \subset \Omega$, the regional gradient transfer cost in ω is smaller than the regional transfer cost in ω. Indeed, let $y_d \in H^1(\Omega)$ and consider

$$\mathcal{W}_S = \{u \in U \text{ such that } y_u(T) = y_d \text{ in } \omega\}$$

and

$$\mathcal{W}_G = \{u \in U \text{ such that } \chi_\omega \nabla y_u(T) = \nabla y_d \text{ in } \omega\}.$$

Then $\mathcal{W}_S \subset \mathcal{W}_G$ and hence

$$\min_{\mathcal{W}_G} J(u) \leq \min_{\mathcal{W}_S} J(u).$$

6. One can find various systems that are G-controllable but which are not controllable. This is illustrated in the following counter-example.

2.3. Counter-Example

Let $\Omega =]0, 1[$ and consider the system described by the parabolic equation

$$\begin{cases} \dfrac{\partial y}{\partial t}(x,t) = \dfrac{\partial^2 y}{\partial x^2}(x,t) + \chi_{[0,b]} u(t) & \text{in } \Omega \times]0, T[, \\ y(0,t) = y(1,t) = 0 & \text{in }]0, T[, \\ y(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{4}$$

which is excited by one actuator located in a subdomain $[0,b] \subset]0,1[$ and $f = 1$ constitutes the spatial distribution of the control in $[0,b]$. This system is equivalent to (1) with $A = \partial^2(\cdot)/\partial x^2$, $y_0 = 0$ and $Bu = \chi_{[0,b]}u$. A generates a semi-group $(S(t))_{t\geq 0}$ on $L^2(\Omega)$ given by

$$S(t)y = \sum_{i=1}^{\infty} e^{\lambda_i t}\langle y, \varphi_i\rangle \varphi_i,$$

where

$$\lambda_i = -i^2\pi^2 \quad \text{and} \quad \varphi_i(x) = \sqrt{2}\sin(i\pi x).$$

The operator H is such that

$$(H^*y)(t) = B^*S^*(T-t)y = \sum_{i=1}^{\infty} e^{\lambda_i(T-t)}\langle y, \varphi_i\rangle\langle \chi_{[0,b]}, \varphi_i\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\Omega)$.

For any $b \in \mathbb{Q}$, system (4) is not weakly controllable on Ω ($\operatorname{Ker} H^* \neq \{0\}$) (El Jai and Pritchard, 1988). But we have the following result:

Proposition 1. *System (4) is not controllable on Ω but is G-controllable in Ω.*

Proof. Let us now show that there exists a state that is not reachable on Ω but its gradient is reachable on Ω . Suppose that $(\varphi_j)_{j\in J}$ are the eigenfunctions of A which are in $\operatorname{Ker} H^*$. Then $\operatorname{Ker} H^* = \operatorname{span}\{(\varphi_j)_{j\in J}\}$ and we have

$$J = \{j \mid jb \in 2\mathbb{N}\} \neq \emptyset,$$

since $b \in \mathbb{Q}$. If $j_0 \in J$, then $\varphi_{j_0} \in \operatorname{Ker} H^*$ and hence φ_{j_0} is not reachable on Ω. Let us show, however, that φ_{j_0} is gradient reachable on Ω, i.e. $\varphi_{j_0} \notin \operatorname{Ker} H^*\nabla^*$. We have

$$\begin{aligned} H^*\nabla^*(\varphi_{j_0}) &= \sum_{k=1}^{\infty} e^{\lambda_k(T-t)}\langle \nabla^*\varphi_{j_0}, \varphi_k\rangle\langle \chi_{[0,b]}, \varphi_k\rangle \\ &= \sum_{k\notin J} e^{\lambda_k(T-t)}\langle \nabla^*\varphi_{j_0}, \varphi_k\rangle\langle \chi_{[0,b]}, \varphi_k\rangle \neq 0 \end{aligned}$$

in general. Otherwise $\langle \nabla^*\varphi_{j_0}, \varphi_k\rangle = 0 \ \forall\, k \notin J$.

A calculation shows that for any $k_0 \notin J$ the condition $\langle \nabla^*\varphi_{j_0}, \varphi_{k_0}\rangle = 0$ is equivalent to

$$(j_0 - k_0)\big(1 - (-1)^{j_0+k_0}\big) = (j_0 + k_0)\big(1 - (-1)^{j_0-k_0}\big)$$

This is not true in general (consider e.g. $b = 1/2$, $j_0 = 4$, $k_0 = 3$). Hence, φ_{j_0} is gradient reachable on Ω. ■

Proposition 2. *We have the following characterizations:*

1) *System (1) is exactly regionally G-controllable on ω iff*

$$\operatorname{Ker}\chi_\omega + \operatorname{Im}\chi_\omega^*\chi_\omega\nabla H = (L^2(\Omega))^n.$$

2) *System (1) is weakly regionally G-controllable on ω iff*

$$\operatorname{Ker}\chi_\omega + \overline{\operatorname{Im}\chi_\omega^*\chi_\omega\nabla H} = (L^2(\Omega))^n.$$

Proof. (Part 1) If $y \in (L^2(\Omega))^n$, we have $y = y_1 + y_2$ with $y_1 = 0$ in ω and $y_2 = 0$ in $\Omega\backslash\omega$. As $y_2 \in (L^2(\omega))^n$, there exists $u \in U$ such that $y_2 = \chi_\omega\nabla Hu$, so $\operatorname{Ker}\chi_\omega + \operatorname{Im}\chi_\omega^*\chi_\omega\nabla H = (L^2(\Omega))^n$.

Conversely, let $y \in (L^2(\omega))^n$. We have $\bar{y} = \chi_\omega^* y \in (L^2(\Omega))^n$, and hence there exist $y_1 \in \operatorname{Ker}\chi_\omega$ and $y_2 \in \operatorname{Im}\chi_\omega^*\chi_\omega\nabla H$ such that $\bar{y} = y_1 + y_2$. Consequently, there exists $u \in U$ such that $\chi_\omega\bar{y} = \chi_\omega\nabla Hu$, so $y = \chi_\omega\nabla Hu$. Hence $\operatorname{Im}\chi_\omega\nabla H = (L^2(\Omega))^n$.

(Part 2) If system (1) is weakly regionally G-controllable on ω, then for $y \in (L^2(\Omega))^n$ we can write $y = y_1 + y_2$ with $y_1 = 0$ in ω and $y_2 = 0$ in $\Omega\backslash\omega$. As $y_2 \in (L^2(\omega))^n$, $\forall\ \varepsilon > 0$ there exists $u \in U$ such that $\|y_2 - \chi_\omega\nabla Hu\|_{(L^2(\omega))^n} \leq \varepsilon$ and we have $\|\chi_\omega^* y_2 - \chi_\omega^*\chi_\omega\nabla Hu\|_{(L^2(\omega))^n} \leq \varepsilon$. Therefore $\chi_\omega^* y_2 \in \overline{\operatorname{Im}\chi_\omega^*\chi_\omega\nabla H}$ and consequently $\operatorname{Ker}\chi_\omega + \overline{\operatorname{Im}\chi_\omega^*\chi_\omega\nabla H} = (L^2(\Omega))^n$.

Conversely, let $y \in (L^2(\omega))^n$. We have $\bar{y} = \chi_\omega^* y \in (L^2(\Omega))^n$, so there exist $y_1 \in \operatorname{Ker}\chi_\omega$ and $y_2 \in \overline{\operatorname{Im}\chi_\omega^*\chi_\omega\nabla H}$ such that $\bar{y} = y_1 + y_2$. Then $\forall\,\varepsilon > 0$ there exists $u \in U$ such that $\|y_2 - \chi_\omega^*\chi_\omega\nabla Hu\|_{(L^2(\Omega))^n} \leq \varepsilon$. Hence $\|\chi_\omega y_2 - \chi_\omega\nabla Hu\|_{(L^2(\omega))^n} \leq \varepsilon$ and then $\|y - \chi_\omega\nabla Hu\|_{(L^2(\omega))^n} \leq \varepsilon$. Finally, $\overline{\operatorname{Im}\chi_\omega\nabla H} = (L^2(\omega))^n$. ■

3. Regional Gradient Controllability and Actuators

In this section, we develop results that link regional gradient controllability to actuator structures. Consider system (1) excited by p zone actuators $(D_i, f_i)_{1\leq i\leq p}$, where $D_i \subset \Omega$ and $f_i \in L^2(D_i)$ for $i = 1, p$, i.e.

$$\begin{cases} \dfrac{\partial y}{\partial t}(x,t) = Ay(x,t) + \displaystyle\sum_{i=1}^{p}\chi_{D_i} f_i u_i(t) & \text{in } Q, \\ y(\xi,t) = 0 & \text{on } \Sigma, \\ y(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{5}$$

Definition 2. A sequence of actuators is said to be G-strategic on $\omega \subset \Omega$ if the excited system is weakly regionally G-controllable on ω.

Consider a set $(\varphi_{m_j})_{\substack{m=1,\dots,\infty \\ j=1,\dots,r_m}}$ of eigenfunctions of A^* in $H^1(\Omega)$ orthonormal in $L^2(\omega)$, associated with eigenvalues λ_m of multiplicities r_m. If A has constant coefficients, we have the following result.

Proposition 3. *Assume that* $\sup r_m = r < \infty$. *If the sequence of actuators* $(D_i, f_i)_{1\le i\le p}$ *is G-strategic on* ω, *then:*

1. $p \ge r$, *and*

2. $\operatorname{rank} M_m = r_m$,

where

$$(M_m)_{ij} = \begin{cases} \displaystyle\sum_{k=1}^{n} \left\langle \frac{\partial \varphi_{m_j}}{\partial x_k}, f_i \right\rangle_{L^2(D_i)} & \textit{(zonal case)},\\ \displaystyle\sum_{k=1}^{n} \frac{\partial \varphi_{m_j}}{\partial x_k}(b_i) & \textit{(pointwise case)}, \end{cases}$$

for $1 \le i \le p$, $1 \le j \le r_m$.

Proof. The proof is developed for the zonal case. The actuators $(D_i, f_i)_{1\le i\le p}$ are G-strategic on ω iff $\forall\, g \in (L^2(\omega))^n$ $\forall\, u \in U$, $\langle \chi_\omega \nabla H u, g\rangle_{(L^2(\omega))^n} = 0 \Longrightarrow g = 0$.

If there exists $m \in \mathbb{N}$ such that rank $M_m \neq r_m$, then there exists a vector

$$h_m = \begin{pmatrix} h_{m_1} \\ \vdots \\ h_{m_{r_m}} \end{pmatrix} \neq 0$$

such that $M_m h_m = 0$. Let $h_0 = \sum_{j=1}^{r_m} h_{m_j}\varphi_{m_j}$, $H_0 = (h_0, \ldots, h_0)$ and $\varphi_{0l} = \widetilde{\chi}_\omega^* h_0$. Assume that the system

$$\begin{cases} \dfrac{\partial \varphi}{\partial t}(x,t) = -A^*\varphi(x,t) & \text{in } Q,\\ \varphi(\xi, t) = 0 & \text{on } \Sigma,\\ \varphi(x,T) = \varphi_{0l} & \text{in } \Omega, \end{cases} \tag{6}$$

has a unique solution $\varphi \in L^2(0,T; H_0^2(\Omega)) \cap C^3(\Omega\times]0,T[)$.

Multiplying (5) by $\partial\varphi/\partial x_k$ and integrating the result over the cylinder Q, we have

$$\int_Q \frac{\partial \varphi}{\partial x_k}(t) y'(t)\,\mathrm{d}t\,\mathrm{d}x = \int_Q \frac{\partial \varphi}{\partial x_k}(t) A y(t)\,\mathrm{d}t\,\mathrm{d}x + \sum_{i=1}^{p} \int_Q \frac{\partial \varphi}{\partial x_k}(t) \chi_{D_i} f_i u_i(t)\,\mathrm{d}t\,\mathrm{d}x$$

and therefore

$$\int_\Omega \left[\frac{\partial \varphi}{\partial x_k}(t) y(t)\right]_0^T \mathrm{d}x - \int_Q \left(\frac{\partial \varphi}{\partial x_k}\right)'(t) y(t)\,\mathrm{d}t\,\mathrm{d}x = \int_Q \frac{\partial \varphi}{\partial x_k}(t) A y(t)\,\mathrm{d}t\,\mathrm{d}x$$
$$+ \sum_{i=1}^{p} \int_0^T \left\langle f_i, \frac{\partial \varphi}{\partial x_k} \right\rangle_{L^2(D_i)} u_i(t)\,\mathrm{d}t.$$

Thus

$$\int_\Omega \frac{\partial\varphi}{\partial x_k}(T)y(T) - \frac{\partial\varphi}{\partial x_k}(0)y(0)\,\mathrm{d}x = \int_Q \left[\frac{\partial\varphi}{\partial x_k}(t)Ay(t) - y(t)A^*\frac{\partial\varphi}{\partial x_k}(t)\right]\,\mathrm{d}t\,\mathrm{d}x$$
$$+\sum_{i=1}^{p}\int_0^T \left\langle f_i, \frac{\partial\varphi}{\partial x_k}\right\rangle_{L^2(D_i)} u_i(t)\,\mathrm{d}t$$

Integrating by parts and making use of the Green formula, we obtain

$$-\int_\Omega \varphi_{0l}\frac{\partial y}{\partial x_k}(T)\,\mathrm{d}x = \int_\Sigma \left(\frac{\partial\varphi}{\partial x_k}(t)\frac{\partial y}{\partial \nu_A}(t) - y(t)\frac{\partial}{\partial\nu_{A^*}}\left(\frac{\partial\varphi}{\partial x_k}\right)(t)\right)\,\mathrm{d}t\,\mathrm{d}\xi$$
$$+\sum_{i=1}^{p}\int_0^T \left\langle f_i, \frac{\partial\varphi}{\partial x_k}\right\rangle_{L^2(D_i)} u_i(t)\,\mathrm{d}t$$

The boundary conditions give

$$-\int_\Omega \widetilde{\chi}_\omega\frac{\partial y}{\partial x_k}(T)h_0\,\mathrm{d}x = \sum_{i=1}^{p}\int_0^T \left\langle f_i, \frac{\partial\varphi}{\partial x_k}\right\rangle_{L^2(D_i)} u_i(t)\,\mathrm{d}t$$

with

$$\left\langle f_i, \frac{\partial\varphi}{\partial x_k}\right\rangle_{L^2(D_i)} = \sum_{m=1}^{\infty} e^{\lambda_m(T-t)}\sum_{j=1}^{r_m}\langle h_0,\varphi_{m_j}\rangle_{L^2(\omega)}\left\langle f_i, \frac{\partial\varphi_{m_j}}{\partial x_k}\right\rangle_{L^2(D_i)}.$$

Now, $M_m h_m = 0$ is equivalent to

$$\sum_{j=1}^{r_m}\langle h_0,\varphi_{m_j}\rangle_{L^2(\omega)}\langle F_i,\nabla\varphi_{m_j}\rangle_{(L^2(D_i))^n} = 0$$

for $1 \le i \le p$ with $F_i = (f_i,\ldots,f_i)$, which gives $\langle\chi_\omega\nabla Hu, H_0\rangle_{(L^2(\omega))^n} = 0$, since otherwise there exists $H_0 \neq 0$ such that $\langle\chi_\omega\nabla Hu, H_0\rangle_{(L^2(\omega))^n} = 0$ for all $u \in U$ and so the system is not regionally G-controllable on ω. ∎

Example 1. If we consider the case of a one-dimensional system defined on $\Omega =]0,1[$ by the parabolic equation

$$\begin{cases} \dfrac{\partial y}{\partial t}(x,t) = \dfrac{\partial^2 y}{\partial x^2}(x,t) + \delta(x-b)u(t) & \text{in }]0,1[\times]0,T[,\\ y(0,t) = y(1,t) = 0 & \text{on }]0,T[,\\ y(x,0) = 0 & \text{in }]0,1[. \end{cases} \tag{7}$$

System (7) is not controllable on $]0,1[$ if

$$b \in S = \left\{\frac{k}{n} \mid 1 \le k \le n-1,\ \ \forall\, n \in \mathbb{N}^*\right\}$$

Since it is not G-controllable on $]0,1[$ if

$$b \in S_G = \left\{ \frac{2k+1}{2n} \mid 0 \leq k < \frac{2n-1}{2}, \ \forall\, n \in \mathbb{N}^* \right\},$$

we have $S_G \subset S$, so there exist actuators which are G-strategic, but not strategic. ♦

4. Regional Gradient Target Control

The purpose of this section is to explore an approach devoted to the computation of an optimal control for system (1) to a given gradient in the subregion ω. Consider (1) with A containing only constant coefficients and suppose that $g_d \in (L^2(\omega))^n$ is given. Set

$$G = \left\{ g \in L^2(\Omega) \text{ such that } g = 0 \text{ on } \omega \right\}.$$

The problem is as follows: Does there exist a minimum-norm control $u \in U$ such that $(\nabla y_u(T) - \chi_\omega^* g_d) \in G^n$?

Let

$$\bar{G} = \left\{ g \in L^2(\Omega) \text{ such that } g = 0 \text{ on } \Omega \backslash \omega \right\}.$$

The method is similar to the one for the internal regional controllability developed in (El Jai *et al.*, 1995) (it is an extension of the HUM method, see (Lions, 1988)) and will be developed for various types of controls.

4.1. Case of Zone Actuator

If we consider system (1) and assume that in the case of a control applied by means of a zone actuator (D, f), where $D \subset \Omega$ is the actuator support and $f \in L^2(D)$ defines the spatial distribution of the control on D, then we have $Bu(t) = \chi_D f(x) u(t)$ and the system may be written down in the form

$$\begin{cases} \dfrac{\partial y}{\partial t}(x,t) = Ay(x,t) + \chi_D f(x) u(t) & \text{in } Q, \\ y(\xi, t) = 0 & \text{on } \Sigma, \\ y(x,0) = y_0(x) & \text{in } \Omega. \end{cases} \tag{8}$$

For $\varphi_0 \in \bar{G}$, consider the system

$$\begin{cases} \dfrac{\partial \varphi}{\partial t}(x,t) = -A^* \varphi(x,t) & \text{in } Q, \\ \varphi(\xi, t) = 0 & \text{on } \Sigma, \\ \varphi(x,T) = \varphi_0(x) & \text{in } \Omega. \end{cases} \tag{9}$$

We assume that it has a unique solution $\varphi \in L^2(0,T; H_0^2(\Omega)) \cap C^3(\Omega \times]0,T[)$.

For a given $\varphi_0 \in \bar{G}$, we consider system (9) and define the mapping

$$\varphi_0 \in \bar{G} \quad \to \|\varphi_0\|^2_{\bar{G}} = \int_0^T \left(\sum_{i=1}^{n} \left\langle f, \frac{\partial \varphi}{\partial x_i} \right\rangle_{L^2(D)} \right)^2 \mathrm{d}t \tag{10}$$

which is a semi-norm on $\bar{G}$.

Lemma 1. *If the actuator (D, f) is G-strategic on ω, then (10) is a norm on $\bar{G}$.*

Proof. Consider a basis (φ_i) of the eigenfunctions of A^*. Without loss of generality we suppose that the associated eigenvalues λ_i are of multiplicity one. Since

$$\|\varphi_0\| = 0 \iff \sum_{i=1}^{n} \left\langle f, \frac{\partial \varphi}{\partial x_i} \right\rangle_{L^2(D)} = 0 \text{ a.e. on } [0, T],$$

the equation

$$\sum_{j=1}^{\infty} e^{\lambda_j (T-t)} \langle \varphi_0, \varphi_j \rangle_{L^2(\omega)} \sum_{i=1}^{n} \left\langle f, \frac{\partial \varphi_j}{\partial x_i} \right\rangle_{L^2(D)} = 0 \text{ a.e. on } [0, T]$$

implies

$$\langle \varphi_0, \varphi_j \rangle_{L^2(\omega)} \sum_{i=1}^{n} \left\langle f, \frac{\partial \varphi_j}{\partial x_i} \right\rangle_{L^2(D)} = 0 \ \forall j.$$

If (D, f) is G-strategic on ω, then

$$\sum_{i=1}^{n} \left\langle f, \frac{\partial \varphi_j}{\partial x_i} \right\rangle_{L^2(D)} \neq 0 \ \forall j,$$

and therefore $\varphi_0 = 0$ on Ω. ■

We denote the completion of the set $\bar{G}$ with respect to the norm (10) again by $\bar{G}$ and consider the system

$$\begin{cases} \dfrac{\partial \Psi}{\partial t}(x,t) = A\Psi(x,t) + \displaystyle\sum_{i=1}^{n} \left\langle f, \frac{\partial \varphi}{\partial x_i} \right\rangle_{L^2(D)} f(x)\chi_D & \text{in } Q, \\ \Psi(\xi, t) = 0 & \text{on } \Sigma, \\ \Psi(x, 0) = y_0(x) & \text{in } \Omega, \end{cases} \tag{11}$$

which may be decomposed in the following two systems:

$$\begin{cases} \dfrac{\partial \Psi_1}{\partial t}(x,t) = A\Psi_1(x,t) & \text{in } Q, \\ \Psi_1(\xi, t) = 0 & \text{on } \Sigma, \\ \Psi_1(x, 0) = y_0(x) & \text{in } \Omega \end{cases} \tag{12}$$

and

$$\begin{cases} \dfrac{\partial \Psi_2}{\partial t}(x,t) = A\Psi_2(x,t) + \displaystyle\sum_{i=1}^{n} \left\langle f, \frac{\partial \varphi}{\partial x_i} \right\rangle_{L^2(D)} f(x)\chi_D & \text{in } Q, \\ \Psi_2(\xi,t) = 0 & \text{on } \Sigma, \\ \Psi_2(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{13}$$

Let $\wedge$ be the operator defined by

$$\begin{aligned} \wedge : \bar{G} &\longrightarrow \bar{G}^* \\ \varphi_0 &\longrightarrow -\widetilde{P}\left(\sum_{i=1}^{n} \frac{\partial \Psi_2}{\partial x_i}(T)\right), \end{aligned} \tag{14}$$

where $\widetilde{P} = \widetilde{\chi}_\omega^* \widetilde{\chi}_\omega$. With this notation the regional gradient control problem on ω leads to solving the equation

$$\wedge \varphi_0 = -\sum_{i=1}^{n} \left(\widetilde{\chi}_\omega^* (g_d)_i - \widetilde{P} \frac{\partial \Psi_1}{\partial x_i}(T) \right). \tag{15}$$

Proposition 4. *If the actuator (D, f) is G-strategic on ω, then (15) has a unique solution $\varphi_0 \in \bar{G}$ and*

$$u^*(t) = \langle F, \nabla\varphi \rangle_{(L^2(D))^n} \tag{16}$$

controls the gradient of system (8) to g_d at time T in ω, where $F = (f, f, \dots, f)$. Moreover, this control minimizes the cost function

$$J(u) = \frac{1}{2} \int_0^T u^2(t)\, \mathrm{d}t.$$

Proof. First, we prove that (15) has a unique solution. For that purpose, multiplying (13) by $\partial\varphi/\partial x_i$ and integrating the result over Q, we get

$$\int_Q \frac{\partial \varphi}{\partial x_i}(t) \Psi_2'(t)\, \mathrm{d}t\, \mathrm{d}x = \int_Q \frac{\partial \varphi}{\partial x_i}(t) A\Psi_2(t)\, \mathrm{d}t\, \mathrm{d}x + \int_Q \frac{\partial \varphi}{\partial x_i}(t) \chi_D f \langle F, \nabla\varphi \rangle_{(L^2(D))^n}\, \mathrm{d}t\, \mathrm{d}x,$$

which gives

$$\int_\Omega \left[\frac{\partial \varphi}{\partial x_i}(t) \Psi_2(t) \right]_0^T \mathrm{d}x - \int_Q \left(\frac{\partial \varphi}{\partial x_i} \right)'(t) \Psi_2(t)\, \mathrm{d}t\, \mathrm{d}x = \int_Q \frac{\partial \varphi}{\partial x_i}(t) A\Psi_2(t)\, \mathrm{d}t\, \mathrm{d}x + \int_0^T \left\langle f, \frac{\partial \varphi}{\partial x_i} \right\rangle_{L^2(D)} \langle F, \nabla\varphi \rangle_{(L^2(D))^n}\, \mathrm{d}t.$$

Thus

$$\int_\Omega (\frac{\partial\varphi}{\partial x_i}(T)\Psi_2(T) - \frac{\partial\varphi}{\partial x_i}(0)\Psi_2(0))\,\mathrm{d}x = \int_Q \left[\frac{\partial\varphi}{\partial x_i}(t)A\Psi_2(t) - \Psi_2(t)A^*\frac{\partial\varphi}{\partial x_i}(t)\right]\mathrm{d}t\,\mathrm{d}x$$
$$+\int_0^T \left\langle f, \frac{\partial\varphi}{\partial x_i}\right\rangle_{L^2(D)} \langle F, \nabla\varphi\rangle_{(L^2(D))^n}\,\mathrm{d}t.$$

Integrating by parts and making use of the Green formula, we obtain

$$-\int_\Omega \varphi_0 \frac{\partial\Psi_2}{\partial x_i}(T)\,\mathrm{d}x = \int_\Sigma \left(\frac{\partial\varphi}{\partial x_i}(t)\,\frac{\partial\Psi_2}{\partial\nu_A}(t) - \Psi_2(t)\frac{\partial}{\partial\nu_{A^*}}(\frac{\partial\varphi}{\partial x_i})(t)\right)\mathrm{d}t\,\mathrm{d}\xi$$
$$+\int_0^T \left\langle f, \frac{\partial\varphi}{\partial x_i}\right\rangle_{L^2(D)} \langle F, \nabla\varphi\rangle_{(L^2(D))^n}\,\mathrm{d}t.$$

Taking account of the boundary conditions gives

$$-\int_\Omega \varphi_0 \frac{\partial\Psi_2}{\partial x_i}(T)\,\mathrm{d}x = \int_0^T \left\langle f, \frac{\partial\varphi}{\partial x_i}\right\rangle_{L^2(D)} \langle F, \nabla\varphi\rangle_{(L^2(D))^n}\,\mathrm{d}t.$$

Hence

$$\langle\Lambda\varphi_0, \varphi_0\rangle = \int_0^T \left(\sum_{i=1}^n \left\langle f, \frac{\partial\varphi}{\partial x_i}\right\rangle_{L^2(D)}\right)^2 \mathrm{d}t = \|\varphi_0\|^2_{\tilde{G}}\,.$$

Consequently Λ is one-to-one and (15) has a unique solution.

Now, let $U_{\text{ad}} = \{u \in U \mid \chi_\omega \nabla y_u(T) = g_d\}$. For $v \in U_{\text{ad}}$ we have

$$\left\langle \frac{\partial\varphi}{\partial x_i}(T), y_v(T) - y_{u^*}(T)\right\rangle_{L^2(\Omega)} - \left\langle \frac{\partial\varphi}{\partial x_i}(0), y_v(0) - y_{u^*}(0)\right\rangle_{L^2(\Omega)}$$
$$= \int_Q \left[(\frac{\partial\varphi}{\partial x_i})'(t)(y_v(t) - y_{u^*}(t)) + \frac{\partial\varphi}{\partial x_i}(t)(y_v'(t) - y_{u^*}'(t))\right]\mathrm{d}t\,\mathrm{d}x$$
$$= \int_Q \left[\frac{\partial\varphi}{\partial x_i}(t)A\,(y_v(t) - y_{u^*}(t)) - A^*\frac{\partial\varphi}{\partial x_i}(t)\,(y_v(t) - y_{u^*}(t))\right]\mathrm{d}t\,\mathrm{d}x$$
$$+\int_Q \chi_D f\frac{\partial\varphi}{\partial x_i}(t)(v - u^*)\,\mathrm{d}t\,\mathrm{d}x.$$

From the Green formula we obtain

$$\left\langle \frac{\partial\varphi}{\partial x_i}(T), y_v(T) - y_{u^*}(T)\right\rangle_{L^2(\Omega)} - \left\langle \frac{\partial\varphi}{\partial x_i}(0), y_v(0) - y_{u^*}(0)\right\rangle_{L^2(\Omega)}$$
$$= \int_\Sigma \left[\frac{\partial\varphi}{\partial x_i}(t)(\frac{\partial y_v}{\partial\nu_A}(t) - \frac{\partial y_{u^*}}{\partial\nu_A}(t)) - \frac{\partial}{\partial\nu_{A^*}}(\frac{\partial\varphi}{\partial x_i})(t)\,(y_v(t) - y_{u^*}(t))\right]\mathrm{d}t\,\mathrm{d}\xi$$
$$+\int_0^T \left\langle f, \frac{\partial\varphi}{\partial x_i}\right\rangle_{L^2(D)} (v - u^*)\,\mathrm{d}t\,.$$

The initial and boundary conditions give

$$-\left\langle \varphi_0, \frac{\partial}{\partial x_i}(y_v(T) - y_{u^*}(T)) \right\rangle_{L^2(\omega)} = \int_0^T \left\langle f, \frac{\partial \varphi}{\partial x_i} \right\rangle_{L^2(D)} (v - u^*)\, dt.$$

The summation gives

$$-\sum_{i=1}^{n} \left\langle \varphi_0, \frac{\partial}{\partial x_i}(y_v(T) - y_{u^*}(T)) \right\rangle_{L^2(\omega)} = \int_0^T \sum_{i=1}^{n} \left\langle f, \frac{\partial \varphi}{\partial x_i} \right\rangle_{L^2(D)} (v - u^*)\, dt.$$

Hence $J'(u^*)(v - u^*) = 0$ and this establishes the optimality of u^*. ■

4.2. Case of a Pointwise Actuator

Let us consider system (1) with one pointwise internal control applied at $b \in \Omega$, where b denotes the location of the actuator and $u \in U$. System (1) may be rewritten in the form

$$\begin{cases} \dfrac{\partial y}{\partial t}(x,t) = Ay(x,t) + \delta(x-b)u(t) & \text{in } Q, \\ y(\xi,t) = 0 & \text{on } \Sigma, \\ y(x,0) = y_0(x) & \text{in } \Omega. \end{cases} \tag{17}$$

We assume that the solution to (17) is such that $y_u(T) \in H^1(\Omega)$.

Now, for a given φ_0, we consider system (9) and define the mapping

$$\varphi_0 \in \bar{G} \to \|\varphi_0\|^2_{\bar{G}} = \int_0^T \left(\sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(b,t) \right)^2 dt \tag{18}$$

which is a semi-norm on $\bar{G}$. Consider the system

$$\begin{cases} \dfrac{\partial \Psi}{\partial t}(x,t) = A\Psi(x,t) + \displaystyle\sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(b,t)\delta(x-b) & \text{in } Q, \\ \Psi(\xi,t) = 0 & \text{on } \Sigma, \\ \Psi(x,0) = y_0(x) & \text{in } \Omega. \end{cases} \tag{19}$$

For $\varphi_0 \in \bar{G}$ system (9) gives φ and system (19) produces $\Psi(T)$.

We can consider a decomposition of (19) into

$$\begin{cases} \dfrac{\partial \Psi_1}{\partial t}(x,t) = A\Psi_1(x,t) & \text{in } Q, \\ \Psi(\xi,t) = 0 & \text{on } \Sigma, \\ \Psi_1(x,0) = y_0(x) & \text{in } \Omega \end{cases} \tag{20}$$

and

$$\begin{cases} \dfrac{\partial \Psi_2}{\partial t}(x,t) = A\Psi_2(x,t) + \displaystyle\sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(b,t)\delta(x-b) & \text{in } Q, \\ \Psi_2(\xi,t) = 0 & \text{on } \Sigma, \\ \Psi_2(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{21}$$

For $\varphi_0 \in \bar{G}$, let $\wedge$ be the operator defined by

$$\wedge \varphi_0 = -\widetilde{P}\left(\sum_{i=1}^{n} \frac{\partial \Psi_2}{\partial x_i}(T)\right), \tag{22}$$

where $\widetilde{P} = \widetilde{\chi}_\omega^* \widetilde{\chi}_\omega$. Then the regional gradient controllability problem on ω is equivalent to solving

$$\wedge \varphi_0 = -\sum_{i=1}^{n}\left(\widetilde{\chi}_\omega^*(g_d)_i - \widetilde{P}\frac{\partial \Psi_1}{\partial x_i}(T)\right). \tag{23}$$

Consequently, we have the following result:

Proposition 5. *If system (17) is weakly regionally G-controllable on ω, then (23) has a unique solution $\varphi_0 \in \bar{G}$ and the control*

$$u^*(t) = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(b,t) \tag{24}$$

gets the system to the desired gradient g_d on ω.

Moreover, this control minimizes the cost function

$$J(u) = \frac{1}{2}\int_0^T u^2(t)\,dt$$

With some minor technical differences, the proof is similar to the zonal case.

Remark 1.

- The developed method controls the system to the desired gradient in the subregion ω and the residual gradient on $\Omega\backslash\omega$ will depend on the control applied.
- The same problem can be considered with more than one actuator. A similar approach leads to a vector control whose each component is associated with one of the actuators.

4.3. Summary

The control to achieve regional gradient controllability depends on both the subregion and nature of the action (actuators). For system (1) under hypotheses of Section 2, the corresponding expressions are summarized in Table 1, where φ is the solution of (9). The respective formulae are derived from the associated subsystems and relations given in the previous sections.

Table 1. Formulae to calculate minimum-norm controls.

Actuator	Control
Pointwise (b, δ_b)	$\sum_{k=1}^{n} \frac{\partial \varphi}{\partial x_k}(b, t)$
Zone (D, f)	$\sum_{k=1}^{n} \left\langle f, \frac{\partial \varphi}{\partial x_k} \right\rangle_{L^2(D)}$

5. Numerical Approach

The numerical approach is realized very easily when one can calculate the eigenfunctions of the system. This case will be discussed in the next subsection. In the general case, an adapted technique is given and detailed later on.

We have seen that the solution to the regional gradient controllability problem is obtained by the solution to the equation

$$\Lambda \varphi_0 = -\sum_{i=1}^{n} \left(\widetilde{\chi}_\omega^* (g_d)_i - \widetilde{P} \frac{\partial \Psi_1}{\partial x_i}(T) \right). \tag{25}$$

In the next section, we give an implementable approach for solving the above equation.

Assume that there exits a basis (φ_i) of eigenfunctions of A^*. Without loss of generality we suppose that the eigenvalues λ_i are of multiplicity one.

5.1. Important Case

Here, the idea is to calculate the components Λ_{ij} of Λ in a suitable basis (φ_i). The problem will be approximated by the solution to the linear system

$$\sum_{j=1}^{M} \Lambda_{ij} \varphi_{0,j} = z_i, \quad i = 1, M, \tag{26}$$

where M is the order of approximation and z_i's are the components of

$$z = -\sum_{i=1}^{n} \left(\widetilde{\chi}_\omega^* (g_d)_i - \widetilde{P} \frac{\partial \Psi_1}{\partial x_i}(T) \right)$$

in the considered basis (φ_i).

As the actuator is pointwise, we have

$$\langle \Lambda\varphi_0, \varphi_0\rangle = \int_0^T \left(\sum_{k=1}^{n} \frac{\partial \varphi}{\partial x_k}(b,t)\right)^2 \mathrm{d}t\,.$$

For

$$\frac{\partial \varphi}{\partial x_k}(b,t) = \sum_{j=1}^{\infty} e^{\lambda_j (T-t)} \langle \varphi_0, \varphi_j\rangle_{L^2(\omega)} \frac{\partial \varphi_j}{\partial x_k}(b)$$

we get

$$\langle \Lambda\varphi_0, \varphi_0\rangle = \sum_{i,j=1}^{\infty} \langle \varphi_0, \varphi_i\rangle_{L^2(\omega)} \langle \varphi_0, \varphi_j\rangle_{L^2(\omega)} \sum_{k,l=1}^{n} \frac{\partial \varphi_i}{\partial x_k}(b) \frac{\partial \varphi_j}{\partial x_l}(b) \frac{(e^{(\lambda_i+\lambda_j)T}-1)}{(\lambda_i+\lambda_j)}.$$

Finally, the components of Λ are given by

$$\Lambda_{ij} = \sum_{k,l=1}^{n} \frac{\partial \varphi_i}{\partial x_k}(b) \frac{\partial \varphi_j}{\partial x_l}(b) \frac{e^{(\lambda_i+\lambda_j)T}-1}{(\lambda_i+\lambda_j)}.$$

In the case of many pointwise actuators $(b_q)_{q=1,p}$, the components of Λ are given by

$$\Lambda_{ij} = \sum_{q=1}^{p} \sum_{k,l=1}^{n} \frac{\partial \varphi_i}{\partial x_k}(b_q) \frac{\partial \varphi_j}{\partial x_l}(b_q) \frac{e^{(\lambda_i+\lambda_j)T}-1}{(\lambda_i+\lambda_j)}.$$

Remark 2. In the zonal case the same developments as in the pointwise case lead to the following components of Λ:

$$\Lambda_{ij} = \frac{e^{(\lambda_i+\lambda_j)T}-1}{\lambda_i+\lambda_j} \sum_{k,l=1}^{n} \left\langle f, \frac{\partial \varphi_i}{\partial x_k}\right\rangle_{L^2(D)} \left\langle f, \frac{\partial \varphi_j}{\partial x_l}\right\rangle_{L^2(D)}.$$

5.2. General Case

In the general case, it is not easy to calculate the eigenfunctions of the operator A^*. Here we give a direct approch which allows us to overcome this difficulty and leads to the desired gradient in ω.

We have seen that the regional gradient controllability is equivalent to solving (25). Consequently, the problem amounts naturally to finding φ_0 which is the solution to the problem

$$\begin{cases} \text{Min } \|\chi_\omega \nabla y_{u^*}(T) - g_d\|_{(L^2(\omega))^n}, \\ \varphi_0 \in \bar{G}. \end{cases} \tag{27}$$

This can easily be achieved by the direct minimization algorithm

1. Initial data ω, g_d, actuator, ϵ.
2. Choose φ_0 in $\bar{G}$.
3. Solve system (9) $\rightarrow \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(b,t)$.
4. Solve system (20) $\rightarrow \Psi_1$.
5. Solve system (21) $\rightarrow \Psi_2$.
6. If $||\chi_\omega \nabla \Psi(T) - g_d||_{(L^2(\omega))^n} > \epsilon$, go to Step 2.
7. The optimal control is given by $u^* = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(b,t)$.

6. Numerical Example

In this section, we consider a numerical example that leads to some conjectures about the difficult problem of the best actuator location for a given subregion. The results are related to the choice of the subregion and the desired gradient to be reached.

Consider the one-dimensional diffusion system described by

$$\begin{cases} \frac{\partial y}{\partial t}(x,t) = 0.01\frac{\partial^2 y}{\partial x^2}(x,t) + \delta(x-b)u(t) & \text{in }]0,1[\times]0,T[, \\ y(0,t) = y(1,t) = 0 & \text{on }]0,T[, \\ y(x,0) = 0 & \text{in }]0,1[. \end{cases} \tag{28}$$

We consider $T = 2$ and the actuator located at $b = 0.59$. The subregion under consideration is $\omega =]0.1, 0.7[$, so that $\Omega\backslash\omega$ is not connected. Let $g_d(x) = (x-0.9)(x-0.6)(x-0.8)/6$ (see Fig. 1) be the desired regional gradient in ω.

The approach of Section 5.1 consists here of the following steps:

Step 1. Solve (26) ($\rightarrow \varphi_0$).

Step 2. Solve (9) ($\rightarrow \varphi(b,t)$).

Step 3. Apply the control $u^*(t) = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(b,t)$ (see (24)).

The final regional gradient is reached with error $||\chi_\omega \nabla y_{u^*}(T) - g_d||^2_{L^2(\omega)} = 0.169\times10^{-4}$ and transfer cost $||u^*||^2 = 0.144$.

Remark 3. Figure 1 shows how the reached final gradient is very close to the desired gradient in the subregion ω.

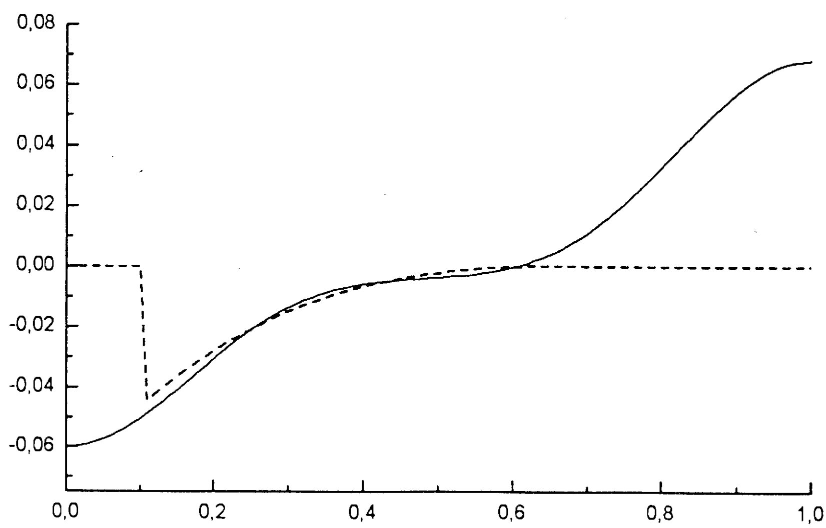

Fig. 1. Desired (dashed line) and final (solid line) gradient in ω.

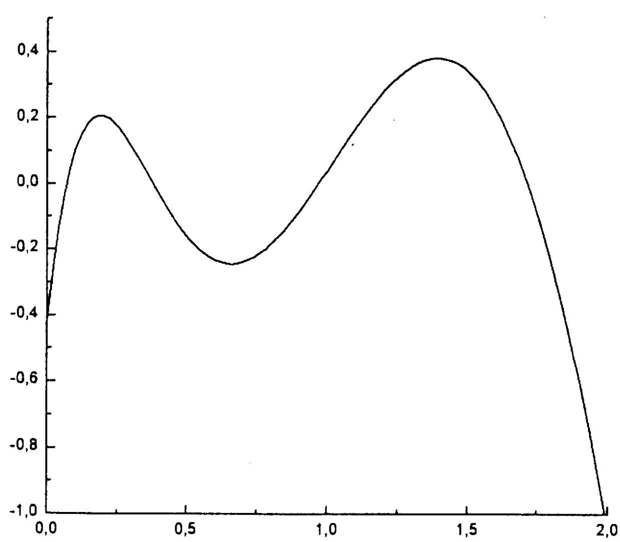

Fig. 2. Control function. The control is calculated via (24).

6.1. Relation between the Subregion and Location of the Pointwise Actuator

The following simulation results show the evolution of the reached gradient error with respect to the actuator location. Figure 3 shows that:

- For a given subregion and a desired gradient, there is an optimal actuator location (optimal in the sense that it leads to a solution which is very close to the desired gradient).
- When the actuator is located sufficiently far from the subregion ω, the estimated gradient error is constant for any location.
- The worst locations correspond to non G-strategic actuators in $\omega =]0, 1[$, as developed in the previous sections, where $b \in S_G = \{(2k+1)/2n \mid 0 \leq k < n - 1/2,\ 0 < n \leq 5\}$ (the order of approximation of the system is five).

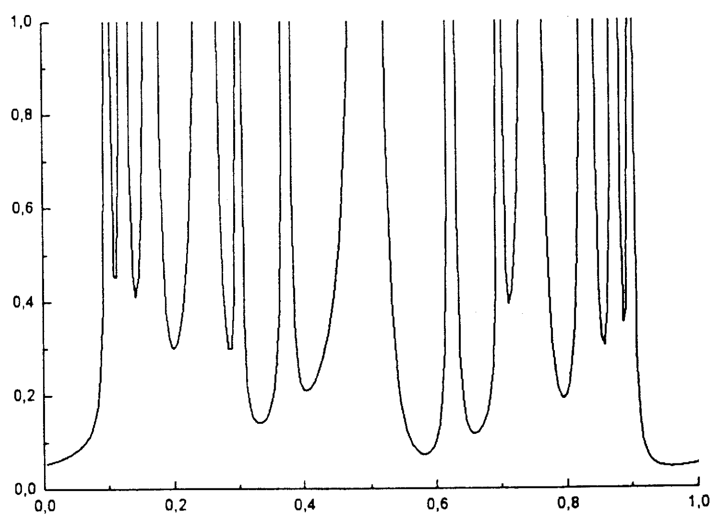

Fig. 3. Evolution of the reached regional gradient error with respect to the actuator location.

Figure 4 shows that, for a given subregion and a desired gradient, there is an optimal actuator location in the sense that it leads to a smaller transfer cost.

6.2. Relation between the Subregion Area and Reached Gradient Error

The reached gradient error depends on the area of the subregion where the gradient has to be given. This error grows with the subregion area. This means that the larger the region, the greater the error is. This is illustrated in Table 2. G-controllability is realized by means of one pointwise actuator located at $b = 0.59$, but the results are similar for any type of actuator.

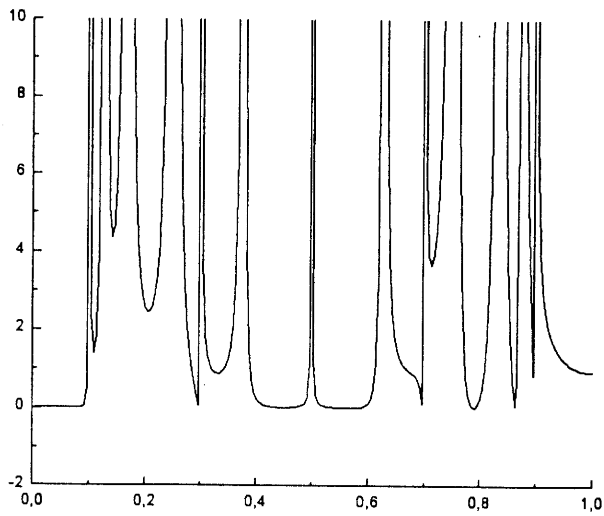

Fig. 4. Evolution of the transfer cost with respect to the actuator locations.

Table 2. Evolution of the gradient controllability error with respect to the subregion area.

Subregion ω	$\|\|g_d - \chi_\omega \nabla y_{u^*}(T)\|\|^2_{L^2(\omega)}$
$]0.4, 0.6[$	0.953×10^{-5}
$]1/4, 0.7[$	0.928×10^{-5}
$]1/2, 3/4[$	0.401×10^{-5}
$]0.1, 0.7[$	0.169×10^{-4}
$]0.12, 0.9[$	0.101×10^{-3}

7. Conclusion

In this paper, we have extented the results of (El Jai *et al.*, 1995; Zerrik, 1993) on regional controllability to a realistic situation encountered in various applications where the gradient control must achieve a certain objective in the subregion of the geometric domain where the system is considered. Moreover, we have explored an approach which allows for implementation of such a control.

Simulation results for real applications are now under consideration. The dual concept of observability which concerns the problem of gradient reconstruction in a given subregion of the domain has also been studied and is based on similar techniques (the results are to be published).

Various open questions are still under consideration. This is the case of the regional boundary gradient target, as well as the structure (location of the support and spatial distribution) and number of the actuators which realize such an objective.

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