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OPTIMIZATION PROBLEMS WITH CONVEX EPIGRAPHS. APPLICATION TO OPTIMAL CONTROL†

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For a class of infinite-dimensional minimization problems with nonlinear equality constraints, an iterative algorithm for finding global solutions is suggested. A key assumption is the convexity of the "epigraph", a set in the product of the image spaces of the constraint and objective functions. A convexification method involving randomization is used. The algorithm is based on the extremal shift control principle due to N.N. Krasovskii. An application to a problem of optimal control for a bilinear control system is described.

Keywords: nonconvex optimization, global optimization methods

1. Introduction

Our goal is to describe and justify a method for approximating a global solution to an optimization problem of the form

$$\begin{aligned} &\text{minimize } J(x),\\ &x \in X,\\ &F(x) = 0. \end{aligned} \tag{1}$$

Here X is a metric space further called the *domain* (of problem (1)), J is a scalar function on X, and F is a function defined on X and taking values in a separable Hilbert space Y. We assume that the feasible set in problem (1) is nonempty. In what follows, $\mathrm{dist}_X(\cdot,\cdot)$ stands for the distance in X, $\langle \cdot,\cdot \rangle_Y$ denotes the scalar product in Y, and $|\cdot|_Y$ stands for the (strong) norm in Y. We introduce the set

$$E = \big\{ (F(x), y) \colon \; x \in X, \; y \geq J(x) \big\} \tag{2}$$

and call it the *epigraph* (in problem (1)).

Our basic assumptions are the following:

(i) function $J \colon X \mapsto \mathbb{R}^1$ is bounded and lower semicontinuous, and

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(ii) function $F\colon\ X \mapsto Y$ is bounded and weakly continuous, i.e. continuous as a map from X to Y equipped with the weak topology (or a weak norm; see, e.g. (Warga, 1975, IV.1)).

We denote by X^0 the set of all solutions to problem (1) and write J^0 for the optimal value in (1). Obviously, $J^0 > -\infty$. Below, we introduce further assumptions implying that X^0 is nonempty. For example, in Section 2 we assume that domain X is compact.

We are interested in finding an algorithm to generate a sequence (x_k) in X that converges to the solution set X^0. A sequence (x_k) in X is said to *converge* to a nonempty set $X_0 \subset X$ in X if $\mathrm{dist}_X(x_k, X_0) \to 0$ where $\mathrm{dist}_X(x, X_0) = \inf\{\mathrm{dist}_X(x, x_0)\colon\ x_0 \in X_0\}\quad (x \in X)$.

Since, in general, problem (1) is not a problem of convex optimization, standard, say, gradient type, successive convex optimization techniques (Vasiliev, 1981) may fail to be applicable. Our analysis is based on the observation that if domain X is a compactum, the objective function J is continuous and epigraph E is convex, then problem (1) is reducible to an (extended) problem of convex optimization. A reduction technique shown in Section 2 employs the extension of domain X to the set of all Borel probability measures on X.

In Section 3 the basic successive solution approximation algorithm for the extended problem is described. The algorithm starts at an arbitrary probability measure μ_0 on X. In each iteration, the current solution approximation, μ_k, is "shifted" towards a measure ν_{k+1} minimizing the (extended) objective function penalized by a certain linear term; the latter is determined by μ_k and intended to control the discrepancy associated with the equality constraint. The algorithm can be attributed to the class of nongradient-type global optimization algorithms which includes penalty (Bertsekas, 1982), path-following (Zangwill and Garcia, 1981) and interior-point homotopy methods (Sonnevend, 1985). It develops the idea of "constraint aggregation" in convex optimization (Ermoliev *et al.*, 1997; Kryazhimskii, 1999). The "constraint aggregation" method relies, in turn, on Krasovskii's extremal shift feedback control principle (Krasovskii, 1985; Krasovskii and Subbotin, 1988).

In Section 4 we use the method described in Section 3 to construct a sequence convergent to the solution set X_0 of problem (1). In Section 5 we extend the field of the applicability of the method. We no longer assume that the metric space X is compact and the objective function J is continuous. Here, we use the extension of domain X to the set of all finite convex combinations of point-concentrated probability measures on X. Basic elements of the technique developed earlier still work.

In Section 6, based on the results of Section 5, we apply the method to a problem of optimal control for a bilinear system with state constraints. This study pertains to the theory of approximate solutions of optimal control problems (Fedorenko, 1978; Gabasov *et al.*, 1984; Matveyev and Yakubovich, 1998); earlier an application of a similar technique to a convex optimal control problem was given in (Kryazhimskii and Maksimov, 1998). We provide a sufficient condition for the applicability of the method and specify it using the structure of the problem analyzed. It is remarkable that the key operation in each iteration is represented as a family of independent finite-dimensional optimization problems.

2. Compact Domain. Reduction to Convex Optimization

In this section and in Sections 3 and 4 we assume that X, the domain of problem (1), is compact and the objective function J is continuous. Then, due to the weak continuity of the constraint function F, the solution set of problem (1), X^0, is nonempty.

Our goal in this section is to show that problem (1) is reducible to a problem of convex optimization if epigraph E in (2) is convex.

Remark 1. One can easily see that E is convex if and only if $F(X) = \{F(x)\colon\ x \in X\}$ is convex and the function $w \mapsto \inf\{J(x)\colon\ x \in X,\ F(x) = w\} : F(X) \mapsto \mathbb{R}^1$ is convex.

Remark 2. If X is a closed, convex and bounded subset of a separable Hilbert space, F is linear and J is convex (i.e. (1) is a convex optimization problem), then epigraph E is convex.

A reduction technique which is standard enough employs the extension of the argument set X to the set of all Borel probability measures on X. We argue as follows. Let $\mathrm{m}(X)$ denote the linear space of all finite Borel measures on X. For every $\mu \in \mathrm{m}(X)$ and every continuous $f\colon\ X \mapsto \mathbb{R}^1$, we adopt the notation

$$f(\mu) = \int_X f(x)\,\mu(\mathrm{d}x) \tag{3}$$

and set

$$F(\mu) = \int_X F(x)\,\mu(\mathrm{d}x). \tag{4}$$

Here the integral is understood in the Bochner sense, with Y endowed with a weak norm (see, e.g. (Warga, 1975, I.4.33)). Using (3) and (4), we extend f and F to $\mathrm{m}(X)$. We equip $\mathrm{m}(X)$ with a $*$ weak norm, $|\cdot|_{\mathrm{m}(X)}$, and treat it as a normed space. Thus, a sequence (μ_i) converges to a μ in $\mathrm{m}(X)$ if $f(\mu_i) \to f(\mu)$ for every continuous $f\colon\ X \mapsto \mathbb{R}^1$ (Warga, 1975, IV.1).

Remark 3. Note that $\mathrm{m}(X)$ can be treated as a Hilbert space if we define the $*$ weak norm, $|\cdot|_{\mathrm{m}(X)}$, in $\mathrm{m}(X)$ by

$$|\mu|_{\mathrm{m}(X)} = \left(\sum_{i=1}^{\infty} 2^{-i}|f_i(\mu)|^2\right)^{1/2} \qquad (\mu \in \mathrm{m}(X)).$$

Here $\{f_1(\cdot), f_2(\cdot), \dots\}$ is a dense subset of the unit ball in $C(X)$, the space of all continuous scalar functions on X. The associated scalar product, $\langle\cdot,\cdot\rangle_{\mathrm{m}(X)}$, in $\mathrm{m}(X)$ is given by

$$\langle\mu,\nu\rangle_{\mathrm{m}(X)} \sum_{i=1}^{\infty} 2^{-i} f_i(\mu) f_i(\nu) \quad (\mu,\nu \in \mathrm{m}(X)).$$

Remark 4. Clearly, the function $\mu \mapsto J(\mu)\colon\ \mathrm{m}(X) \mapsto \mathbb{R}^1$ is linear and continuous.

Lemma 1. *Let X be compact. Then the function $\mu \mapsto F(\mu)\colon\ \mathrm{m}(X) \mapsto Y$ is linear and weakly continuous, i.e. if $\mu_i \to \mu$ in $\mathrm{m}(X)$, then $F(\mu_i) \to F(\mu)$ weakly in Y.*

Proof. The linearity of $\mu \mapsto F(\mu)$ is obvious. Let us prove that $\mu \mapsto F(\mu)$ is weakly continuous. To this end, suppose that the assertion is false. Then there exists a sequence (μ_i) in $\mathrm{m}(X)$ such that $\mu_i \to \mu$ in $\mathrm{m}(X)$ and $F(\mu_i) \not\to F(\mu)$ weakly in Y. With no loss of generality, we assume that for some $y \in Y$ and some $\varepsilon > 0$ we have

$$|\langle y, F(\mu_i)\rangle_Y - \langle y, F(\mu)\rangle_Y| \geq \varepsilon \quad (i = 1, \dots). \tag{5}$$

The set $F(X) = \{F(x)\colon\ x \in X\}$ is bounded, since X is compact and F is weakly continuous. Therefore, the sequence $(F(\mu_i))$ taking values in the closed convex hull of $F(X)$ is bounded, i.e. weakly compact in Y. With no loss of generality, we assume that $F(\mu_i) \to z$ weakly in Y. Then

$$\langle y, F(\mu_i)\rangle_Y \to \langle y, z\rangle_Y \tag{6}$$

and, by (5),

$$|\langle y, z\rangle_Y - \langle y, F(\mu)\rangle_Y| \geq \varepsilon. \tag{7}$$

We have

$$\begin{aligned}\langle y, F(\mu_i)\rangle_Y &= \left\langle y, \int_X F(x)\,\mu_i(\mathrm{d}x)\right\rangle_Y \\ &= \int_X \langle y, F(x)\rangle_Y\,\mu_i(\mathrm{d}x) \to \int_X \langle y, F(x)\rangle_Y\,\mu(\mathrm{d}x) \\ &= \left\langle y, \int_X F(x)\,\mu(\mathrm{d}x)\right\rangle_Y = \langle y, F(\mu)\rangle_Y.\end{aligned}$$

Now (6) shows that $\langle y, z\rangle_Y = \langle y, F(\mu)\rangle_Y$, which contradicts (7). The contradiction completes the proof. ∎

Let $\mathrm{pm}(X)$ be the set of all Borel probability measures on X. As usual, a probability measure $\mu \in \mathrm{pm}(X)$ is said to be *concentrated at an* $x \in X$ if $\mu(\{x\}) = 1$.

Remark 5. The set $\mathrm{pm}(X)$ is a convex compactum in $\mathrm{m}(X)$ (Warga, 1975, Theorem IV.2.1).

Introduce the *extended* problem

$$\begin{aligned}&\text{minimize } J(\mu),\\ &\mu \in \mathrm{pm}(X),\\ &F(\mu) = 0.\end{aligned} \tag{8}$$

Once the functions $\mu \mapsto J(\mu)\colon\ \mathrm{m}(X) \mapsto \mathbb{R}^1$ and $\mu \mapsto F(\mu)\colon\ \mathrm{m}(X) \mapsto Y$ are linear and continuous (cf. Remark 4 and Lemma 1), and $\mathrm{pm}(X)$ is a convex compactum (cf. Remark 5), the extended problem (8) falls into the class of problems of convex optimization (with linear objective functions). Moreover, Remark 4, Lemma 1 and Remark 5 imply that the extended problem (8) has a solution. We denote by $\hat{X}^0$ the set of all solutions to (8) and write $\hat{J}^0$ for the optimal value in (8).

Now consider the case where epigraph E in (2) is convex. The following observation will be used in our argument.

Lemma 2. *Let X be a compactum, J be continuous and epigraph E be convex. Then for every $\mu \in \mathrm{pm}(X)$, the set*

$$r(\mu) = \{x \in X\colon\ F(x) = F(\mu),\ J(x) \leq J(\mu)\} \tag{9}$$

is nonempty.

Proof. Let $E* = \{(F(x), J(x)) :\ \ x \in X\}$. Obviously, $E_* \subset E$ and $(F(\mu), J(\mu))$ belongs to $\mathrm{conv}(E_*)$, the closed convex hull of E_* in $Y \times \mathbb{R}^1$ where Y is equipped with the weak norm. Lemma 1 and Remark 4 imply that E is closed in $Y \times \mathbb{R}^1$. Due to the convexity of E, we have $\mathrm{conv}(E_*) \subset E$. Consequently, $(F(\mu), J(\mu)) \in E$. By the definition of E there exists an $x \in X$ such that $F(x) = F(\mu)$ and $J(\mu) \geq J(x)$; obviously, $x \in r(\mu)$. The lemma is proved. ■

The next theorem reduces the original problem (1) to the extended problem (8).

Theorem 1. *Let X be a compactum, J be continuous and epigraph E be convex. Then problems (1) and (8) are equivalent in the following sense:*

(i) *the optimal values in the original problem (1) and extended problem (8) coincide, $J^0 = \hat{J}^0$;*

(ii) *if x^0 solves the original problem (1), then the probability measure $\mu^0 \in \mathrm{pm}(X)$ concentrated at x^0 solves the extended problem (8).*

Proof. For every $x \in X$ admissible in the original problem (1), i.e. satisfying $F(x) = 0$, the measure $\mu \in \mathrm{pm}(X)$ concentrated at x satisfies $F(\mu) = 0$. Hence, μ is admissible in the extended problem (8). Moreover, $J(\mu) = J(x)$. Hence, the optimal value in the original problem (1) is not greater than that in the extended problem (8), $J^0 \geq \hat{J}^0$. Therefore, if (ii) is proved, (i) holds automatically. Let us prove (ii). Let x^0 be a solution to (1) and $\mu^0 \in \mathrm{pm}(X)$ be concentrated at x^0. Then $J(\mu^0) = J(x^0)$. Take arbitrary $\mu \in \mathrm{pm}(X)$ admissible in problem (8), i.e. satisfying $F(\mu) = 0$. By Lemma 2 the set $r(\mu)$ in (9) is nonempty; hence there exists an $x \in X$ such that $F(x) = F(\mu)$ and $J(x) \leq J(\mu)$. Equality $F(\mu) = 0$ implies that x is admissible in (1). Consequently, $J(x^0) \leq J(x)$ and $J(\mu^0) = J(x^0) \leq J(\mu)$. Recalling that μ is an arbitrary element admissible in problem (8), we deduce that $J(\mu^0) = \hat{J}^0$. Thus μ^0 solves problem (8). Statement (ii) (implying (i)) is proved. ■

Remark 6. For solving the extended problem (8), methods of convex optimization can, in principle, be used. However, the structure of space $\mathrm{m}(X)$ may lead to considerable difficulties in the implementation of these methods. Consider, for example, the gradient projection method for Hilbert spaces (Vasiliev, 1981, Sec. 4). Referring to Remark 3, we treat $\mathrm{m}(X)$ as a Hilbert space. For problem (8), a gradient projection sequence (μ_k) convergent to the solution set $\hat{X}^0$ in $\mathrm{m}(X)$ (see the corresponding definition in Section 3) is determined by $\mu_{k+1} = \pi(\mu_k - \delta_k J'(\mu_k))$ $(k = 0, 1, \dots)$ where $\delta_k > 0$, $J'(\mu_k)$ is the gradient of J at μ_k and $\pi(\mu)$ is the projection of $\mu \in \mathrm{m}(X)$ onto the admissible set $M = \{\mu \in \mathrm{pm}(X):\ F(\mu) = 0\}$. Since J is linear and continuous on $\mathrm{m}(X)$, there is a $\nu \in \mathrm{m}(X)$ such that $J(\mu) = \langle \mu, \nu \rangle_{\mathrm{m}(X)}$ $(\mu \in \mathrm{m}(X))$, and we have $J'(\mu) = \nu$ for each $\mu \in \mathrm{m}(X)$. Therefore, $\mu_{k+1} = \pi(\mu_k - \delta_k \nu)$ $(k = 0, 1, \dots)$. We see that the major operations are the identification of element ν and finding the projections of the elements $\mu_k - \delta_k \nu$ onto M. Unfortunately, it is not clear whether any constructive algorithm to perform these operations can be suggested.

3. Compact Domain. Solution Method for the Extended Problem

In this section we describe a successive solution approximation method for the extended problem (8). As mentioned in the Introduction, the method refers to the principle of extremal shift, originally proposed by N.N. Krasovskii (Krasovskii and Subbotin, 1988; Krasovskii, 1985). Like in Section 2, we assume that domain X is a compactum and the objective function J is continuous.

Fix positive $\delta_1, \delta_2, \dots$ less than 1 and such that for

$$\tau_k = \sum_{i=1}^{k} \delta_i \quad (k = 1, \dots) \tag{10}$$

we have

$$\tau_k \to \infty, \quad \delta_k \tau_k \to 0. \tag{11}$$

Remark 7. Let $\delta_k = c/k$ $(k = 1, \dots)$ where $c > 0$. Then (11) holds. Indeed, $\tau_k \to \infty$ holds obviously. Let us show that $\delta_k \tau_k \to 0$. For arbitrary $\varepsilon > 0$, let natural k_0 and $k_1 \ge k_0$ be such that

$$\sum_{i=k_0}^{\infty} \left(\frac{c}{i}\right)^2 < \frac{\varepsilon}{2}, \quad \frac{c}{k_1} \sum_{i=0}^{k_0} \frac{c}{i} < \frac{\varepsilon}{2}.$$

Then for every $k \in \{k_1, \dots\}$ we have

$$\delta_k \tau_k = \frac{c}{k} \sum_{i=0}^{k} \frac{c}{i} \le \frac{c}{k} \sum_{i=0}^{k_0} \frac{c}{i} + \frac{c}{k} \sum_{i=0}^{k} \frac{c}{i} \le \frac{c}{k} \sum_{i=0}^{k_0} \frac{c}{i} + \sum_{i=0}^{k} \left(\frac{c}{i}\right)^2$$

$$\le \frac{c}{k_1} \sum_{i=0}^{k_0} \frac{c}{i} + \sum_{i=0}^{\infty} \left(\frac{c}{i}\right)^2 < \varepsilon.$$

Thus $\delta_k \tau_k \to 0$.

We also fix nonnegative $\sigma_1, \sigma_2, \dots$ such that

$$\sigma_k \tau_k \to 0. \tag{12}$$

Take arbitrary

$$\mu_0 \in \mathrm{pm}(X) \tag{13}$$

for a zero approximaton to a solution of problem (8). If a k-th approximation, $\mu_k \in \mathrm{pm}(X)$, is defined $(k \in \{0, 1, \dots\})$, we set

$$\mu_{k+1} = (1 - \delta_{k+1})\mu_k + \delta_{k+1}\nu_{k+1}, \tag{14}$$

where ν_{k+1} is a σ_{k+1}-minimizer of

$$\hat{\varphi}_k(\nu) = 2(1 - \delta_{k+1})\langle F(\mu_k), F(\nu)\rangle_Y + \frac{J(\nu)}{\tau_{k+1}} \quad (\nu \in \mathrm{pm}(X)), \tag{15}$$

in $\mathrm{pm}(X)$, i.e.

$$\nu_{k+1} \in \mathrm{pm}(X), \; \hat{\varphi}_k(\nu_{k+1}) \le \inf\{\hat{\varphi}_k(\nu) : \; \nu \in \mathrm{pm}(X)\} + \sigma_{k+1}. \tag{16}$$

Every sequence (μ_k) satisfying (13)–(16) will be called an *extended extremal shift sequence.*

Remark 8. By Lemma 1, the function $\hat{\varphi}_k(\cdot)$ is continuous on $\mathrm{m}(X)$. Therefore, it is admissible to set $\sigma_k = 0$ $(k = 1, \dots)$. We will need positive σ_k in Section 5, where we will analyze a case where the domain X is, generally, noncompact.

Remark 9. Observing the recurrent formula (14) and taking into account the fact that $\nu_{k+1} \in \mathrm{pm}(X)$ and $0 < \delta_{k+1} < 1$, we easily find that $\mu_k \in \mathrm{pm}(X)$ $(k = 0, 1, \dots)$.

Remark 10. As (14) shows, μ_{k+1} is found through slightly shifting μ_k towards the "target" point ν_{k+1}. The definition of ν_{k+1} (see (16)) has an extremely clear geometrical interpretation if J is zero: ν_{k+1} minimizes the projection of $F(\nu)$ onto $F(\mu_k)$, i.e. "shifts" $F(\mu_{k+1})$ towards 0 at a maximum "speed". If J is nonzero, (16) corresponds to a penalized extremal shift (Kryazhimskii, 1999).

We say that a sequence (μ_k) from $\mathrm{m}(X)$ *converges* to a (nonempty) set $M \subset \mathrm{m}(X)$ in $\mathrm{m}(X)$ if $\mathrm{dist}_{\mathrm{m}(X)}(\mu_k, M) \to 0$ where $\mathrm{dist}_{\mathrm{m}(X)}(\mu, M) = \inf\{|\mu - \mu|_{\mathrm{m}(X)} : \mu \in M\}$ $(\mu \in \mathrm{m}(X))$.

Remark 11. Let a sequence (η_k) take values in $\mathrm{pm}(X)$, and a set $E \subset \mathrm{m}(X)$ be closed in $\mathrm{m}(X)$. Then (η_k) converges to E in $\mathrm{m}(X)$ if and only if E contains every accumulation point of (η_k) in $\mathrm{m}(X)$ (an $\eta \in \mathrm{m}(X)$ is said to be an accumulation point of (η_k) in $\mathrm{m}(X)$ if there exists a subsequence (η_{k_j}) convergent to η in $\mathrm{m}(X)$). This observation follows directly from the fact that $\mathrm{pm}(X)$ is a compactum in $\mathrm{m}(X)$ (Remark 5).

The next theorem generalizes Theorem 8.2 of (Kryazhimskii, 1999), which holds for problems of convex programming in finite-dimensional spaces.

Theorem 2. *Let* X *be a compactum and* J *be continuous. Then every extended extremal shift sequence* (μ_k) *converges in* $\mathrm{m}(X)$ *to the solution set* $\hat{X}^0$ *of the extended problem (8), and the sequence* $(J(\mu_k))$ *converges to the optimal value* $\hat{J}^0$ *in this problem.*

We use three lemmas in our proof.

Lemma 3. *Let* X *be a compactum,* J *be continuous,* (μ_k) *be a sequence in* $\mathrm{pm}(X)$*,*

$$|F(\mu_k)|_Y \to 0 \tag{17}$$

and

$$\limsup_{k\to\infty} J(\mu_k) \le \hat{J}^0. \tag{18}$$

Then (μ_k) *converges to* $\hat{X}^0$ *in* $\mathrm{m}(X)$ *and* $J(\mu_k) \to \hat{J}^0$.

Proof. Let us prove that (μ_k) converges to $\hat{X}^0$ in $\mathrm{m}(X)$. Take an arbitrary subsequence (μ_{k_j}) of (μ_k) which converges to a $\mu \in \mathrm{pm}(X)$ in $\mathrm{m}(X)$. It is sufficient to argue that $\mu \in \hat{X}^0$ (Remark 11). Convergence (17) and the weak continuity of F (Lemma 1) imply $F(\mu) = 0$. Hence μ is admissible in problem (8). The continuity of J (Remark 4) and relation (18) yield $J(\mu) \le \hat{J}^0$. Consequently, μ is a solution to (8), $\mu \in \hat{X}^0$. We stated that (μ_k) converges to $\hat{X}^0$ in $\mathrm{m}(X)$. This convergence and the fact that $J(\mu^0) = \hat{J}^0$ for all $\mu^0 \in \hat{X}^0$ imply $J(\mu_k) \to \hat{J}^0$. The proof is thus completed. ■

The next lemma is a modification of Lemma 2 in (Ermoliev *et al.*, 1997).

Lemma 4 *Let*

$$\zeta_{k+1} \le \zeta_k(1 - \varepsilon_{k+1}) + \gamma_k \quad (k = 0, 1, \dots), \tag{19}$$

$$\varepsilon_{k+1} > 0 \quad (k \ge k_0), \quad \sum_{k=0}^{\infty} \varepsilon_{k+1} = \infty \tag{20}$$

and

$$\frac{\gamma_k}{\varepsilon_{k+1}} \to 0. \tag{21}$$

Then

$$\limsup_{k\to\infty} \zeta_k \le 0. \tag{22}$$

Proof. Let us show that $\liminf_{k\to\infty}\zeta_k \le 0$. Indeed, if $\liminf_{k\to\infty}\zeta_k = \zeta > 0$, then assumptions (19), $\varepsilon > 0$ $(k \ge k_0)$ (see (20)) and (21) yield

$$\zeta_{k+1} \le \zeta_k - \varepsilon_{k+1}\left(\zeta_k - \frac{\gamma_k}{\varepsilon_{k+1}}\right) \le \zeta_k - \varepsilon_{k+1}\frac{\zeta}{2}$$

for all $k \ge k_1$ where k_1 is sufficiently large. Hence

$$\sum_{k_1}^{\infty}\varepsilon_{k+1} \le \frac{2}{\zeta}\limsup_{n\to\infty}\sum_{k_1}^{n}(\zeta_k - \zeta_{k+1}) = \limsup_{n\to\infty}\frac{2}{\zeta}(\zeta_n - \zeta_{k_1}) = \frac{2}{\zeta}(\zeta - \zeta_{k_1}),$$

which contradicts (20). Thus, there exists a subquence (ζ_{k_j}) such that $\lim_{j\to\infty}\zeta_{k_j} \le 0$. Now suppose that (22) is false, i.e. there is a subsequence (ζ_{s_j}) such that $\lim_{j\to\infty}\zeta_{s_j} \ge \zeta > 0$. With no loss of generality assume that $k_1 < s_1 < k_2 < s_2 \ldots$. Then for every j large enough there is an $r_j \in \{k_j, \ldots, s_j\}$ such that

$$\zeta_{r_j+1} > \zeta_{r_j} > \zeta/2. \tag{23}$$

Then for j large, (19), inequality $\varepsilon_{r_j} > 0$ (see (20)) and (21) yield

$$\zeta_{r_j+1} \le \zeta_{r_j}(1 - \varepsilon_{r_j+1}) + \gamma_k \le \zeta_{r_j} - \varepsilon_{r_j+1}\left(\frac{\zeta}{2} - \frac{\gamma_k}{\varepsilon_{r_j+1}}\right) \le \zeta_{r_j},$$

which contradicts (23). The proof is complete. ■

The next lemma (employing Lemma 4) plays a key role.

Lemma 5. *Let X be a compactum and J be continuous. Then for every extended extremal shift sequence (μ_k), relations (17) and (18) hold.*

Proof. By assumption, F and J are bounded on X. Inclusions $\mu_k \in \mathrm{pm}(X)$ and $\nu_{k+1} \in \mathrm{pm}(X)$ $(k = 1, \ldots)$, and Remark 9 imply

$$|F(\mu_k)| \le K_F, \quad |F(\nu_{k+1})| \le K_F, \tag{24}$$

$$|J(\nu_{k+1})| \le K_J \tag{25}$$

$(k = 1, \ldots)$, where

$$K_F = \sup\{|F(x)|_Y\colon\ x \in X\}, \quad K_J = \sup\{|J(x)|\colon\ x \in X\}.$$

Let

$$\lambda_0 = |F(\mu_0)|_Y^2, \quad \lambda_k = |F(\mu_k)|_Y^2 + \frac{J(\mu_k)}{\tau_k} - \frac{\hat{J}^0}{\tau_k} \quad (k = 1, \ldots). \tag{26}$$

Recall that τ_k $(k = 1, \ldots)$ are defined in (10). We also set $\tau_0 = 0$. Owing to (14), for arbitrary $k \in \{0, 1, \ldots\}$ we have

$$\lambda_{k+1} = |(1 - \delta_{k+1})F(\mu_k) + \delta_{k+1}F(\nu_{k+1})|_Y^2$$
$$+ \frac{(1 - \delta_{k+1})J(\mu_k) + \delta_{k+1}J(\nu_{k+1})}{\tau_{k+1}} - \frac{(1 - \delta_{k+1})\hat{J}^0 + \delta_{k+1}\hat{J}^0}{\tau_{k+1}}$$

or, equivalently,

$$\lambda_{k+1} = a_k + b_k - c_k, \tag{27}$$

where

$$a_k = |(1-\delta_{k+1})F(\mu_k) + \delta_{k+1}F(\nu_{k+1})|_Y^2,$$

$$b_k = \frac{(1-\delta_{k+1})J(\mu_k) + \delta_{k+1}J(\nu_{k+1})}{\tau_{k+1}},$$

$$c_k = \frac{(1-\delta_{k+1})\hat{J}^0 + \delta_{k+1}\hat{J}^0}{\tau_{k+1}}.$$

Obviously,

$$\begin{aligned} a_k &= (1-\delta_{k+1})^2|F(\mu_k)|_Y^2 + 2(1-\delta_{k+1})\delta_{k+1}\langle F(\mu_k), F(\nu_{k+1})\rangle_Y \\ &\quad + \delta_{k+1}^2|F(\nu_{k+1})|_Y^2. \end{aligned}$$

Noticing that $(1-\delta_{k+1})^2 \le (1-\delta_{k+1}) + \delta_{k+1}^2$ and taking into account (24), we get the estimate

$$\begin{aligned} a_k &\le (1-\delta_{k+1})|F(\mu_k)|_Y^2 + 2(1-\delta_{k+1})\delta_{k+1}\langle F(\mu_k), F(\nu_{k+1})\rangle_Y \\ &\quad + 2K_F^2\delta_{k+1}^2. \end{aligned}$$

Now, represent b_k as

$$b_k = \frac{(1-\delta_{k+1})J(\mu_k)}{\tau_k} - \frac{\delta_{k+1}(1-\delta_{k+1})J(\mu_k)}{\tau_k\tau_{k+1}} + \frac{\delta_{k+1}J(\nu_{k+1})}{\tau_{k+1}}.$$

Using (25), we arrive at

$$b_k \le \frac{(1-\delta_{k+1})J(\mu_k)}{\tau_k} + \frac{\delta_{k+1}J(\nu_{k+1})}{\tau_{k+1}} + \frac{\delta_{k+1}}{\tau_k\tau_{k+1}}K_J.$$

Similarly, we deduce that

$$c_k \ge \frac{(1-\delta_{k+1})\hat{J}^0}{\tau_k} + \frac{\delta_{k+1}\hat{J}^0}{\tau_{k+1}} - \frac{\delta_{k+1}}{\tau_k\tau_{k+1}}K_J.$$

Substituting the estimates for a_k, b_k and c_k into (27), we get

$$\begin{aligned} \lambda_{k+1} \le & \left[(1-\delta_{k+1})|F(\mu_k)|_Y^2 + \frac{(1-\delta_{k+1})J(\mu_k)}{\tau_k} - \frac{(1-\delta_{k+1})\hat{J}^0}{\tau_k}\right] \\ & + 2(1-\delta_{k+1})\delta_{k+1}\langle F(\mu_k), F(\nu_{k+1})\rangle_Y + \frac{\delta_{k+1}J(\nu_{k+1})}{\tau_{k+1}} - \frac{\delta_{k+1}\hat{J}^0}{\tau_{k+1}} \\ & + 2K_F^2\delta_{k+1}^2 + 2\frac{\delta_{k+1}}{\tau_k\tau_{k+1}}K_J. \end{aligned}$$

The expression in the square brackets equals $(1-\delta_{k+1})\lambda_k$ (cf. (26)). In a shorter notation, we have

$$\lambda_{k+1} \le (1-\delta_{k+1})\lambda_k + \alpha_k + \beta_k, \tag{28}$$

where

$$\alpha_k = \left[2(1-\delta_{k+1})\langle F(\mu_k), F(\nu_{k+1})\rangle_Y + \frac{J(\nu_{k+1})}{\tau_{k+1}} - \frac{\hat{J}^0}{\tau_{k+1}}\right]\delta_{k+1},$$

$$\beta_k = 2K_F^2\delta_{k+1}^2 + 2\frac{\delta_{k+1}}{\tau_k\tau_{k+1}}K_J. \tag{29}$$

Let us estimate α_k from above. Note that a solution μ^0 to the extended problem (8) satisfies $F(\mu^0)=0$ and $J(\mu^0)=\hat{J}^0$. Then

$$\begin{aligned}\alpha_k &= \left[2(1-\delta_{k+1})\langle F(\mu_k), F(\nu_{k+1})\rangle_Y + \frac{J(\nu_{k+1})}{\tau_{k+1}}\right]\delta_{k+1} \\ &\quad - \left[2(1-\delta_{k+1})\langle F(\mu_k), F(\mu^0)\rangle_Y + \frac{J(\mu^0)}{\tau_{k+1}}\right]\delta_{k+1} \\ &= [\hat{\varphi}_k(\nu_{k+1}) - \hat{\varphi}_k(\mu^0)]\delta_{k+1}\end{aligned}$$

(cf. (15)). Owing to (16), we have $\hat{\varphi}_k(\nu_{k+1}) - \hat{\varphi}_k(\mu^0) \le \sigma_{k+1}$. Therefore

$$\alpha_k \le \sigma_{k+1}\delta_{k+1}$$

and (28) is specified into

$$\lambda_{k+1} \le (1-\delta_{k+1})\lambda_k + \sigma_{k+1}\delta_{k+1} + \beta_k.$$

Now we apply Lemma 4 to

$$\zeta_k = \lambda_k\tau_k = |F(\mu_k)|_Y^2\tau_k + J(\mu_k) - \hat{J}^0 \quad (k=1,\dots) \tag{30}$$

(see (26)). We have

$$\zeta_{k+1} \le \frac{\tau_{k+1}}{\tau_k}(1-\delta_{k+1})\zeta_k + \gamma_k,$$

where

$$\gamma_k = (\sigma_{k+1}\delta_{k+1} + \beta_k)\tau_{k+1}. \tag{31}$$

The last inequality is specified as follows:

$$\begin{aligned}\zeta_{k+1} &\le \left(1 + \frac{\delta_{k+1}}{\tau_k}\right)(1-\delta_{k+1})\zeta_k + \gamma_k \\ &= \left(1 + \frac{\delta_{k+1}}{\tau_k} - \delta_{k+1} - \frac{\delta_{k+1}^2}{\tau_k}\right)\zeta_k + \gamma_k \\ &= (1-\varepsilon_{k+1})\zeta_k + \gamma_k,\end{aligned}$$

where

$$\varepsilon_{k+1} = \delta_{k+1}\eta_{k+1}, \quad \eta_{k+1} = 1 - \frac{1}{\tau_k} + \frac{\delta_{k+1}}{\tau_k}. \tag{32}$$

Recalling that $\tau_k = \sum_{i=1}^{k} \delta_i \to \infty$ (see (10) and (11)), we deduce that assumptions (19) and (20) of Lemma 4 hold. Using (31) and (29), we get

$$\gamma_k \le \sigma_{k+1}\delta_{k+1}\tau_{k+1} + K\left(\delta_{k+1}^2\tau_{k+1} + \frac{\delta_{k+1}}{\tau_k}\right),$$

where $K = \max\{2K_F^2, 2K_J\}$. Due to (32), (12) and (11), we have

$$\frac{\gamma_k}{\varepsilon_{k+1}} = \frac{\gamma_k}{\delta_{k+1}\eta_{k+1}} \le \frac{1}{\eta_{k+1}}\left[\sigma_{k+1}\tau_{k+1} + K\left(\delta_{k+1}\tau_{k+1} + \frac{1}{\tau_k}\right)\right] \to 0.$$

Thus assumption (21) of Lemma 4, holds. By Lemma 4, $\limsup_{k\to\infty} \zeta_k \le 0$. From (30), we see that $\limsup_{k\to\infty} J(\mu_k) \le \hat{J}^0$ and $\limsup_{k\to\infty} \lambda_k \le 0$. The latter inequality implies $|F(\mu_k)|_Y \to 0$ (see (26)). The lemma is proved. ■

Proof of Theorem 2. Let (μ_k) be an extended extremal shift sequence. By Lemma 5 relations (17) and (18) hold. Hence, by Lemma 3, (μ_i) converges to $\hat{X}^0$ in $\mathrm{m}(X)$ and $J(\mu_k) \to \hat{J}^0$. The theorem is proved. ■

4. Compact Domain. Solution Method for the Original Problem

Like in Sections 2 and 3, we assume that X is a compactum and J is continuous. We shall show that if the epigraph E is convex (i.e. the extended problem (8) is equivalent to the original problem (1), see Theorem 1), then elements $x_k \in r(\mu_k)$ (see (9)) associated with an extremal shift sequence (μ_k) converge to the solution set X^0 of the original problem (1).

Remark 12. If X is a compactum, then a sequence (x_k) in X converges to a closed set $X_0 \subset X$ in X if and only if X_0 contains every accumulation point of (x_k).

Theorem 3. *Let X be a compactum, J be continuous, the epigraph E be convex, (μ_k) be an extended extremal shift sequence, and $x_k \in r(\mu_k)$ $(k = 1, \dots)$. Then the sequence (x_k) converges to the solution set X^0 of the original problem (1), and each of the sequences $(J(\mu_k))$ and $(J(x_k))$ converges to the optimal value J^0 in this problem.*

Remark 13. The definition of x_k $(k = 1, \dots)$ is correct, since by Lemma 2 we have $r(\mu_k) \ne \emptyset$.

Proof of Theorem 3. Let us show that (x_k) converges to X^0 in X. Take an arbitrary subsequence (x_{k_j}) of (x_k) which converges to some $x \in X$ in X. It is sufficient to show that $x \in X^0$ (see Remark 12). By Theorem 3, (μ_{k_j}) converges to $\hat{X}^0$ in $\mathrm{m}(X)$ and

$$J(\mu_{k_j}) \to \hat{J}^0. \tag{33}$$

By assumption, $x_{k_j} \in r(\mu_{k_j})$. Hence, by the definition of $r(\mu_{k_j})$ (cf. (9)), $F(x_{k_j}) = F(\mu_{k_j})$ and

$$J(x_{k_j}) \le J(\mu_{k_j}) \quad (j = 1, \dots). \tag{34}$$

By Lemma 5, $|F(\mu_{k_j})|_Y \to 0$. Due to the weak continuity of F, we have

$$0 = \lim_{j\to\infty} |F(\mu_{k_j})|_Y = \lim_{j\to\infty} |F(x_{k_j})|_Y \ge |F(x)|_Y.$$

Thus x is admissible in problem (1). We have $J(x_{k_j}) \to J(x)$ by the continuity of J and $\hat{J}^0 = J^0$ by Theorem 1. Then, in view of (34) and (33),

$$J(x) = \lim_{j\to\infty} J(x_{k_j}) \le \lim_{j\to\infty} J(\mu_{k_j}) = \hat{J}^0 = J^0.$$

Consequently, x is a solution to problem (1), $x \in X^0$. We showed that (x_k) converges to X^0 in X. This convergence and the fact that $J(x^0) = J^0$ for all $x^0 \in X^0$ lead easily to the convergence $J(x_k) \to J^0$. Finally, $J(\mu_k) \to J^0 = \hat{J}^0$ by Theorem 3. The proof is complete. ■

Now we provide a more constructive form of the suggested solution approximation method for the original problem (1). Namely, we specify the definition of the sequence (x_k) described in Theorem 3 in terms of elements of X only, without employing any probability measures. The specification is based on the representation of the elements μ_k of the extended extemal shift sequence as finite convex combinations of point-concentrated probability merasures. We achieve such a representation if we define the "target" measures ν_{k+1} in (16) to be point-concentrated.

Thus we define an extended extremal shift sequence (μ_k) of a particular form. In what follows, we use notation $\mu(x)$ for the probability measure from $\mathrm{pm}(X)$ which is concentrated at an $x \in X$.

We fix arbitrary $v_0 \in X$ and set

$$\mu_0 = \mu(v_0). \tag{35}$$

If the k-th approximation, $\mu_k \in \mathrm{pm}(X)$, is found $(k \in \{0, 1, \dots\})$, we define μ_{k+1} by

$$\mu_{k+1} = (1 - \delta_{k+1})\mu_k + \delta_{k+1}\nu_{k+1}, \tag{36}$$

where

$$\nu_{k+1} = \mu(v_{k+1}) \tag{37}$$

and v_{k+1} is a σ_{k+1}-minimizer of

$$\varphi_k(v) = 2(1 - \delta_{k+1})\langle F(\mu_k), F(v)\rangle_Y + \frac{J(v)}{\tau_{k+1}} \quad (v \in X) \tag{38}$$

in X,

$$v_{k+1} \in X, \quad \varphi_k(v)(v_{k+1}) \le \inf\{\varphi_k(v) \colon \ v \in X\}. \tag{39}$$

Remark 14. Since F is continuous, the function $\varphi_k(\cdot)$ is continuous on compact X. Therefore it is admissible to assume $\sigma_k = 0$ $(k = 1, \dots)$.

The given definition of the sequence (μ_i) differs from the definition (13)–(16) on two details: the initial element μ_0 (13) is determined by (35) and the "target" elements by $\nu_{k+1} = \mu(v_{k+1})$ instead of (16). However, the following is true:

Lemma 6. *Let X be a compactum and J be continuous. Then every sequence (μ_k) satisfying (35)–(39) is an extended extremal shift sequence.*

Proof. Let (μ_k) be defined by (35)–(39). We must show that (μ_k) satisfies (13)–(16). Obviously, μ_0 in (35) satisfies (13). Given μ_k $(k \in \{0, 1, \dots\})$, μ_{k+1} is determined by (36), which is the same as (14). Furthermore, note that the minimum value of $\hat{\varphi}_k(\cdot)$ in (15) over $\mathrm{pm}(X)$ coincides with the minimum value of $\varphi_k(\cdot)$ in (38) over X. Hence $\nu_{k+1} = \mu(v_{k+1})$ with v_{k+1} given by (39) minimizes $\hat{\varphi}_k(\cdot)$ in $\mathrm{pm}(X)$, i.e. it satisfies (16). Thus (μ_k) is an extended extremal shift sequence. ■

The k-th element μ_k of the extremal shift sequence given by (35)–(39) is a convex combination of the point-concentrated probability measures $\nu_i = \mu(v_i)$ $(i = 1, \dots, k)$. Namely, the next observation follows from (35)–(39) in a straightforward manner.

Remark 15. Let (μ_k) be an extended extremal shift sequence defined by (35)–(39). Then $\mu_k = \sum_{i=0}^{k-1} \alpha_{ki}\mu(v_i)$ $(k = 1, \dots)$, where $\alpha_{00} = 1$, $\alpha_{k+1\ k+1} = \delta_{k+1}$, $\alpha_{k+1\ i} = \alpha_{ki}(1 - \delta_{k+1})$ $(i = 0, \dots, k)$.

Let (μ_k) be an extended extremal shift sequence defined by (35)–(39). Write $F_k = F(\mu_k)$ and $J_k = J(\mu_k)$ $(k = 0, 1, \dots)$. By (35) we have $F_0 = F(u_0)$ and $J_0 = J(u_0)$; (36) and (37) imply the recurrent formulas

$$F_{k+1} = (1 - \delta_{k+1})F_k + \delta_{k+1}F(v_{k+1}),$$

$$J_{k+1} = (1 - \delta_{k+1})J_k + \delta_{k+1}J(v_{k+1}).$$

The values $\varphi_k(v)$ in (38) are represented as

$$\psi_k(v) = 2(1 - \delta_{k+1})\langle f_k, F(v)\rangle_Y + \frac{J(v)}{\tau_{k+1}}.$$

The inclusion $x_k \in r(\mu_k)$ that determines the k-th approximation to a solution (see Theorem 3 and Definition (9)) can be rewritten as $x_k \in R_k$, where $R_k = \{x \in X : F(x) = F_k,\ J(x) \le J_k\}$ $(k = 0, 1, \dots)$.

Instead of the extended extremal shift sequences (μ_k) (35)–(39) taking values in $\mathrm{pm}(X)$, we will deal with the associated sequences $((v_k, F_k, J_k, x_k))$ taking values in $X \times Y \times \mathbb{R}^1 \times X$. The resulting algorithm for constructing $((v_k, F_k, J_k, x_k))$ is as follows. We set

$$v_0 \in X, \quad F_0 = F(v_0), \quad J_0 = J(v_0), \quad x_0 \in R_0, \tag{40}$$

where

$$R_0 = \{x \in X\colon\ F(x) = F_0,\ J(x) \le J_0\}. \tag{41}$$

Given $v_k \in X$, $F_k \in Y$, $J_k \in \mathbb{R}^1$ and $x_k \in X$, where $k \in \{0, 1 \ldots\}$, we find

$$v_{k+1} \in X, \quad \psi_k(v_{k+1}) \le \inf\{\psi_k(v)\colon \ v \in X\} + \sigma_{k+1}, \tag{42}$$

where

$$\psi_k(v) = 2(1 - \delta_{k+1})\langle F_k, F(v)\rangle_Y + \frac{J(v)}{\tau_{k+1}}. \tag{43}$$

Compute

$$F_{k+1} = (1 - \delta_{k+1})F_k + \delta_{k+1}F(v_{k+1}),$$
$$J_{k+1} = (1 - \delta_{k+1})J_k + \delta_{k+1}J(v_{k+1}), \tag{44}$$

and find

$$x_{k+1} \in R_{k+1} = \{x \in X\colon \ F(x) = F_{k+1}, \ J(x) \le J_{k+1}\}. \tag{45}$$

Every sequence $((v_k, F_k, J_k, x_k))$ in $X \times Y \times \mathbb{R}^1 \times X$ satisfying (40)–(45) will be called an *extremal shift sequence.*

Remark 16. Similarly to Remark 14, we note that the case $\sigma_k = 0$ $(k = 1, \ldots)$ is admissible.

The next theorem is a specification of Theorem 3.

Theorem 4. *Let X be a compactum, J be continuous, the epigraph E be convex and $((v_k, F_k, J_k, x_k))$ be an extremal shift sequence. Then the sequence (x_k) converges to the solution set X^0 of the original problem (8), and each of the sequences (J_k) and $(J(x_k))$ converges to the optimal value J^0 in this problem.*

Proof. We reverse the argument that led us to the definition of the sequence $((v_k, F_k, J_k, x_k))$. Namely, define sequences (μ_k) and (ν_k) in $\mathrm{pm}(X)$ by (35)–(37). Obviously, $F_k = F(\mu_k)$ and $J_k = J(\mu_k)$. Hence v_{k+1} given by (42), (43) satisfies (39), (38). By Lemma 6, (μ_k) is an extended extremal shift sequence. Then, by Theorem 3, the sequence (x_k) converges to X^0, and each of the sequences $(J(\mu_k)) = (J_k)$ and $(J(x_k))$ converges to J^0. The theorem is proved. ∎

5. General Case. Solution Method

Now, assuming that the epigraph E is convex, we extend the field of the applicability of the method described in Section 4. We no longer assume that the metric space X is a compactum and the objective function J is continuous. Recall that those assumptions (and the weak continuity of F) allowed us to employ a standard randomization technique and reduce problem (1) to a problem of convex optimization. Here, we make a weaker assumption that does not allow us to use the randomization technique in full. However, its basic elements still work.

We call a fuction $G\colon \ X \mapsto Y$ a *compactifier* if every sequence (x_k) in X such that $|G(x_k)|_Y \to 0$ is compact in X (i.e. (x_k) has an accumulation point in X).

Here, we assume that F is a compactifier. (Recall that F is bounded and weakly continuous, and J is bounded and lower semicontinuous; see the Introduction, assumptions (i) and (ii)).

Remark 17. If X is a compactum, then F is a compactifier.

Lemma 7. *Let F be a compactifier. Then*

(i) *the admissible set in problem (1), $\{x \in X\colon\ F(x) = 0\}$, is a compactum;*

(ii) *the solution set in problem (1), X^0, is a nonempty compactum;*

(iii) *every sequence (x_k) in X such that $|F(x_i)|_Y \to 0$ is compact in X; it converges to X^0 in X if and only if X^0 contains every accumulation point of (x_k).*

Proof. Statement (i) follows from the definition of a compactifier. Statement (ii) follows from statement (i), the weak continuity of F and the lower semicontinuity of J. Finally, statement (iii) follows from the definition of a compactifier and the fact that X^0 is a compactum. ■

Our goal is to show that the solution method described in Section 4 is still applicable. Essentially, we argue as in Sections 2, 3 and 4. Again, we start with the formulation of an extension of problem (1). However, unlike in our previous argument, we do not extend domain X to the set $\mathrm{pm}(X)$ of all Borel probability measures on X. For an extension of X, we take the set of all finite convex combinations of point-concentrated probability measures on X. The advantage of this (weaker) extension is that F and J are integrable with respect to any finite convex combination of point-concentrated probability measures on X.

Let us give accurate definitions. As previously, $\mathrm{pm}(X)$ stands for the set of all Borel probability measures on X, and $\mu(x)$ denotes the Borel probability measure on X which is concentrated at an $x \in X$: $\mu(\{x\}) = 1$. We use the notation $\mathrm{cpm}(X)$ for the set of all finite convex combinations of point-concentrated measures from $\mathrm{pm}(X)$. More precisely, $\mathrm{cpm}(X)$ is the set of all $\mu \in \mathrm{pm}(X)$ of the form

$$\mu = \sum_{i=1}^{k} \alpha_i \mu_i(x_i), \tag{46}$$

where $k \in \{1, \dots\}$, $x_i \in X$, $\alpha_i \geq 0$ $(i = 1, \dots, k)$ and $\sum_{i=1}^{k} \alpha_i = 1$. For every $\mu \in \mathrm{cmp}(X)$ of form (46), functions F and J are μ-integrable and

$$\int_X F(x)\,\mu(\mathrm{d}x) = \sum_{i=1}^{k} \alpha_i F(x_i), \qquad \int_X J(x)\,\mu(\mathrm{d}x) = \sum_{i=1}^{k} \alpha_i J(x_i)$$

(the fact that F is a compactifier does not matter here). As earlier, we write

$$F(\mu) = \int_X F(x)\,\mu(\mathrm{d}x), \quad J(\mu) = \int_X J(x)\,\mu(\mathrm{d}x)$$

and define the set $r(\mu)$ by (9). The following counterpart of Lemma 2 holds obviously.

Lemma 8. *Let the epigraph E be convex. Then for every $\mu \in \operatorname{cpm}(X)$, the set $r(\mu)$ is nonempty.*

Introduce the *weakly extended* problem

$$\begin{aligned} &\text{minimize } J(\mu),\\ &\mu \in \operatorname{cpm}(X),\\ &F(\mu) = 0. \end{aligned} \tag{47}$$

We denote by $\bar{J}$ the optimal value in the weakly extended problem (47). Note that the weakly extended problem (47) may not have a solution (unlike the extended problem (8) in the case where X is a compactum and J is continuous). However, Lemma 8 yields the following equivalence theorem.

Theorem 5. *Let the epigraph E be convex. Then problems (1) and (47) are equivalent in the following sense:*

(i) *the optimal values in the original problem (1) and the weakly extended problem (47) coincide, $J^0 = \bar{J}^0$;*

(ii) *if x^0 solves the original problem (1), $x^0 \in X^0$, then $\mu(x^0)$ solves the weakly extended problem (47).*

Similarly to Section 3, we define a successive solution approximation method for the weakly extended problem (47). Again, we fix positive $\delta_1, \delta_2, \ldots$ less than 1 and such that for τ_k given by (10) $(k = 1, \ldots)$ relations (11) hold. We also fix positive $\sigma_1, \sigma_2, \ldots$ satisfying (12).

Now we repeat the definition of an extended extremal shift sequence (see (13)–(16)), where we replace $\operatorname{pm}(X)$ by $\operatorname{cpm}(X)$. Take

$$\mu_0 \in \operatorname{cpm}(X). \tag{48}$$

Given a $\mu_k \in \operatorname{cmp}(X)$, $(k \in \{0, 1, \ldots\})$, set

$$\mu_{k+1} = (1 - \delta_{k+1})\mu_k + \delta_{k+1}\nu_{k+1}, \tag{49}$$

where ν_{k+1} is a σ_{k+1}-minimizer of

$$\bar{\varphi}_k(\nu) = 2(1 - \delta_{k+1})\langle F(\mu_k), F(\nu)\rangle_Y + \frac{J(\nu)}{\tau_{k+1}} \quad (\nu \in \operatorname{cpm}(X)) \tag{50}$$

in $\operatorname{cpm}(X)$:

$$\nu_{k+1} \in \operatorname{cpm}(X),\ \bar{\varphi}_k(\nu_{k+1}) \le \inf\{\bar{\varphi}_k(\nu)\colon\ \nu \in \operatorname{cpm}(X)\} + \sigma_{k+1}. \tag{51}$$

Every sequence (μ_k) satisfying (48)–(51) will be called a *weakly extended extremal shift sequence.*

Remark 18. Observing the definition of ν_k, (51), we notice that the continuous function $\bar{\varphi}_k(\cdot)$ may have no minimizers in X if X is noncompact. Therefore, unlike in the situation treated in Section 3 (see Remark 8), we do not admit $\sigma_k = 0$ $(k = 1, \dots)$.

Theorem 6. *Let F be a compactifier, the epigraph E be convex, (μ_k) be a weakly extended extremal shift sequence and $x_k \in r(\mu_k)$ $(k = 0, 1, \dots)$. Then the sequence (x_k) converges in X to the solution set X^0 of the original problem (8), and the sequences $(J(\mu_k))$ and $(J(x_k))$ converge to the optimal value J^0 in this problem.*

Remark 19. The definition of x_k is correct, since by Lemma 8, $r(\mu_k)$ is nonempty $(k = 0, 1, \dots)$.

The proof of Theorem 6 follows, generally, the proof of Theorem 2. The next lemma is a counterpart of Lemma 3.

Lemma 9. *Let F be a compactifier, the epigraph E be convex, (μ_k) be a sequence in* $\mathrm{cpm}(X)$,

$$|F(\mu_k)|_Y \to 0, \tag{52}$$

$$\limsup_{k\to\infty} J(\mu_k) \le J^0, \tag{53}$$

and $x_k \in r(\mu_k)$ $(k = 0, 1, \dots)$. Then the sequence (x_k) converges to X^0 in X, and the sequences $(J(\mu_k))$ and $(J(x_k))$ converge to J^0.

Proof. From (9) and inclusion $x_k \in r(\mu_k)$, we have $F(x_k) = F(\mu_k)$ $(k = 0, 1, \dots)$. Hence, by (52), $|F(x_k)|_Y \to 0$. Then by Lemma 7, (iii), the sequence (x_k) is compact in X, and (x_k) converges to X^0 in X if and only if X^0 contains every accumulation point of (x_k). Let $x \in X$ be an arbitrary accumulation point of (x_k), i.e. there is a subsequence (x_{k_j}) convergent to x in X. Let us state that $x \in X^0$, which will prove that (x_k) converges to X^0. The weak continuity of F, convergence $x_{k_j} \to x$ (in X) and convergence $|F(x_{k_j})|_Y \to 0$ imply $F(x) = 0$. Hence, x is admissible in problem (1). By (9) and inclusion $x_{k_j} \in r(\mu_{k_j})$, we have $J(x_{k_j}) \le J(\mu_{k_j})$ $(j = 1, \dots)$. Then (53) and the lower semicontinuity of J yield $J(x) \le J^0$. Since x is admissible in problem (1), we conclude that $x \in X^0$. We proved that (x_k) converges to X^0 in X. Now let a subsequence (x_{k_j}) be chosen so that

$$\lim_{k_j\to\infty} J(\mu_{k_j}) = \liminf_{k\to\infty} J(\mu_k).$$

Then using the lower semicontinuity of J, inequalities $J(x_{k_j}) \le J(\mu_{k_j})$ $(j = 1, \dots)$ and relation (53), we get

$$J^0 = J(x) \le \liminf_{k_j\to\infty} J(x_{k_j}) \le \lim_{k_j\to\infty} J(\mu_{k_j}) = \liminf_{k\to\infty} J(\mu_k)$$

$$\le \limsup_{k\to\infty} J(\mu_k) \le J^0.$$

Hence $J(\mu_k) \to J^0$. Furthermore,

$$J^0 = J(x) \le \liminf_{k_j \to \infty} J(x_{k_j}) \le \limsup_{k \to \infty} J(x_k) \le \limsup_{k \to \infty} J(\mu_k) = J^0.$$

Hence $J(x_k) \to J^0$, and the lemma follows. ■

The next lemma is a counterpart of Lemma 5.

Lemma 10. *Let F be a compactifier. Then for every extended extremal shift sequence (μ_k), relations (52) and (53) hold.*

We do not give the proof of Lemma 10, which is identical to the proof of Lemma 5. There are only two details on which the proofs differ. First, everywhere in the proof (starting from (26) defining index λ_k) we replace $\hat{J}^0$ by J^0. Second, while estimating α_k, we replace μ^0, a solution to the extended problem (8), by x^0, a solution to the original problem (1).

Proof of Theorem 6. Let the assumptions of Theorem 6 be satisfied and (ν_k) be the sequence associated with (μ_k) through (51) $(k = 0, 1, \dots)$. Then the assumptions of Lemma 10 are satisfied. By Lemma 10, the conditions (17) and (18) of Lemma 9 hold. Hence, the statement of Lemma 9 holds true. The theorem is proved. ■

Now we define a weakly extended extremal shift sequence of a particular form. We repeat the definition given in Section 4. Namely, we fix arbitrary $v_0 \in X$ and define μ_0 by (35). If $\mu_k \in \mathrm{cpm}(X)$ is found $(k \in \{0, 1, \dots\})$, we define μ_{k+1} by (36)–(39). The next lemma is a reformulation of Lemma 6.

Lemma 11. *A sequence (μ_k) defined by (35)–(39) is a weakly extended extremal shift sequence.*

We omit the proof of Lemma 11, which is identical to the proof of Lemma 6.

Like in Section 4, we associate a weakly extended extremal shift sequence (μ_k) defined by (35)–(39) with a sequence in $X \times Y \times \mathbb{R}^1 \times X$. Namely, every sequence $((v_k, F_k, J_k, x_k))$ in $X \times Y \times \mathbb{R}^1 \times X$ that satisfies (40)–(45) will be called an *extremal shift sequence* as earlier.

The next theorem is a counterpart of Theorem 4 and a specification of Theorem 6.

Theorem 7. *Let F be a compactifier, the epigraph E be convex and $((v_k, F_k, J_k, x_k))$ be an extremal shift sequence. Then the sequence (x_k) converges to the solution set X^0 of the original problem (1), and each of the sequences (J_k) and $(J(x_k))$ converges to the optimal value J^0 in this problem.*

Proof. Define sequences (μ_k) and (ν_k) in $\mathrm{pm}(X)$ by (35)–(37). Obviously, $F_k = F(\mu_k)$ and $J_k = J(\mu_k)$. Hence v_{k+1} given by (42), (43) satisfies (39), (38). By Lemma (11), (μ_k) is a weakly extended extremal shift sequence. Then, by Theorem 6, the sequence (x_k) converges to X^0, and each of the sequences $(J(\mu_k)) = (J_k)$ and $(J(x_k))$ converges to J^0. This is the desired conclusion. ■

6. Application to Optimal Control (a Bilinear System)

Now we apply the solution method described in the previous section to an optimal control problem (Pontryagin *et al.*, 1969) for the n-dimensional bilinear system

$$\dot{z}^{(i)}(t) = \sum_{j=1}^{n}[a_{ij}(t)z^{(j)}(t) + b_{ij}(t)u^{(j)}(t)z^{(j)}(t)] \quad (i = 1,\dots,n). \tag{54}$$

The system operates on a bounded time interval $[t_0,\vartheta]$ under pointwise constraints on controls and states:

$$u^{(i)}(t) \in U = \Pi_{i=1}^{n}[u_-^{(i)}, u_+^{(i)}],$$

$$z(t) = (z^{(1)}(t),\dots,z^{(n)}(t)) \in Z \quad (t \in [t_0,\vartheta]). \tag{55}$$

Here $u_-^{(i)} \le u_+^{(i)}$ $(i = 1,\dots,n)$ and Z is a convex compactum in $\mathbb{R}^n$. The system's initial state $\bar{z} = (\bar{z}^{(1)},\dots,\bar{z}^{(n)}) \in \mathbb{R}^n$ is fixed:

$$z(t_0) = \bar{z}. \tag{56}$$

In (54), $a_{ij}(\cdot)$ and $b_{ij}(\cdot)$ $(i,j = 1,\dots,n)$ are bounded measurable scalar functions on $[t_0,\vartheta]$. Introducing the notation

$$f_i(t,u,z) = \sum_{j=1}^{n}[a_{ij}(t)z^{(j)} + b_{ij}(t)u^{(j)}z^{(j)}] \quad (i = 1,\dots,n), \tag{57}$$

$$f(t,u,z) = (f_1(t,u,z),\dots,f_n(t,u,z)), \tag{58}$$

$$(u = (u^{(1)},\dots,u^{(n)}) \in \mathbb{R}^n, \quad z = (z^{(j)},\dots,z^{(j)}) \in \mathbb{R}^n), \tag{59}$$

we rewrite (54) as

$$\dot{z}(t) = f(t,u(t),z(t)). \tag{60}$$

Any measurable function $u(\cdot) : t \mapsto u(t) = (u^{(1)}(t),\dots,u^{(j)}(t)) : [t_0,\vartheta] \mapsto U$ is called the *control*. A *motion* corresponding to a control $u(\cdot)$ is defined to be a Caratheodory solution $z(\cdot) : t \mapsto z(t) = (z^{(j)}(t),\dots,z^{(j)}(t)) : [t_0,\vartheta] \mapsto \mathbb{R}^n$ of the Cauchy problem (60), (56) on $[t_0,\vartheta]$. For every control $u(\cdot)$ there exists a unique motion corresponding to $u(\cdot)$. A pair $(u(\cdot),z(\cdot))$ where $u(\cdot)$ is a control and $z(\cdot)$ is the motion corresponding to $u(\cdot)$ is called the *control process*. A control process $(u(\cdot),z(\cdot))$ is said to be *admissible* if $z(t) \in Z$ for all $t \in [t_0,\vartheta]$. The set of all admissible control processes will further be denoted by $\mathcal{P}$. We assume that $\mathcal{P}$ is nonempty.

Let $\omega(\cdot)\colon (t,u,z) \mapsto \omega(t,u,z)$ be a bounded scalar function on $[t_0,\vartheta]\times\mathbb{R}^n\times\mathbb{R}^n$ such that for every $t \in [t_0,\vartheta]$ the function $(u,z) \mapsto \omega(t,u,z)$ is continuous, for every $(u,z) \in \mathbb{R}^n \times \mathbb{R}^n$ the function $t \mapsto \omega(t,u,z)$ is measurable, and for every $(t,z) \in [t_0,\vartheta] \times \mathbb{R}^n$ the function $u \mapsto \omega(t,u,z)$ is convex. For every control process $(u(\cdot),z(\cdot))$ we set

$$I(u(\cdot),z(\cdot)) = \int_{t_0}^{\vartheta} \omega(t,u(t),z(t))\,\mathrm{d}t. \tag{61}$$

The optimal control problem under consideration is the following:

$$\begin{aligned} &\text{minimize } I(u(\cdot), z(\cdot)),\\ &\quad (u(\cdot), z(\cdot)) \in \mathcal{P}. \end{aligned} \tag{62}$$

Every solution to (62) is called an *optimal control process*. We denote by $\mathcal{P}^*$ the set of all optimal control processes in (62), and by J^* the optimal value in problem (62).

Let us reduce problem (62) to form (1) (the application of the proposed solution method will require some further assumptions which will be formulated later). We use notation L_n^2 for the Hilbert space $L^2([t_0, \vartheta], \mathbb{R}^n)$ equipped with the standard scalar product $\langle \cdot, \cdot \rangle_{L_n^2}$ and the (strong) norm $| \cdot |_{L_n^2}$ (Warga, 1975, I.5.B). We use notation $L_{n,w}^2$ for the space L_n^2 equipped with the weak norm $| \cdot |_{L_{n,w}^2}$. In what follows, $\mathcal{U}$ denotes the set of all controls, and $\mathcal{Z}$ the set of all $z(\cdot) \in L_n^2$ such that $z(t) \in Z(t)$ for all $t \in [t_0, \vartheta]$.

Remark 20. Obviously, $\mathcal{U}$ and $\mathcal{Z}$ are convex weak compacta in L_n^2.

We set

$$X = \mathcal{U} \times \mathcal{Z} \tag{63}$$

and treat X as a metric subspace of $L_{n,w}^2 \times L_n^2$. (Thus the distance between the controls is measured with respect to the weak norm $|\cdot|_{L_{n,w}^2}$, and the distance between motions is measured with respect to the strong norm $| \cdot |_{L_n^2}$.) Let $Y = L_n^2$. We define F: $X \mapsto Y$ by

$$F(x)(t) = z(t) - \bar{z} - \int_{t_0}^{t} f(s, u(s), z(s))\, \mathrm{d}s \tag{64}$$

$$(t \in [t_0, \vartheta], \quad x = (u(\cdot), z(\cdot)) \in X)$$

(see (57) and (58)), and J: $X \mapsto \mathbb{R}^1$ by

$$J(x) = \int_{t_0}^{\vartheta} \omega(t, u(t), z(t)) dt \tag{65}$$

$$(x = (u(\cdot), z(\cdot)) \in X).$$

In this section we deal with problem (1) with X, F and J given by (63)–(65).

Remark 21. For problem (1), assumptions (i) and (ii) given in the Introduction are satisfied. Indeed, function J is, clearly, bounded; the lower semicontinuity of J follows from the continuity of the functions $z \mapsto \omega(t, u, z)$ and $z \mapsto |f(t, u, z)|^2$ and the convexity of the function $u \mapsto \omega(t, u, z)$. Finally, it is clear that F is bounded and weakly continuous (moreover, F is strongly continuous, i.e. continuous as a map from X to $Y = L_n^2$ equipped with the strong norm $| \cdot |_{L_n^2}$).

The following equivalence theorem is obvious.

Theorem 8. *The optimal control problem (62) is equivalent to (1) in the following sense:*

(i) *the optimal values in problems (62) and (1) coincide,* $J^* = J^0$*;*

(ii) *every optimal control process solves problem (1);*

(iii) *for every solution* $(x^0 = (u^0(\cdot), z^0(\cdot))$ *to problem (1) there exists an optimal control process* $(x^* = (u^*(\cdot), z^*(\cdot))$ *such that* $(u^*(t), z^*(t)) = (u^0(t), z^0(t))$ *for almost all* $t \in [t_0, \vartheta]$*.*

The next lemma justifies the applicability of the theory presented in the beginning of Section 4.

Lemma 12. *Function* F *is a compactifier.*

Proof. Let $(x_k) = ((u_k(\cdot), z_k(\cdot)))$ be a sequence in X such that $|F(x_k)|_Y \to 0$. By Remark 20, the set $\mathcal{U}$ is a weak compactum in L^2_n. Therefore the sequence $(u_k(\cdot))$ taking values in $\mathcal{U}$ has a subsequence $(u_{k_j}(\cdot))$ convergent to some $u(\cdot) \in \mathcal{U}$ weakly in L^2_n. Now it remains to establish that $(z_{k_j}(\cdot))$ (or its subsequence) converges to some $z(\cdot)$ strongly in L^2_n. By Remark 20, $\mathcal{Z}$ is a weak compactum in L^2_n. As long as $z_k(\cdot) \in \mathcal{Z}$ $(k = 1, \dots)$, we assume, with no loss of generality, that $(z_{k_j}(\cdot))$ converges to some $z(\cdot) \in \mathcal{Z}$ weakly in L^2_n. Let

$$g_{k_j}(t) = \bar{z} + \int_{t_0}^{t} f(s, u(s), z(s))\, \mathrm{d}s \quad (t \in [t_0, \vartheta]). \tag{66}$$

Obviously, sequence $(g_{k_j}(\cdot))$ is compact in $C_n = C([t_0, \vartheta], \mathbb{R}^n)$. With no loss of generality, we assume that $(g_{k_j}(\cdot))$ converges to some $g(\cdot)$ in C_n. Consequently, $g_{k_j}(\cdot) \to g(\cdot)$ strongly in L^2_n. By (64), we get

$$F(x_{k_j}) = F(u_{k_j}(\cdot), z_{k_j}(\cdot)) = z_{k_j}(\cdot) - g_{k_j}(\cdot).$$

Since $z_{k_j}(\cdot) \to z(\cdot)$ weakly in L^2_n, and $g_{k_j}(\cdot) \to g(\cdot)$ strongly in L^2_n,

$$F(x_{k_j}) = z_{k_j}(\cdot) - g_{k_j}(\cdot) \to z(\cdot) - g(\cdot) \quad \text{weakly in} \quad L^2_n.$$

By assumption, $F(x_{k_j}) \to 0$ strongly in L^2_n. Therefore

$$z_{k_j}(\cdot) - g_{k_j}(\cdot) \to z(\cdot) - g(\cdot) \quad \text{strongly in} \quad L^2_n.$$

Recalling that $g_{k_j}(\cdot) \to g(\cdot)$ strongly in L^2_n, and $z_{k_j}(\cdot) \to z(\cdot)$ weakly in L^2_n, we conclude that $z_{k_j}(\cdot) \to z(\cdot) = g(\cdot)$ strongly in L^2_n. The proof is complete. ■

Lemmas 12, 7(ii) and 8 yield the following results:

Corollary 1. *The solution set of problem (1),* X^0*, and the set of all optimal control processes,* $\mathcal{P}^*$*, are nonempty.*

By Theorem 7, the concavity of the epigraph E in (2) insures that every extremal shift sequence provides convergent approximations to X^0 and J^0 (equivalently, to $\mathcal{P}^*$ and J^*). Now our goal is to show that the condition given below is sufficient for the convexity of E.

Condition 1. The following assumptions hold:

(i) $u_-^{(i)} > 0$ and $z_-^{(i)} > 0$ $(i = 1, \dots, n)$ where $z_-^{(i)} = \inf\{z^{(i)}:\ z \in Z\}$;

(ii) $\omega(\cdot)$ has the form

$$\omega(t, u, z) = \sum_{i=1}^{n} [c_i(t) u^{(i)2} + h(t, z^{(i)})] \quad (t \in [t_0, \vartheta],\ u, z \in \mathbb{R}^n), \tag{67}$$

where, for every $i \in \{1, \dots, n\}$, $c_i(\cdot)$ is a bounded positive measurable function on $[t_0, \vartheta]$, and $h_i(\cdot)$ is a bounded scalar function on $[t_0, \vartheta] \times \mathbb{R}^1$ such that $t \mapsto h_i(t, z^{(i)})$ is measurable for each $z^{(i)} \in \mathbb{R}^1$ and $z^{(i)} \mapsto h_i(t, z^{(i)})$ is continuous for each $t \in [t_0, \vartheta]$;

(iii) for every $i \in \{1, \dots, n\}$ and every $t \in [t_0, \vartheta]$, the function $z^{(i)} \mapsto h_i(t, z^{(i)})$ is twice continuously differentiable and

$$\inf\{h_i''(t, z^{(i)}) z^{(i)2}:\ z^{(i)} \in [z_-^{(i)}, z_+^{(i)}])\} \geq 2 c_i(t) u_+^{(i)+}, \tag{68}$$

where

$$z_+^{(i)} = \sup\{z^{(i)} : z \in Z\}, \quad h_i''(t, z^{(i)}) = \frac{\partial h_i(t, z^{(i)})}{\partial z^{(i)2}}.$$

Theorem 9. *Let Condition 1 be satisfied. Then the epigraph E of (2) in problem (1) is convex.*

Proof. Let $(w_1, y_1), (w_2, y_2) \in E$, i.e. $w_1 = F(x_1)$, $w_2 = F(x_2)$ and $y_1 \geq J(x_1)$, $y_2 \geq J(x_2)$ for some $x_1 = (u_1(\cdot), z_1(\cdot)) \in X$ and $x_2 = (u_2(\cdot), z_2(\cdot)) \in X$. Let $\lambda \in [0, 1]$. We must show that $(\lambda w_1 + (1-\lambda) w_2,\ \lambda y_1 + (1-\lambda) y_2) \in E$. It is sufficient to establish that there exists an $x = (u(\cdot), z(\cdot)) \in X$ such that

$$F(x) = \lambda F(x_1) + (1-\lambda) F(x_2) \tag{69}$$

and

$$J(x) \leq \lambda J(x_1) + (1-\lambda) J(x_2). \tag{70}$$

Define $x = (u(\cdot), z(\cdot))$ by

$$u^{(i)}(t) = \frac{\lambda u_1^{(i)}(t) z_1^{(i)}(t) + (1-\lambda) u_2^{(i)}(t) z_2^{(i)}(t)}{\lambda z_1^{(i)}(t) + (1-\lambda) z_2^{(i)}(t)}, \tag{71}$$

$$z^{(i)}(t) = \lambda z_1^{(i)}(t) + (1-\lambda) z_2^{(i)}(t) \tag{72}$$

($i = 1, \ldots, n,\ t \in [t_0, \vartheta]$). Assumption (ii) in Condition 1 implies that for every $t \in [t_0, \vartheta]$, $u^{(i)}(t)$ is a convex combination of $u_1^{(i)}(t)$ and $u_2^{(i)}(t)$, belonging to $[u_-^{(i)}, u_+^{(i)}]$. Hence $u^{(i)}(\cdot)$ takes values in $[u_-^{(i)}, u_+^{(i)}]$ $(i = 1, \ldots, n)$. Therefore $u(\cdot) \in \mathcal{U}$. We have $z(\cdot) \in \mathcal{Z}$ since $z_1(\cdot), z_2(\cdot) \in \mathcal{Z}$ and $\mathcal{Z}$ is convex. Thus $x = (u(\cdot), z(\cdot)) \in \mathcal{U} \times \mathcal{Z} = X$. Let

$$q_1^{(i)}(t) = u_1^{(i)}(t) z_1^{(i)}(t), \quad q_2^{(i)}(t) = u_2^{(i)}(t) z_2^{(i)}(t), \quad q^{(i)}(t) = u^{(i)}(t) z^{(i)}(t) \tag{73}$$

($i = 1, \ldots, n,\ t \in [t_0, \vartheta]$). From (64), (58) and (57) we have

$$(F(x_1)(t))^{(i)} = z_1^{(i)}(t) - \bar{z}_1^{(i)} - \int_{t_0}^{t} \sum_{j=1}^{n} [a_{ij}(s) z_1^{(j)} + b_{ij}(s) q_1^{(j)}(s)]\, \mathrm{d}s,$$

$$(F(x_2)(t))^{(i)} = z_2^{(i)}(s) - \bar{z}_2^{(i)} - \int_{t_0}^{t} \sum_{j=1}^{n} [a_{ij}(s) z_2^{(j)} + b_{ij}(s) q_2^{(j)}(s)]\, \mathrm{d}s,$$

$$(F(x)(t))^{(i)} = z^{(i)}(s) - \bar{z}^{(i)} - \int_{t_0}^{t} \sum_{j=1}^{n} [a_{ij}(s) z^{(j)} + b_{ij}(s) q^{(j)}(s)]\, \mathrm{d}s$$

($i = 1, \ldots, n,\ t \in [t_0, \vartheta]$). From (71)

$$q^{(i)}(\cdot) = \lambda q_1^{(i)}(\cdot) + (1-\lambda) q_2^{(i)}(\cdot) \quad (i = 1, \ldots, n). \tag{74}$$

Then, using (72), we see that

$$(F(x)(t))^{(i)} = \lambda (F(x_1)(t))^{(i)} + (1-\lambda)(F(x_2)(t))^{(i)}$$

($i = 1, \ldots, n,\ t \in [t_0, \vartheta]$). Thus (69) holds.

Let us prove (70). Fix arbitrary $i \in \{1, \ldots, n\}$ and arbitrary $t \in [t_0, \vartheta]$. We have (see Condition 1, (ii))

$$c_i(t) u_1^{(i)2}(t) + h(t, z_1^{(i)}(t)) = l(z_1^{(i)}(t), q_1^{(i)}(t)),$$

$$c_i(t) u_2^{(i)2}(t) + h(t, z_2^{(i)}(t)) = l(z_2^{(i)}(t), q_2^{(i)}(t)),$$

$$c_i(t) u^{(i)2}(t) + h(t, z^{(i)}(t)) = l(z^{(i)}(t), q^{(i)}(t)),$$

where

$$l(p, r) = c_i(t) \frac{r^2}{p^2} + h(t, p).$$

For arbitrary positive r and p the matrix of the second-order derivatives of $l(\cdot)$ at (r, p) has the form

$$D(p, r) = \begin{pmatrix} \dfrac{\partial^2 l(p,r)}{\partial p^2} & \dfrac{\partial^2 l(p,r)}{\partial p \partial r} \\ \dfrac{\partial^2 l(p,r)}{\partial r \partial p} & \dfrac{\partial^2 l(p,r)}{\partial r^2} \end{pmatrix} = \begin{pmatrix} 6c_i(t)\dfrac{r^2}{p^4} + h_i''(t,p) & -4c_i(t)\dfrac{r}{p^3} \\ -4c_i(t)\dfrac{r}{p^2} & 2c_i(t)\dfrac{1}{p^2} \end{pmatrix}.$$

The diagonal minors of $D(p,r)$ are given by

$$d_1(p,r) = 2c_i(t)\frac{r}{p^3} + h_i''(t,p),$$

$$d_2(p,r) = \frac{2c_i(t)}{p^4}\left(h_i''(t,p)p^2 - c_i(t)\frac{r^2}{p^2}\right).$$

Let

$$H = \{(p,r)\colon\ p \in [z_-^{(i)}, z_+^{(i)}], r \in [u_-^{(i)}p, u_+^{(i)}p]\}.$$

For $(p,r) \in H$ we have

$$d_1(p,r) > 0, \quad d_2(p,r) \geq \frac{2c_i(t)}{p^4}(h_i''(t,p)p^2 - c_i(t)u_+^{(i)2}) \geq 0$$

owing to (68). Then for every $(p,r) \in H$, the matrix $D(p,r)$ is nonnegative definite. Therefore $l(\cdot)$ is convex in H. Noticing that $(z_1^{(i)}(t), q_1^{(i)}(t))$, $(z_2^{(i)}(t), q_2^{(i)}(t))$ and $(z^{(i)}(t), q^{(i)}(t))$ lie in H (see (73)), and recalling (74), (72), we deduce that

$$l(z^{(i)}(t), q^{(i)}(t)) \leq \lambda l(z_1^{(i)}(t), q_1^{(i)}(t)) + (1-\lambda) l(z_2^{(i)}(t), q_2^{(i)}(t)),$$

or

$$c_i(t)u^{(i)}(t) + h(t, z^{(i)}(t)) \leq \lambda[c_i(t)u_1^{(i)}(t) + h(t, z_1^{(i)}(t))] + (1-\lambda)[c_i(t)u_2^{(i)}(t) + h(t, z_2^{(i)}(t))].$$

This holds for every $i \in \{1,\ldots,n\}$ and every $t \in [t_0, \vartheta]$. Therefore, from (67) we get

$$\omega(t, u(t), z(t)) \leq \lambda\omega(t, u_1(t), z_1(t)) + (1-\lambda)\omega(t, u_2(t), z_2(t))$$

($t \in [t_0, \vartheta]$). Hence, from (65) we obtain

$$J(u(\cdot), z(\cdot)) \leq \lambda J(u_1(\cdot), z_1(\cdot)) + (1-\lambda) J(u_2(\cdot), z_2(\cdot)),$$

which is equivalent to (70). The proof is complete. ■

Applying Theorem 9 and Lemma 12 (saying that F is a compactifier), we conclude that under Condition 1, the assumptions of Theorem 7 hold. By this theorem, solutions to problem (1) (or, equivalently, the optimal control problem (62)) can be approximated by extremal shift sequences. Recall that in Section 5 we defined the extremal shift sequences $((v_k, F_k, J_k, x_k))$ by relations (35)–(39).

Let us specify the key relation (42) defining the correction term v_{k+1}. Denote by $\langle\cdot,\cdot\rangle$ the scalar product in $\mathbb{R}^n$. Using the form of F, cf. (64), and J, cf. (65), we represent $\psi_k(v)$ in (43) for $v=(v^u(\cdot),v^z(\cdot))\in\mathcal{U}\times\mathcal{Z}$ as follows:

$$\begin{aligned}
\psi_k(v) &= \psi_k(v^u(\cdot),v^z(\cdot))\\
&= 2(1-\delta_{k+1})\int_{t_0}^{\vartheta}\langle F_k(t),F(v^u(\cdot),v^z(\cdot))\rangle\,\mathrm{d}t+\frac{1}{\tau_{k+1}}\int_{t_0}^{\vartheta}\omega(t,v^u(t),v^z(t))\,\mathrm{d}t\\
&= 2(1-\delta_{k+1})\int_{t_0}^{\vartheta}\left\langle F_k(t),v^z(t)-\bar z-\int_{t_0}^{t}f(s,v^u(s),v^z(s))\,\mathrm{d}s\right\rangle\mathrm{d}t\\
&\quad+\frac{1}{\tau_{k+1}}\int_{t_0}^{\vartheta}\omega(t,v^u(t),v^z(t))\,\mathrm{d}t\\
&= 2(1-\delta_{k+1})\int_{t_0}^{\vartheta}\langle F_k(t),v^z(t)-\bar z\rangle\,\mathrm{d}t\\
&\quad-2(1-\delta_{k+1})\int_{t_0}^{\vartheta}\left\langle\int_{s}^{\vartheta}F_k(t)\,\mathrm{d}t,f(s,v^u(s),v^z(s))\right\rangle\mathrm{d}s\\
&\quad+\frac{1}{\tau_{k+1}}\int_{t_0}^{\vartheta}\omega(t,v^u(t),v^z(t))\,\mathrm{d}t\\
&= \int_{t_0}^{\vartheta}\Big[\langle\bar F_k(t),v^z(t)\rangle-\langle\bar G_k(t),f(t,v^u(t),v^z(t))\rangle\\
&\quad+\frac{1}{\tau_{k+1}}\omega(t,v^u(t),v^z(t))\Big]\,\mathrm{d}t-g_k,
\end{aligned}$$

where

$$\bar F_k(t)=2(1-\delta_{k+1})F_k(t),$$

$$\bar G_k(t)=2(1-\delta_{k+1})\int_{t}^{\vartheta}F_k(s)\,\mathrm{d}s,$$

$$g_k=2(1-\delta_{k+1})\int_{t_0}^{\vartheta}\langle F_k(t),\bar z\rangle\,\mathrm{d}t.$$

Setting

$$p_k(t,v^u,v^z)=\langle\bar F_k(t),v^z\rangle-\langle\bar G_k(t),f(t,v^u,v^z)\rangle+\frac{1}{\tau_{k+1}}\omega(t,v^u,v^z)$$

($t\in[t_0,\vartheta]$, $v^u,v^z\in\mathbb{R}^n$), we get

$$\psi_k(v^u(\cdot),v^z(\cdot))=\int_{t}^{\vartheta}p_k(t,v^u(t),v^z(t))\,\mathrm{d}t-g_k.$$

Recall that $\mathcal{U}$ is the set of all $u(\cdot) \in L_n^2$ such that $u(t) \in U = \Pi_{i=1}^n [u_-^{(i)}(t), u_+^{(i)}(t)]$ for all $t \in [t_0, \vartheta]$, and $\mathcal{Z}$ is the set of all $z(\cdot) \in L_n^2$ such that $z(t) \in Z$ for all $t \in [t_0, \vartheta]$. Hence

$$\inf\{\psi_k(v^u(\cdot), v^z(\cdot)):\ (v^u(\cdot), v^z(\cdot)) \in \mathcal{U} \times \mathcal{Z}\} = \int_t^\vartheta \bar{p}_k(t)\,\mathrm{d}t - g_k,$$

where

$$\bar{p}_k(t) = \min\{p_k(t, v^u, v^z) : (v^u, v^z) \in U \times Z\} \quad (t \in [t_0, \vartheta]).$$

Note that $p_k(\cdot)$ is bounded on $[t_0, \vartheta] \times U \times Z$, $t \mapsto p_k(t, v^u, v^z)$ is measurable, $(v^u, v^z) \mapsto p_k(t, v^u, v^z)$ and is continuous. Then $t \mapsto \bar{p}_k(t)$ is bounded and measurable (Warga, 1975, Theorem I.4.21), i.e. integrable. By the Filippov theorem (Warga, 1975, Theorem I.7.10), there exist measurable $v_{k+1}^u(\cdot):\ [t_0, \vartheta] \mapsto U$ and $v_{k+1}^z(\cdot):\ [t_0, \vartheta] \mapsto Z$ such that $p_k(t, v_{k+1}^u(t), v_{k+1}^z(t)) = \bar{p}_k(t)$ $(t \in [t_0, \vartheta])$. Obviously, $v_{k+1} = (v_{k+1}^u(\cdot), v_{k+1}^z(\cdot)) \in \mathcal{U} \times \mathcal{Z}$ satisfies (42). We established the definition of v_{k+1} in (42).

Algorithm (35)–(39) to construct an extremal shift sequence $((v_k, F_k, J_k, x_k))$ takes the following form: We set

$$v_0 = (v_0^u(\cdot), v_0^z(\cdot)) \in \mathcal{U} \times \mathcal{Z}, \quad F_0 = F(v_0), \quad J_0 = J(v_0),$$

$$x_0 = (u_0(\cdot), z_0(\cdot)) \in R_0, \tag{75}$$

where

$$R_0 = \{x \in \mathcal{U} \times \mathcal{Z} : F(x) = F_0,\ J(x) \le J_0\}. \tag{76}$$

Given $v_k = (v_k^u(\cdot), v_k^z(\cdot)) \in \mathcal{U} \times \mathcal{Z}$, $F_k \in Y = L_n^2$, $J_k \in \mathbb{R}^1$ and $x_k = (u_k(\cdot), z_k(\cdot)) \in \mathcal{U} \times \mathcal{Z}$ where $k \in \{0, 1, \dots\}$, we define $v_{k+1} = (v_{k+1}^u(\cdot), v_{k+1}^z(\cdot)) \in \mathcal{U} \times \mathcal{Z}$ by

$$p_k(t, v_{k+1}^u(t), v_{k+1}^z(t)) = \min\{p_k(t, v^u, v^z):\ (v^u, v^z) \in U \times Z\} \quad (t \in [t_0, \vartheta]), \tag{77}$$

where

$$p_k(t, v^u, v^z) = \langle \bar{F}_k(t), v^z\rangle - \langle \bar{G}_k(t), f(t, v^u, v^z)\rangle + \frac{1}{\tau_{k+1}}\omega(t, v^u, v^z), \tag{78}$$

$$\bar{F}_k(t) = 2(1 - \delta_{k+1})F_k(t), \quad \bar{G}_k(t) = 2(1 - \delta_{k+1}) \int_t^\vartheta F_k(s)\,\mathrm{d}s. \tag{79}$$

Finally, we compute

$$F_{k+1} = (1 - \delta_{k+1})F_k + \delta_{k+1} F(v_{k+1}^u(\cdot), v_{k+1}^z(\cdot)), \tag{80}$$

$$J_{k+1} = (1 - \delta_{k+1})J_k + \delta_{k+1} J(v_{k+1}^u(\cdot), v_{k+1}^z(\cdot)), \tag{81}$$

and find

$$x_{k+1} = (u_k(\cdot), z_k(\cdot)) \in R_{k+1} \tag{82}$$

$$= \{(u(\cdot), z(\cdot)) \in \mathcal{U} \times \mathcal{Z}:\ F((u(\cdot), z(\cdot)) = F_{k+1},\ J(u(\cdot), z(\cdot)) \le J_{k+1}\}. \tag{83}$$

Now Theorem 7 is specified as follows (here we also refer to Theorem 8):

Theorem 10. *Let Condition 1 be satisfied and the sequence* $((v_k, F_k, J_k, x_k))$, *where* $v_k = (v_k^u(\cdot), v_k^z(\cdot)) \in \mathcal{U} \times \mathcal{Z}$, $F_k \in L_n^2$, $J_k \in \mathbb{R}^1$ *and* $x_k = (u_k(\cdot), z_k(\cdot)) \in \mathcal{U} \times \mathcal{Z}$ $(k = 0, 1, \dots)$, *be defined by (75)–(83). Then the sequence* (x_k) *converges to the solution set* $\mathcal{P}^*$ *of the optimal control problem (62) in* $L_{n,w}^2 \times L_n^2$, *i.e.*

$$\inf\{|x_k - x|_{L_{n,w}^2 \times L_n^2} : \ x \in \mathcal{P}^*\} \to 0,$$

and each of the sequences (J_k) *and* $(J(x_k))$ *converges to the optimal value* J^* *in this problem.*

Remark 22. Relation (77) shows that the key operation in the method (75)–(83), namely, the computation of the correction term $v_{k+1} = (v_{k+1}^u(\cdot), v_{k+1}^z(\cdot))$, is reduced to a family of independent finite-dimensional optimization problems parametrized by time t (the solutions to these problems, $(v_{k+1}^u(t), v_{k+1}^z(t))$, are interconnected through the mesurability of $v_{k+1}^u(\cdot)$ and $v_{k+1}^z(\cdot)$ only). In many typical cases, v_{k+1} can be found explicitly.

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