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RELATIVE SETS AND ROUGH SETS

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In this paper, by defining a pair of classical sets as a relative set, an extension of the classical set algebra which is a counterpart of Belnap's four-valued logic is achieved. Every relative set partitions all objects into four distinct regions corresponding to four truth-values of Belnap's logic. Like truth-values of Belnap's logic, relative sets have two orderings; one is an order of inclusion and the other is an order of knowledge or information. By defining a rough set as a pair of definable sets, an integrated approach to relative sets and rough sets is obtained. With this definition, we are able to define an approximation of a rough set in an approximation space, and so we can obtain sequential approximations of a set, which is a good model of communication among agents.

Keywords: rough sets, set theory, data analysis, multi-valued logic, interval sets, knowledge representation

1. Introduction

The classical set algebra is based on the conventional two-valued logic, which lacks the capability of modelling incomplete and inconsistent information (Lin and Lin, 1981). To deal with such situations, Belnap added two new values (*None, Both*) to the two classical ones (t, f), making a four-valued logic (Belnap, 1977a; 1977b). Thereby, he actually provided a new order (to represent the level of knowledge), in which a variable, regardless of its measure of truth, receives a state of determination, moving from the state of being undetermined (lack of knowledge) to the state of being over determined (over-knowledge). The former describes the incompleteness of information, while the latter describes its inconsistencies (it might be caused by the contradiction between different informational resources). Ginsberg (1986; 1988) tried to generalize Belnap's structure (FOUR) in the form of a new algebraic structure named bilattice. The elements of this structure are arranged in two related partial orders, each forming a lattice, thus making a bilattice, which in its simplest form can be considered as Belnap's four-valued logic. As discussed in the paper, we can call every truth-value of Belnap's logic a *relative truth-value*, which is determined by evaluating a sentence relative to some evidence or knowledge (Mousavi and Jabedar-Maralani, 1999).

To extend these notions into set algebra, we decided to present every concept through a pair of sets, which is called a *relative set* (Mousavi and Jabedar-Maralani,

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1999). In fact, to construct a set according to the available information, one might not be able to classify every arbitrary element into a well-defined boundary. Hence, we tried to present each concept by means of two positive and negative regions: the collection of all elements with a piece of evidence for their belonging to the concept, and the collection of all elements such that there was some evidence for their not satisfying the concept. The intersection of these two regions models the contradictory information, whereas the elements not included in either of them will correspond to our lack of knowledge. We also considered two orderings among these pairs of sets, corresponding to those of a bilattice. The first is an extension of the classical set inclusion ($\subseteq_I$), and the other constitutes a newly defined order which can be considered as an order of knowledge or information.

The theory of rough sets is an extension of the classical set theory in which the inadequacy of available information plays the central role (Pawlak, 1982; 1991; 1996; 1997; 1998a; 1998b; Yao, 1996a). The underlying philosophy is the assumption that with every object of the universe some information is associated, and objects are described only by the accessible information. Hence, all distinct objects with the same information are indiscernible in view of the available information and appear as being the same. So, the available information in this theory is represented by an equivalence relation and is called the indiscernibility relation. Each equivalence class consists of all indiscernible objects. In such situations, by using the available information, we may not be able to precisely determine any arbitrary set. However, we can associate with every concept a pair of classical sets, which are called the approximations of the set. One consists of all objects that surely belong to the set, and the other consists of all objects which possibly belong to the set.

By combining these two extensions of the classical set algebra (relative sets and rough sets), we obtain an integrated approach to relative sets and rough sets (Mousavi and Jabedar-Maralani, 2000). Therefore, a new approach to the theory of rough sets is achieved, and the definitions and operations of the relative sets can be extended to the theory of rough sets.

2. Belnap's Four-Valued Logic

Belnap (1977a; 1977b) introduced a logic to deal with inconsistent and incomplete information which has four truth-values. Let V be the set $\{t,\ f,\ Both,\ None\}$, whose elements are the four different truth-values of Belnap's logic. Each truth-value of Belnap's logic shows the 'state of information' a sentence can have (Rodrigues *et al.*, 1998). The intuitive meanings of these values are given as follows:

1. p is stated to be true only (t),
2. p is stated to be false only (f),
3. p is stated to be both true and false (*Both*), e.g. by different sources,
4. p's state is unknown, i.e. it is neither true nor false (*None*).

The truth-values of Belnap's logic mentioned above have two natural orderings. One ordering, $\leq_t$, records the degree of truth. According to this order, f is the minimal element, t is the maximal one, and *Both, None* are two intermediate values that are incomparable. $(V, \leq_t)$ is a lattice with an order reversing involution $\neg$, for which $\neg Both = Both$ and $\neg None = None$. The lattice's meet and join are denoted by $\wedge$ and $\vee$, respectively. The other ordering, $\leq_k$, reflects the degree of information or knowledge. So, we have another lattice with the same set V and the ordering of $\leq_k$. We shall denote the meet and join of the $\leq_k$ by $\otimes$ and $\oplus$ that are called the consensus and gullibility, respectively. Note that the negation operator, $\neg$, is an order preserving operator with respect to $\leq_k$. The structure that consists of these four elements and the five basic operators $(\wedge, \vee, \neg, \otimes, \oplus)$ is usually called 'FOUR' (Arieli and Avron, 1998). A double Hasse diagram of FOUR is depicted in Fig. 1.

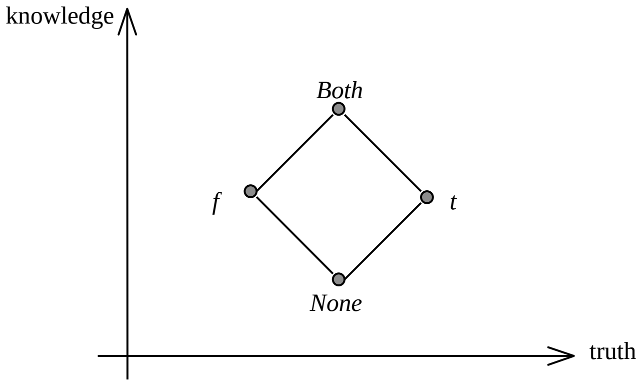

Fig. 1. Double Hasse diagram of FOUR.

3. Bilattice Structure

Ginsberg has introduced into computer science a family of algebraic structures called the bilattices that naturally generalized Belnap's FOUR. The original motivation of Ginsberg for using bilattices was to provide a uniform approach for a diversity of applications in AI. For more details, the interested reader is advised to see Ginsberg's (1988) main paper.

Essentially, a bilattice is a space of generalized truth-values with two orderings:

Definition 1. (Ginsberg, 1988) A bilattice is a structure $\boldsymbol{B} = (B, \leq_t, \leq_k, \neg)$ where B is a nonempty set containing at least two elements, $(B, \leq_t)$ and $(B, \leq_k)$ are complete lattices, and $\neg$ is a unary operation on B with the following properties:

1. if $a \leq_t b$, then $\neg b \leq_t \neg a$,
2. if $a \leq_k b$, then $\neg a \leq_k \neg b$,
3. $\neg\neg a = a$.

It is easy to see that $t, f, Both$ and $None$ are all distinct from each other, and the FOUR structure is the minimal non-degenerated bilattice.

4. Relative Truth-Value

Following (Fitting, 1989; 1990), we will express each value of Belnap's FOUR by a pair (a, b), in which $a, b \in \{0, 1\}$. Here '$a = 1$' means that there is some evidence that a sentence is true, and '$b = 1$' means that there is some evidence that a sentence is false. So, $(0, 0)$ shows that there is no evidence for or against a sentence which corresponds to *None*. In the same way, $(1, 0)$ and $(0, 1)$ correspond to t and f, respectively, and $(1, 1)$ corresponds to *Both*. As a result of this argument, we would like to claim that the truth-value of a sentence is determined by some evidence or knowledge, and every agent uses its own knowledge to determine the value of a sentence. Thus, the value of the sentence is completely dependent on the knowledge of the agent and we call it the relative truth-value of the sentence. For instance, consider a card whose one side is black and the other side is white. We ask four agents to determine the truth-value of the sentence 'card is white'. The first agent has not seen the card. So, he does not have any evidence for or against the sentence, and will say that the truth-value of the sentence is $(0, 0)$ or *None*. The second agent has only seen the white side of the card. So, he will say that the truth-value of the sentence is $(1, 0)$ or t. The third agent has only seen the black side of the card and will choose the $(0, 1)$ or f for the truth-value of the sentence and the fourth agent has seen both sides of the card and will say that the truth-value of the sentence is $(1, 1)$ or *Both*. Therefore, the truth-value of a sentence depends on the agent who evaluates the sentence.

4.1. Algebraic Structure

In this section, we define some operations on the relative truth-values, where $\wedge$, $\vee$ and $\neg$ are useful when only one agent is present:

1. **And:** An agent will have some evidence for '$p \wedge q$' if he has some evidence for both p and q, and will have some evidence against it if he has some evidence against p or against q. So
$$\left(p^+, p^-\right) \wedge \left(q^+, q^-\right) = \left(p^+ \cdot q^+, p^- + q^-\right).$$
2. **Or:** An agent will have some evidence for '$p \vee q$' if he has some evidence for p or for q, and will have some evidence against it if he has some evidence against p and against q. In other words,
$$\left(p^+, p^-\right) \vee \left(q^+, q^-\right) = \left(p^+ + q^+, p^- \cdot q^+\right).$$
3. **Not:** An agent will have some evidence for '$\neg p$' if he has some evidence against p and vice versa. Consequently,
$$\neg \left(p^+, p^-\right) = \left(p^-, p^+\right).$$

Operations $\otimes$ and $\oplus$ can be considered as the operations on the relative truth-values of only one sentence evaluated relatively to different agents.

4. **Consensus:** A consensus of two agents results in an agent who will confirm a sentence if two agents confirm it, and will refute it if two agents refute the sentence:

$$\left(p^{+}, p^{-}\right) \otimes \left(q^{+}, q^{-}\right) = \left(p^{+} \cdot q^{+}, p^{-} \cdot q^{-}\right).$$

5. **Gullibility:** A gullibility of two agents results in an agent who will confirm a sentence if at least one agent confirms it, and will refute it if at least one agent refutes it:

$$\left(p^{+}, p^{-}\right) \oplus \left(q^{+}, q^{-}\right) = \left(p^{+} + q^{+}, p^{-} + q^{-}\right).$$

It can be summarized as an algebraic structure which consists of a set $S = \{(0,0), (1,0), (0,1), (1,1)\}$ and five operations on S as follows (Schoter, 1996):

$$\begin{aligned}
&\forall\left(a^{+}, a^{-}\right), \left(b^{+}, b^{-}\right) \in S, \\
&\left(a^{+}, a^{-}\right) \wedge \left(b^{+}, b^{-}\right) = \left(a^{+} \cdot b^{+}, a^{-} + b^{-}\right), \\
&\left(a^{+}, a^{-}\right) \vee \left(b^{+}, b^{-}\right) = \left(a^{+} + b^{+}, a^{-} \cdot b^{-}\right), \\
&\left(a^{+}, a^{-}\right) \otimes \left(b^{+}, b^{-}\right) = \left(a^{+} \cdot b^{+}, a^{-} \cdot b^{-}\right), \\
&\left(a^{+}, a^{-}\right) \oplus \left(b^{+}, b^{-}\right) = \left(a^{+} + b^{+}, a^{-} + b^{-}\right), \\
&\neg\left(a^{+}, a^{-}\right) = \left(a^{-}, a^{+}\right).
\end{aligned}$$

5. Relative Sets

In this section, we will try to interpret the relative truth-value concept in the form of set algebra. This can be done by using a pair of classical sets (A^{+}, A^{-}) to represent a concept which is called the relative set. A^{+} is the set of all objects such that there is some evidence for their belonging to the concept and is called the positive region of the relative set. A^{-} consists of all objects such that there is some evidence against their belonging to the concept and is called the negative region of the relative set. We will say that:

1. a belongs to (A^{+}, A^{-}) if $a \in A^{+}$ and $a \notin A^{-}$,

2. a does not belong to (A^{+}, A^{-}) if $a \notin A^{+}$ and $a \in A^{-}$,

3. a has a contradicting behaviour in (A^{+}, A^{-}) if $a \in A^{+}$ and $a \in A^{-}$, and

4. belonging of a to (A^{+}, A^{-}) is unknown if $a \notin A^{+}$ and $a \notin A^{-}$.

According to these definitions, a relative set partitions all objects into four distinct regions:

1. The region of all objects that belong to A^+ and do not belong to A^-. This region corresponds to the $(1,0)$ or t value of Belnap's logic.

2. The region of all objects that belong to A^- and do not belong to A^+. This region corresponds to the $(0,1)$ or f value of Belnap's logic.

3. The region of all objects that belong both to A^+ and A^-, which corresponds to the $(1,1)$ or *Both* value of Belnap's logic.

4. The region of all objects that belong neither to A^+ nor A^-, which corresponds to the $(0,0)$ or *None* value of Belnap's logic.

The relative set intersection, union and complement operations are defined like $\wedge$, $\vee$ and $\neg$ relative truth-value operations:

1. Intersection of relative sets

$$\left(A^+, A^-\right) \cap_R \left(B^+, B^-\right) = \left(A^+ \cap B^+, A^- \cup B^-\right),$$

2. Union of relative sets

$$\left(A^+, A^-\right) \cup_R \left(B^+, B^-\right) = \left(A^+ \cup B^+, A^- \cap B^-\right),$$

3. Complement of relative sets

$$\sim \left(A^+, A^-\right) = \left(A^-, A^+\right).$$

Like operations with relative truth-values, two new operations are defined as follows:

1. Consensus of relative sets

$$\left(A^+, A^-\right) \otimes \left(B^+, B^-\right) = \left(A^+ \cap B^+, A^- \cap B^-\right),$$

2. Gullibility of relative sets

$$\left(A^+, A^-\right) \oplus \left(B^+, B^-\right) = \left(A^+ \cup B^+, A^- \cup B^-\right).$$

The consensus of two relative sets can be understood as the intersection of two agents' knowledge about the same concept, and the gullibility of two relative sets can be considered as a union of two agents' knowledge about the same concept.

5.1. Orders of Two Relative Sets

Let U be a finite non-empty set called the universe. If we denote by 2^U the power set of U, then $\left(2^U, \subseteq\right)$ will be a lattice in which meet and join operators are the classical set intersection, $\cap$, and the classical set union, $\cup$, respectively. The order of the lattice is the classical set inclusion, and the classical set complement is an order reversing involution.

Like Belnap's four truth-values, we can distinguish two distinct orders for relative sets:

1) Inclusion ordering of relative sets:

$$\left(A^+, A^-\right) \subseteq_I \left(B^+, B^-\right) \Leftrightarrow A^+ \subseteq B^+,\ B^- \subseteq A^-,$$

2) Knowledge ordering of relative sets:

$$\left(A^+, A^-\right) \subseteq_K \left(B^+, B^-\right) \Leftrightarrow A^+ \subseteq B^+,\ A^- \subseteq B^-.$$

The first one, $\subseteq_I$, is an extension of the classical set inclusion which corresponds to the $\leq_t$ ordering of FOUR. $\left(2^U \times 2^U, \subseteq_I\right)$ is a lattice whose meet and join operators are the relative set intersection, $\cap R$, and the union, $\cup R$, respectively, and the relative set complement is an order reversing involution of this lattice. The second ordering, $\subseteq_K$, is a new one and can be understood as an order of information that each relative set exhibits. It corresponds to the $\leq_k$ ordering of FOUR and $\left(2^U \times 2^U, \subseteq_K\right)$ will be a lattice in which the relative set consensus, $\otimes$, and the relative set gullibility, $\oplus$, are its meet and join operators, respectively. The relative set complement, $\sim$, is an order-preserving operator of this lattice. So, with relative set algebra, the lattice of classical sets has been generalized to bilattice $(2^U \times 2^U, \subseteq_I, \subseteq_K, \sim)$.

5.2. A Simple Example

Suppose that we have a number of cards with their two sides coloured black or white, numbered from 1 to 10. Three agents are gathering information about the colour of different cards, each making a relative set, for the concept of being white, on the basis of their observation as follows:

Agent 1:

white cards: $1, 2, 3$,

black cards: $6, 7, 8, 9$,

$\left(A_1^+, A_1^-\right) = \left(\{1, 2, 3\}, \{6, 7, 8, 9\}\right)$.

Agent 2:

white cards: $1, 2, 3, 4, 6$,

black cards: $5, 6, 7, 8, 9$,

$\left(A_2^+, A_2^-\right) = \left(\{1, 2, 3, 4, 6\}, \{5, 6, 7, 8, 9\}\right)$.

Agent 3:

white cards: $1, 2, 10,$

black cards: $5, 6, 7, 8,$

$$\left(A_3^+, A_3^-\right) = \left(\{1, 2, 10\}, \{5, 6, 7, 8\}\right).$$

According to the definitions, we have:

$$A_1^+ \subseteq A_2^+, A_1^- \subseteq A_2^- \Rightarrow \left(A_1^+, A_1^-\right) \subseteq_K \left(A_2^+, A_2^-\right),$$

$$\left(A_C^+, A_C^-\right) = \left(A_1^+, A_1^-\right) \otimes \left(A_3^+, A_3^-\right) = \left(\{1, 2\}, \{6, 7, 8\}\right),$$

$$\left(A_G^+, A_G^-\right) = \left(A_1^+, A_1^-\right) \oplus \left(A_3^+, A_3^-\right) = \left(\{1, 2, 3, 10\}, \{5, 6, 7, 8, 9\}\right).$$

Some relevant comments are as follows:

1. As far as Agent 2 is concerned, Card No. 6 presents a contradicting behaviour (it is both white and black) and there is no information for Card No. 10.

2. Since $\left(A_1^+, A_1^-\right) \subseteq_K \left(A_2^+, A_2^-\right)$, $\left(A_2^+, A_2^-\right)$ exhibits more knowledge (or information) about different cards. In other words, Agent 2 has more knowledge about white cards than Agent 1. Note that $\left(A_1^+, A_1^-\right)$ and $\left(A_3^+, A_3^-\right)$ are incomparable due to the order of knowledge.

3. If one is going to make deductions on the basis of the knowledge of Agents 1 and 3, he or she can choose either to accept the information confirmed by both Agents, or to accept all proposed information. The former approach insists on a consensus and will form the relative set $\left(A_C^+, A_C^-\right)$, whereas the latter approach that is willing to accept all will form the relative set $\left(A_G^+, A_G^-\right)$.

Here, we only gave an example for the knowledge order, which can be discussed when one is dealing with several agents. Classical set algebra can be considered as a special case when there is only one agent present. The order of inclusion can be clarified if one thinks of a single agent logically comparing two different concepts.

6. Three Distinct Worlds in Relative Sets

As we mentioned earlier, relative sets are constructed relative to some information. Therefore, two new operators (consensus and gullibility) are defined to make manipulations in the knowledge order possible. The strong point about this approach is its capability of dealing with multiple agents' information.

In classical set theory, each set is determined by evaluating a sentence, which is formed by attributing one concept to an object. For example, to determine the set of all 'white cars', we shall evaluate the sentence 'car is white' on all the cars in the desired universe. If the sentence on a specific car is true, then the car will belong to the set, and if the sentence is false, then the car does not belong to the set. In relative sets, however, each relative set is determined by evaluating a sentence on all objects relative to one agent's knowledge. So, different agents may form different relative sets

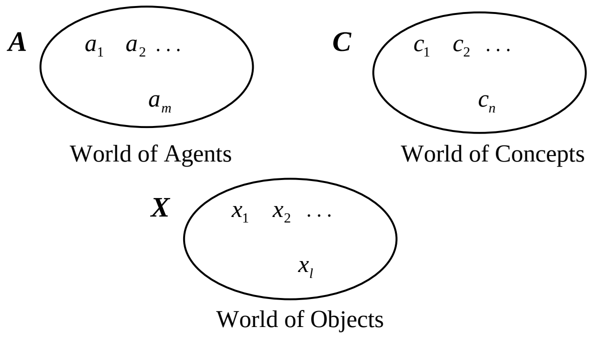

Fig. 2. Three distinct worlds in relative sets.

for the same concept. It will be made clearer by using three distinct worlds:

1. World of objects: it is the set of all desired objects.

2. World of concepts: it is the set of all labels that are attributed to categories of objects, e.g. red, large, etc.

3. World of agents: it is the set of all agents. Each agent made its own relative sets for concepts by using its own knowledge.

We will write the world of objects as $\boldsymbol{X} = \{x_1, x_2, \ldots, x_l\}$, the world of concepts as $\boldsymbol{C} = \{c_1, c_2, \ldots, c_n\}$, and the world of agents as $\boldsymbol{A} = \{a_1, a_2, \ldots, a_m\}$ (Fig. 2). Now, we define a mapping

$$L : A \times C \to 2^X \times 2^X.$$

This mapping relates a pair of two sets to each pair of agent and concept. This pair of sets is called the relative set of a concept with respect to the agent,

$$L\left(a_i, c_j\right) = \left(X_{ij}^+, X_{ij}^-\right), \quad X_{ij}^+, X_{ij}^- \in 2^X.$$

We can equate this mapping to the following two mappings:

$$L^+ : A \times C \to 2^X, \quad L^+\left(a_i, c_j\right) = X_{ij}^+,$$

$$L^- : A \times C \to 2^X, \quad L^-\left(a_i, c_j\right) = X_{ij}^-.$$

Here L^+ is called the positive part of the mapping L and $L^+\left(a_i, c_j\right)$ is the set of all objects that the agent a_i can claim to belong to the concept c_j. Moreover, L^- is called the negative part of L, and $L^-\left(a_i, c_j\right)$ consists of all objects that the agent a_i can claim not to belong to the concept c_j.

6.1. World of Concepts

The world of concepts consists of all the labels that are attributed to categories of objects. We can think of intersection, union and complement as operations in this world, and the concepts are usually compared with the set inclusion order. We define a set as

$$K_{ai} = \{L(a_i, cj) \mid j = 1, 2, \ldots, n\},$$

i.e. K_{ai} is the set of all relative sets which the agent a_i forms for each concept. We call them the equi-knowledge relative sets. The classical sets can be considered as a special case of equi-knowledge relative sets.

6.2. World of Agents

The world of agents is the set of all agents where each of them forms different relative sets with its own knowledge. We can think of them as different persons, each having his or her own opinion and knowledge about objects and concepts that are not necessarily the same. The order of knowledge is an order of agents and we can compare agents with one another with respect to their knowledge about a specific concept. The gullibility and consensus operations can be considered as two operations on agents. We define another set as

$$E_{cj} = \{L(a_i, c_j) \mid i = 1, 2, \ldots, n\}.$$

This set consists of all the relative sets that different agents form for the same concept, and they are called the equi-concept relative sets. An illustrative diagram is shown in Fig. 3.

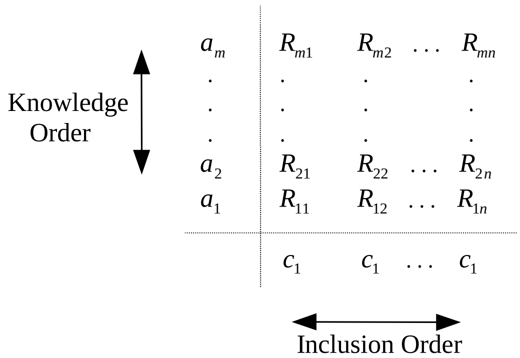

Fig. 3. Entry R_{ij} is equal to $L(a_i, c_j)$. Each row consists of equi-knowledge relative sets and each column consists of equi-concept relative sets.

7. Rough Sets

In rough set theory it is simply assumed that the knowledge is based on the ability to classify objects and is formally represented by an equivalence relation called the indiscernibility relation. The indiscernibility relation induces an approximation space on the universe, which is made of equivalence classes of indiscernible objects. Let U be the universe and R be an equivalence relation on U. A pair (U, R) is called the approximation space. So, one may not be able to determine precisely an arbitrary subset $X \subseteq U$ by means of the available information, i.e. equivalence classes of R. Instead, each set is replaced by two crisp sets which are called the lower and upper approximations, and which are defined as

$$\underline{R}X = \{x \mid [x]_R \subseteq X\}, \quad \overline{R}X = \{x \mid [x]_R \cap X \neq \emptyset\},$$

respectively, where $[x]_R$ is the equivalence class containing x. Using the available information, we can say that the lower approximation is the set of all the objects that surely belong to X, and the upper approximation contains all the objects that possibly belong to X. The approximated set lies between its lower and upper approximations:

$$\underline{R}X \subseteq X \subseteq \overline{R}X.$$

With every pair of approximations we can distinguish three distinct regions on U:

$$\begin{aligned} &POS_R(X) = \underline{R}X, && R\text{-positive region of } X, \\ &BND_R(X) = \overline{R}X - \underline{R}X, && R\text{-boundary region of } X, \\ &NEG_R(X) = U - \overline{R}X, && R\text{-negative region of } X. \end{aligned}$$

A positive region of X is the set of all the objects which can certainly be classified as elements of X. A negative region is the set of all the objects which can certainly be classified as elements of X^c (set complement of X). Finally, the boundary region contains all the objects which cannot be classified as elements of X or X^c. Hence, imprecision in rough sets is due to the boundary region, and higher precision results in a smaller boundary region. Obviously, crisp sets have no boundary region.

8. Integrated Approach to Relative Sets and Rough Sets

In this section, we are going to propose an integrated approach to relative sets and rough sets, and to investigate some of the interrelations between them. The approach is based on the idea that we can think of a rough set as a relative set. In this way, the approximation of a set can be viewed as a concept projection from one agent to another which is more general than the notion of approximation. It is also possible to extend some of the definitions and operations of relative sets into rough sets.

8.1. Rough Sets as Relative Sets

The theory of rough sets provides a formal tool for dealing with imprecise or incomplete information in terms of three-valued logic. Relative sets, however, are based on the Belnap's logic, whose semantics are four-valued. Consequently, we can easily think of rough sets as a special case of relative sets, and the definitions and operations on relative sets can be extended to the theory of rough sets.

Let the pair $apr = (U, R)$ be an approximation space on U, and let U/R denote the set of all the equivalence classes of R. The empty set $\emptyset$ and the elements of U/R are called the elementary sets. A set which is a union of elementary sets is called the definable set. The family of all definable sets in approximation space apr is denoted by $Def(apr)$. We would like to define a rough set as a pair of disjoint definable sets, i.e. given two subsets $A^+, A^- \in Def(apr)$ and $A^+ \cap A^- = \emptyset$, we call the pair (A^+, A^-) a rough set in which A^+ denotes the R-positive region and A^- denotes the R-negative region of the rough set. The R-boundary region will be $U - A^+ \cup A^-$ and, if $A^+ \cup A^- = U$, the pair (A^+, A^-) will have no boundary region and will be a crisp set. So, the three regions of the rough set are completely determined by a pair of sets, which is a relative set. The family of all rough sets in this approximation space is denoted by $Rough\,(apr)$. If we suppose that an approximation space is a knowledge base of an agent, then $Rough\,(apr)$ will be a set of equi-knowledge relative sets. The relative set inclusion can be considered as an order of $Rough\,(apr)$ and interpreted as a rough set inclusion. The knowledge order of relative sets can be used to compare two rough sets when both of them are related to the same concept. Formally, suppose that we are given two approximation spaces $apr_1 = (U, R_1)$ and $apr_2 = (U, R_2)$, and $\left(A_1^+, A_1^-\right)$, $\left(A_2^+, A_2^-\right)$ are two rough sets in these two approximation spaces, both being related to the same concept. Now, we can compare them in the order of information. If

$$\left(A_1^+, A_1^-\right) \in Rough\,(apr_1), \quad \left(A_2^+, A_2^-\right) \in Rough\,(apr_2), \quad \left(A_2^+, A_2^-\right) \subseteq_K \left(A_1^+, A_1^-\right)$$

then $\left(A_1^+, A_1^-\right)$ has a smaller boundary region and is more precise than $\left(A_2^+, A_2^-\right)$. Consequently, if these two sets are two approximations of the same set A in different approximation spaces, we have a better approximation of A in apr_1 than in apr_2.

8.2. Approximation as a Concept Projection

In rough set theory the approximation of a crisp set in an approximation space is obtained by two approximation operators. Each approximation operator associates a definable set with every crisp set. In this section, the notion of an approximation of a crisp set is generalized to an approximation of a rough set from one approximation space to another. Let $apr_1 = (U, R_1)$ and $apr_2 = (U, R_2)$ be two arbitrary approximation spaces, and let $\left(A_1^+, A_1^-\right)$ be a rough set in $Rough\,(apr_1)$. An approximation of $\left(A_1^+, A_1^-\right)$ in the approximation space apr_2 is a rough set in $Rough\,(apr_2)$ which is defined as follows:

$$\left(A_2^+, A_2^-\right) = apr_2 \left(A_1^+, A_1^-\right) = \left(\underline{R_2} A_1^+, \underline{R_2} A_1^-\right),$$

where $\underline{R_2} A_1^+$ and $\underline{R_2} A_1^-$ are lower approximations of A_1^+ and A_1^- in the approximation space apr_2, respectively. Since $\underline{R_2} A_1^+ \subseteq A_1^+$ and $\underline{R_2} A_1^- \subseteq A_1^-$, we have

$$\left(A_2^+, A_2^-\right) \subseteq_K \left(A_1^+, A_1^-\right).$$

In other words, by approximating a rough set from one approximation space to another, we obtain a less informative rough set. Now, consider each approximation space as an agent's knowledge. In this sense, by approximating a rough set, we project a concept from one agent's knowledge to another, and thereby we lose some information or precision of the concept and obtain a less precise rough concept.

8.3. Sequential Approximations of a Set

Suppose that we are given m equivalence relations $\{R_1, R_2, \ldots, R_m\}$ on U. Each equivalence relation represents a piece of knowledge about objects of the universe and results in an approximation space on U. Consequently, we have a family of m approximation spaces defined as follows:

$$POP = \{apr_1, apr_2, \ldots, apr_m\}, \quad apr_i = (U, R_i).$$

Each approximation space can be considered as an agent's knowledge base. So, we have m agents and we call this family of approximation spaces the *population* on U. A family of m rough sets related to one concept will be called the *sequence* on POP and denoted by SEQ_A if

$$SEQ_A = \left\{\left(A_1^+, A_1^-\right), \left(A_2^+, A_2^-\right), \ldots, \left(A_m^+, A_m^-\right)\right\}, \quad \left(A_i^+, A_i^-\right) \in Rough\,(apr_i).$$

As the notion of approximation of a crisp set is generalized to the approximation of a rough set, we are able to make a sequence of approximations of a set (Yao, 1996b). Consider m agents which form a population on U. Every agent is depicted as a node in Fig. 4. Each directed line shows the direction of a concept's projection from one

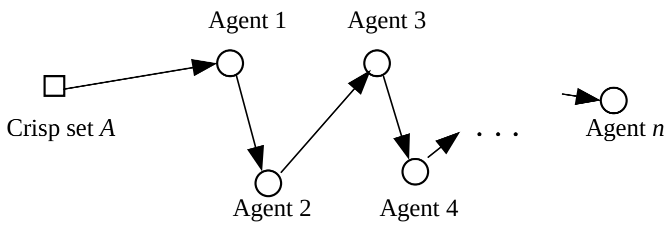

Fig. 4. A sequence on a population.

agent to another. Therefore, according to Fig. 4, Agent 1 has a direct projection of a crisp Concept A. Agent 2 has the projection of Concept A trough Agent 1 and so on, up to Agent n. If the projected concept of Agent n is $(\emptyset, \emptyset)$, then Agent n has no knowledge about Concept A. It is possible to consider different sequences of projections, each forming a sequence on POP. For example,

$$A \to \text{Agent } 2 \to \text{Agent } 3 \to \text{Agent } 1 \to \cdots \to \text{Agent } n,$$

$$A \to \text{Agent } 1 \to \text{Agent } 3 \to \text{Agent } 4 \to \cdots \to \text{Agent } n,$$

$$A \to \text{Agent } 4 \to \text{Agent } 1 \to \text{Agent } 2 \to \cdots \to \text{Agent } n.$$

In each state, Agent n may have a different approximation of the crisp concept. We can also think of the directed lines as an information flow path. Consequently, with different paths, the amount of the missed knowledge is varied.

8.4. Example

To illustrate the forthcoming argument we are going to present an example. Suppose that there are ten toys which form a universe set like

$$U = \{t_1, t_2, \dots, t_{10}\}.$$

Agent 1 classifies the toys according to their colour, Agent 2 classifies them according to their shape, and Agent 3 according to their size:

Knowledge of Agent 1: $\{t_1, t_2\}, \{t_3\}, \{t_4\}, \{t_5, t_6\}, \{t_7, t_8\}, \{t_9, t_{10}\}$,

Knowledge of Agent 2: $\{t_1\}, \{t_2\}, \{t_3, t_4, t_5\}, \{t_6\}, \{t_7\}, \{t_8, t_9\}, \{t_{10}\}$,

Knowledge of Agent 3: $\{t_1, t_2, t_3\}, \{t_4, t_5\}, \{t_6\}, \{t_7\}, \{t_8, t_9\}, \{t_{10}\}$.

A crisp set of toys is given as

$$A = \{t_1, t_2, t_3, t_6, t_7, t_8\}.$$

So, we have a population of three agents and we can form different sequences on it. With $A \to$ Agent 1 $\to$ Agent 2 $\to$ Agent 3, we have

Concept A due to Agent 1: $(\{t_1, t_2, t_3, t_7, t_8\}, \{t_4, t_9, t_{10}\})$,

Concept A due to Agent 2: $(\{t_1, t_2, t_7\}, \{t_{10}\})$,

Concept A due to Agent 3: $(\{t_7\}, \{t_{10}\})$.

With $A \to$ Agent 2 $\to$ Agent 1 $\to$ Agent 3, we have

Concept A due to Agent 2: $(\{t_1, t_2, t_6, t_7\}, \{t_{10}\})$,

Concept A due to Agent 1: $(\{t_1, t_2\}, \emptyset)$,

Concept A due to Agent 3: $(\emptyset, \emptyset)$.

With the first sequence, the information about Concept A is transformed first through Agent 1 and then through Agent 2 to Agent 3. In this way, Agent 3 can claim that t_7 belongs to A and t_{10} does not belong to A. With the second sequence, the information about Concept A is transformed first through Agent 2 and then through Agent 2 to Agent 3. In this case, Agent 3 has no knowledge about A.

9. Relative Sets and Interval Sets

The notion of interval sets is close to what is defined as relative sets. However, we claim that the latter notion is more general than the former. Let U be the universe set and 2^U be its power set. A subset of 2^U of the form

$$\boldsymbol{A} = [A_1, A_2] = \left\{A \in 2^U \mid A_1 \subseteq A \subseteq A_2\right\}$$

is called a closed interval set, where it is assumed that $A_1 \subseteq A_2$ (Yao, 1993). For two interval sets

$$\boldsymbol{A} = [A_1, A_2]\,, \quad \boldsymbol{B} = [B_1, B_2]\,,$$

the intersection, union and complement are defined as follows:

1. intersection

$$\boldsymbol{A} \cap_I \boldsymbol{B} = \{A \cap B \mid A \in \boldsymbol{A},\ \ B \in \boldsymbol{B}\}\,,$$

2. union

$$\boldsymbol{A} \cup_I \boldsymbol{B} = \{A \cup B \mid A \in \boldsymbol{A},\ \ B \in \boldsymbol{B}\}\,,$$

3. complement

$$\sim \boldsymbol{A} = [U - A_2, U - A_1]\,,$$

and the inclusion order of interval sets is defined as

$$\boldsymbol{A} \subseteq_I \boldsymbol{B} \Leftrightarrow A_1 \subseteq B_1,\ A_2 \subseteq B_2.$$

If we suppose that, for an interval set $\boldsymbol{A} = [A_1, A_2]$, A_1 consists of the elements with truth value t, and A_2 consists of the elements with truth value v such that $f \leq_t v(t, Both, None)$, then we have a relative set (A^+, A^-), where $A^+ = A_1$ and $A^- = U - A_2$ (set complement of A_2). In other words, A^+ consists of the elements of membership t, and 1-st consists of the elements of membership f, which is consistent with the definition of relative sets, and the relative set intersection, union and complement are similar to those of interval sets. Furthermore, the interval set inclusion order is similar to the relative set inclusion order. But the other order, which deals with information, has no counterpart in the interval sets.

10. Conclusion

In real-world conditions, we do not normally have access to complete and valid information to base our deductions on, nevertheless, an intelligent system should have the capability to deal with incomplete and sometimes even inconsistent information. An efficient tool in such situations is the four-valued logic of Belnap, where the four values are evaluated according to two distinct orders (knowledge and truth). The bilattice is a mathematical structure that naturally generalizes Belnap's four-valued logic. In this paper, unlike classical set algebra, a pair of sets (a relative set) represented each concept. Thus, based on the bilattice structure, we provided an extension of the classical set algebra. As was illustrated by an example, we are able to deal with several information sources in a set theoretic manner. We also proposed an integrated approach to relative and rough sets which is a new view on the theory of rough sets.

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