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OPTIMIZING THE LINEAR QUADRATIC MINIMUM–TIME PROBLEM FOR DISCRETE DISTRIBUTED SYSTEMS

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With reference to the work of Verriest and Lewis (1991) on continuous finite-dimensional systems, the linear quadratic minimum-time problem is considered for discrete distributed systems and discrete distributed time delay systems. We treat the problem in two variants, with fixed and free end points. We consider a cost functional J which includes time, energy and precision terms, and then we investigate the optimal pair (N, u) which minimizes J.

Keywords: discrete distributed systems, time delay systems, minimum time, optimal control

1. Introduction

The linear quadratic minimum-time problem was considered before (Athans and Falb, 1996; Schwartz and Gourdeau, 1989), but is was not fully exploited. Verriest and Lewis (1991) treat the case of continuous finite-dimensional systems. Discrete systems in the finite-dimensional case were considered later (El Alami *et al.*, 1998). In the present paper, we investigate discrete-time distributed systems. In the first part of this work, we consider systems described by

$$\begin{cases} x(i+1) = Ax(i) + Bu(i), \quad 0 \le i \le N-1, \\ x(0) = x_0, \end{cases} \tag{1}$$

where N is taken to be free, $x(i) \in X$ is the state variable and $u(i) \in U$ is the input variable. X and U are Hilbert spaces, the operators A and B are bounded ($A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(U, X)$).

We consider a cost functional $J(N, u)$ which includes time and energy, that is to say,

$$J(N, u) = \varphi(N) + \sum_{i=0}^{N-1} \langle u(i), Ru(i) \rangle, \tag{2}$$

where $u = (u(0), \dots, u(N-1)) \in U^N$, φ: $\mathbb{N} \to \mathbb{R}^+$ is assumed to be positive and increasing, i.e.

$$\varphi(N) \ge 0, \quad \forall N \in \mathbb{N},$$

$$N \le M \Rightarrow \varphi(N) \le \varphi(M), \quad \forall N, M \in \mathbb{N},$$

and

$$\lim_{N \to +\infty} \varphi(N) = +\infty. \tag{3}$$

$R \in \mathcal{L}(U)$ is a self-adjoint positive definite operator. Then we investigate the optimal pair $(N^*, u^*) \in \mathbb{N}^* \times U^{N^*}$ which minimizes the cost functional $J(N, u)$ under constraints

$$(N, u) \in \{(M, v) \in \mathbb{N} \times U^N: \ x_v(M) = x_d\},$$

where N^* is taken to be as small as possible, x_d is a given desired final state, $x_v(\cdot)$ is the trajectory of system (1) corresponding to the control v, and $\mathbb{N}^*$ is the set of all non-zero integers.

We establish that the optimal solution (N^*, u^*) exists, is unique and is obtained by solving a finite sequence of algebraic equations and by minimizing a time functional over a finite sub-interval of $\mathbb{N}$. An example is given to illustrate the results. The case where the final end point $x(N)$ is free, is also considered. In this case, the functional cost includes time, energy and precision terms, i.e.

$$J(N, u) = \varphi(N) + \sum_{i=0}^{N-1} \left[\langle u(i), Ru(i) \rangle + \langle x(i), Mx(i) \rangle \right] + \langle x(N), Gx(N) \rangle, \tag{4}$$

where $M, G \in \mathcal{L}(X)$ are self-adjoint positive operators. Since J contains both the final time N and quadratic components of $x(i)$ and $u(i)$, we shall call J a linear quadratic minimum-time performance index. In the second part of this paper, we treat the case of discrete distributed time delay systems. To settle the problem, we define a new state variable which satisfies a discrete system without delays.

In what follows, we denote by $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_U$ the inner products defined respectively on X and U. We also denote by $\mathbb{N}^*$ and $\mathbb{R}^*$ the set of non-zero integers and the set of non-zero reals, respectively.

2. The Case of a Fixed End Point

Consider the linear discrete-time system given by

$$\begin{cases} x(i+1) = Ax(i) + Bu(i), \quad 0 \le i \le N-1, \\ x(0) = x_0, \end{cases} \tag{5}$$

where N is free, $x(i) \in X$ is the state variable and $u(i) \in U$ is the input variable. X and U are Hilbert spaces, $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(U, X)$. Let φ be a positive increasing function such that

$$\lim_{N \to +\infty} \varphi(N) = +\infty. \tag{6}$$

The problem can be stated as follows: Given the performance index

$$J(N, u) = \varphi(N) + \sum_{i=0}^{N-1} \langle u(i), Ru(i) \rangle \tag{7}$$

and a desired final state $x_d \in X$, we investigate the optimal pair $(N^*, u^*) \in \mathbb{N}^* \times U^{N^*}$ where N^* is as small as possible and

$$J(N^*, u^*) = \min_{(N,u) \in \mathcal{V}} J(N, u), \tag{8}$$

with $\mathcal{V} = \{(N, u) \in \mathbb{N}^* \times U^N \colon x(N) = x_d\}$.

Definition 1. An integer N is said to be *admissible* if there exists a control sequence $u \in U^N$ such that $x(N) = x_d$.

To determine the optimal sequence (N^*, u^*), we proceed as follows: For each admissible integer N, we determine an optimal control $u^N = (u^N(0), \dots, u^N(N-1))$ which minimizes the cost $J(N, u)$ over all controls $u = (u(0), \dots, u(N-1))$ such that $x(N) = x_d$. The optimal time N^* is the smallest integer which minimizes $J(N, u^N)$ over all admissible integers N.

Let $N \in \mathbb{N}$ be a fixed integer. From (5) it follows that for every control $u = (u(0), \dots, u(N-1)) \in U^N$, we have

$$x(N) = A^N x_0 + H_N u, \tag{9}$$

where H_N is the operator defined by

$$H_N \colon \begin{array}{l} U^N \to X, \\ (u(0), \dots, u(N-1)) \hookrightarrow \displaystyle\sum_{j=0}^{N-1} A^{N-1-j} Bu(j). \end{array} \tag{10}$$

Consider the inner product on U^N given by

$$\langle u, v \rangle_R = \sum_{i=0}^{N-1} \langle u(i), Rv(i) \rangle_U, \tag{11}$$

$u = (u(0), \dots, u(N-1)), v = (v(0), \dots, v(N-1))$, and let H_N^* be the adjoint operator of H_N defined with respect to the inner products $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_R$, i.e.

$$\langle H_N u, x \rangle = \langle u, H_N^* x \rangle_R, \quad \forall\, u \in U^N, \quad \forall\, x \in X. \tag{12}$$

Define the functional $\| \cdot \|_{F_N}$ by

$$\|x\|_{F_N} = \|H_N^* x\|_R, \quad \forall\, x \in X, \tag{13}$$

where $\| \cdot \|_R$ is the norm corresponding to the inner product $\langle \cdot, \cdot \rangle_R$. Then the functional $\| \cdot \|_{F_N}$ describes a semi-norm on X and a norm on F_0, where F_0 is the subspace of X defined by

$$F_0 = \overline{\operatorname{Im} H_N} = (\operatorname{Ker} H_N^*)^\perp. \tag{14}$$

Indeed, if $x \in F_0$ and $\|x\|_{F_N} = 0$, then we deduce that $x \in (\operatorname{Ker} H_N^*) \cap (\operatorname{Ker} H_N^*)^\perp$, which implies that $x = 0$. We denote by $\langle \cdot, \cdot \rangle_N$ the inner product on F_0 given by

$$\langle x, y \rangle_N = \langle H_N^* x, H_N^* y \rangle_R, \quad \forall\, x, y \in F_0. \tag{15}$$

Now, we introduce the operator

$$\Lambda_N \colon \begin{array}{l} F_0 \to F_0, \\ x \hookrightarrow H_N H_N^* x. \end{array} \tag{16}$$

For every $x \in F_0$, we have

$$\begin{aligned} \|\Lambda_N x\|_{F_N} &= \|H_N^* \Lambda_N x\|_R = \|H_N^* H_N H_N^* x\|_R \\ &\le \|H_N^* H_N\| \, \|x\|_{F_N}. \end{aligned}$$

Hence Λ_N is a bounded operator on F_0 endowed with the norm $\| \cdot \|_{F_N}$.

Let F_N be the completion of F_0 with respect to the norm $\| \cdot \|_{F_N}$. Since we have

$$|\langle \Lambda_N x, y \rangle| = |\langle x, y \rangle_N| \le \|x\|_{F_N} \|y\|_{F_N}, \quad \forall\, x, y \in F_0, \tag{17}$$

it is classical that Λ_N has a unique extension denoted also by Λ_N and defined from F_N to its dual F_N' (Lions, 1988). Indeed, for any $x \in F_0$ we define the map ψ_x by

$$\psi_x \colon \begin{array}{l} F_0 \to \mathbb{R}, \\ y \hookrightarrow \langle \Lambda_N x, y \rangle. \end{array} \tag{18}$$

The map ψ_x is linear and continous with respect to the norm $\| \cdot \|_{F_N}$. Since F_0 is dense in F_N, ψ_x can be extended to a bounded linear operator denoted by $\overline{\psi_x}$ which belongs to the space $F_N^{'}$. Now consider the map π defined by

$$\pi\colon \quad \begin{aligned} \Lambda_N(F_0) &\to F_N^{'}, \\ \Lambda_N x &\hookrightarrow \overline{\psi_x}. \end{aligned} \tag{19}$$

We verify that the map π is well defined on $\Lambda_N(F_0)$. Moreover, π is linear and injectif. This allows us to identify the space $\Lambda_N(F_0)$ with a subspace of $F_N^{'}$. Using the operator π, we rewrite the operator Λ_N as follows:

$$\Lambda_N\colon \quad \begin{aligned} F_0 &\to F_N^{'}, \\ x &\hookrightarrow \overline{\psi_x}. \end{aligned} \tag{20}$$

We show that Λ_N is linear and continous with respect to the norm $\|\cdot\|_{F_N}$, which implies that Λ_N has a linear and bounded extension also denoted by Λ_N and defined from F_N to its dual $F_N^{'}$. Moreover, this extension is an isomorphism from F_N to $F_N^{'}$. To show this, we prove that

$$\langle \Lambda_N x, x\rangle_{F_N^{'},F_N} = \|x\|^2_{F_N}, \quad \forall\, x \in F_N, \tag{21}$$

where we denote by $\langle \phi, x\rangle_{F_N^{'},F_N}$ the range of $x \in F_N$ by the operator $\phi \in F_N^{'}$. From (21) it follows that Λ_N is injectif. Consequently, Λ_N is an isomorphism from F_N to $\Lambda_N(F_N)$. This implies that $\Lambda_N(F_N)$ is a closed subspace of $F_N^{'}$, and hence $\overline{\Lambda_N(F_N)} = \Lambda_N(F_N)$. On the other hand, if $A \subset F_N^{'}$, we denote by A° the subspace of F_N given by

$$A^\circ = \{x \in F_N / \langle \phi, x\rangle_{F_N^{'},F_N} = 0,\ \forall\, \phi \in A\}. \tag{22}$$

If $B \subset F_N$, we denote by B° the subspace of $F_N^{'}$ given by

$$B^\circ = \{\phi \in F_N^{'} / \langle \phi, x\rangle_{F_N^{'},F_N} = 0,\ \forall\, x \in B\}. \tag{23}$$

Let $x \in (\Lambda_N(F_N))^\circ$. Then from (22) it follows that

$$\langle \Lambda_N y, x\rangle_{F_N^{'},F_N} = 0, \quad \forall\, y \in F_N. \tag{24}$$

This implies

$$\langle \Lambda_N x, x\rangle_{F_N^{'},F_N} = 0 = \|x\|^2_{F_N}.$$

Hence $x = 0$. Consequently, $(\Lambda_N(F_N))^\circ = \{0\}$. Thus

$$\Lambda_N(F_N) = \overline{\Lambda_N(F_N)} = ((\Lambda_N(F_N))^\circ)^\circ = \{0\}^\circ = F_N^{'}, \tag{25}$$

which implies that Λ_N is an isomorphism from F_N to $F_N^{'}$.

Remark 1. Suppose that $x \in \mathrm{Im} H_N$. Then there exists $u \in U^N$ such that $x = H_N u$. Consider the function φ_x defined by

$$\varphi_x\colon \quad \begin{aligned} F_0 &\to \mathbb{R}, \\ y &\hookrightarrow \langle x, y\rangle. \end{aligned} \tag{26}$$

We have

$$\begin{aligned} |\varphi_x(y)| &= |\langle H_N u, y\rangle| \\ &= |\langle u, H_N^* y\rangle_R| \le \|u\|_R \|y\|_{F_N}, \quad \forall\, y \in F_0. \end{aligned}$$

Hence φ_x is a bounded operator on F_0 endowed with the norm F_N. Using the Hahn-Banach theorem, we deduce that $\varphi_x \in F_N^{'}$. Consequently, we may assume that $\mathrm{Im} H_N \subset F_N^{'}$ since the map i given by

$$\varphi_x\colon \quad \begin{aligned} \mathrm{Im} H_N &\to F_N^{'}, \\ x &\hookrightarrow \varphi_x \end{aligned} \tag{27}$$

is injectif.

Now, we can formulate the following proposition which characterizes the admissible integers.

Proposition 1. *An integer N is admissible if and only if $x_d - A^N x_0 \in F_N^{'}$.*

Proof. If $x_d - A^N x_0 \in F_N^{'}$, then there exists a unique $f \in F_N$ such that $\Lambda_N f = x_d - A^N x_0$. Consider the control $u = H_N^* f$. Then

$$x(N) = A^N x_0 + H_N u = A^N x_0 + \Lambda_N f = x_d \tag{28}$$

and hence N is admissible.

Conversely, if N is admissible, then there exists a control u such that $x(N) = x_d$, which implies $x_d - A^N x_0 = H_N u$. Hence $x_d - A^N x_0 \in \mathrm{Im} H_N \subset F_N^{'}$ (see Remark 1). ■

Proposition 2. *If N is an admissible integer, then the control u^N being a solution to the optimization problem*

$$J(N, u^N) = \min_{u \in U^N} J(N, u)$$

subject to $x(N) = x_d$ is given by $u^N = H_N^* f$, where $f \in F_N$ is the unique solution of the algebraic equation

$$\Lambda_N f = x_d - A^N x_0.$$

Moreover, the corresponding cost is

$$J(N, u^N) = \varphi(N) + \|f\|^2_{F_N}.$$

Proof. Let N be an admissible integer. From Proposition 1 it follows that there exists a unique $f \in F_N$ such that $\Lambda_N f = x_d - A^N x_0$. Define $u = H_N^* f \in U^N$. Then

$$x(N) = A^N x_0 + H_N u = A^N x_0 + \Lambda_N f = x_d.$$

On the other hand, for each control $v \in U^N$ such that $x_v(N) = x_d$, we have

$$x(N) = x_v(N) = x_d,$$

where $x_v(\cdot)$ denotes the trajectory of system (5) corresponding to the control v. Hence

$$H_N u = H_N v,$$

which implies

$$\langle H_N(u - v), f\rangle = 0$$

or

$$\langle u - v, H_N^* f\rangle_R = 0.$$

Since $u = H_N^* f$, we deduce that

$$\langle u, u\rangle_R = \langle v, u\rangle_R \le \|v\|_R \|u\|_R.$$

Thus $\|u\|_R \le \|v\|_R,\ \forall\, v \in U^N$. ■

Remark 2.

(a) By convention, if N is not admissible, we set $J(N, u^N) = +\infty$.

(b) In order to obtain the minimizing control u^N, we have to solve the algebraic equation $\Lambda_N f = x_d - A^N x_0$. However, we do not in general have an explicit expression for the operator Λ_N^{-1}, so we propose the Galerkin method to approximate f (the bilinear form $F_N \times F_N \to \mathbb{R}\colon\ (x, y) \hookrightarrow \langle \Lambda_N x, y\rangle$ is coercive).

Finally, the optimal sequence (N^*, u^*) is given by the following proposition.

Proposition 3. *Let $\mathcal{A}$ be the set of all admissible integers. If $\mathcal{A}$ is bounded, then N^* is the smallest integer that minimizes $J(N, u^N)$ over $\mathcal{A}$. Otherwise, consider $N_0 \in \mathcal{A}$ and $M \in \mathcal{A}$ such that $\varphi(M) > J(N_0, u^{N_0})$. Then N^* is the smallest integer that minimizes $J(N, u^N)$ over the interval $[1, M]$.*

Proof. If $\mathcal{A}$ is bounded, the result is obvious. Suppose that $\mathcal{A}$ is not bounded and consider $N_0, M \in \mathcal{A}$ such that $\varphi(M) > J(N_0, u^{N_0})$. It follows that $N_0 \in [1, M]$. Indeed, if it is not, then $\varphi(N_0) \ge \varphi(M)$, which implies

$$J(N_0, u^{N_0}) \ge \varphi(N_0) \ge \varphi(M) > J(N_0, u^{N_0}),$$

a contradiction. Thus $N_0 \in [1, M]$.

On the other hand, for each $N \in \mathbb{N}$ such that $N > M$, we have $\varphi(N) \ge \varphi(M)$. Consequently,

$$J(N, u^N) \ge \varphi(N) \ge \varphi(M) > J(N_0, u^{N_0}). \qquad ■$$

Example 1. Consider the discrete-time system described by

$$\begin{cases} x(i+1) = Ax(i) + Bu(i), \quad i = 0, \dots, N-1, \\ x(0) = 0, \end{cases} \tag{29}$$

where N is free, $\Omega =]0, 1[$, $x(i) \in L^2(\Omega)$ is the state variable, $u_i \in \mathbb{R}$ is the input variable and

$$A = S(\delta) \in \mathcal{L}\big(L^2(\Omega)\big), \tag{30}$$

$S(t)_{t\ge 0}$ being the strongly continuous semigroup generated by the Laplacian operator Δ, i.e.

$$S(\delta)x = \sum_{i=1}^{\infty} e^{-i^2\pi^2\delta} \langle x, e_i\rangle e_i, \quad \forall\, x \in L^2(\Omega), \tag{31}$$

where $\langle\cdot,\cdot\rangle$ is the usual inner product on $L^2(\Omega)$, $\delta > 0$ and $e_i(s) = \sqrt{2}\sin(i\pi s)$, ($(e_i)_i$ is a basis of $L^2(\Omega)$).

The operator B is defined by

$$B = \int_0^{\delta} S(\sigma) D \,\mathrm{d}\sigma, \tag{32}$$

where

$$D\colon \quad \begin{aligned} &\mathbb{R} \to L^2(\Omega) \\ &u \hookrightarrow u e_1(\cdot). \end{aligned}$$

Remark 3. The difference equation (29) can be interpreted as the sampling version ot the following continuous diffusion system:

$$\begin{cases} \dfrac{\partial x}{\partial t} - \Delta x = g(s)u(t), & s \in \Omega, t \in [0, T], \\ x(0, \cdot) = x_0(\cdot) & \text{in } \Omega, \\ x(t, s) = 0 & \text{in } \partial\Omega \times]0, T[, \end{cases} \tag{33}$$

where $g = e_1$.

The linear quadratic minimum-time problem consists in determining the optimal pair (N^*, u^*) which minimizes the cost functional

$$J(N, u) = N^2 + \sum_{i=0}^{N-1} R u^2(i) \tag{34}$$

while driving the system from $x_0 = 0$ to $x_d = \alpha e_1$, where $\alpha \in \mathbb{R}^*$ is given.

Lemma 1. *The space* F_0 *defined by* $F_0 = \overline{\mathrm{Im} H_N}$ *is given by*

$$F_0 = E(e_1),$$

where $E(e_1)$ *is the subspace of* $L^2(\Omega)$ *spanned by the vector* e_1.

Proof. For every $N \geq 1$ and every $u \in \mathbb{R}^N$, we have

$$\begin{aligned} H_N u &= \sum_{i=0}^{N-1} A^{N-1-i} B u(i) \\ &= \sum_{i=0}^{N-1} S((N-1-i)\delta) u(i) \int_0^\delta S(\sigma) e_1 \, \mathrm{d}\sigma \\ &= \sum_{i=0}^{N-1} u(i) \int_0^\delta S((N-1-i)\delta + \sigma) e_1 \, \mathrm{d}\sigma \\ &= \sum_{i=0}^{N-1} u(i) \int_0^\delta \sum_{j=1}^{\infty} e^{-j^2\pi^2((N-1-i)\delta+\sigma)} \langle e_1, e_j \rangle e_j \, \mathrm{d}\sigma \\ &= \sum_{i=0}^{N-1} u(i) \int_0^\delta e^{-\pi^2((N-1-i)\delta+\sigma)} \, \mathrm{d}\sigma e_1 \\ &= (c \sum_{i=0}^{N-1} u(i) e^{-\pi^2((N-1-i)\delta)}) e_1, \end{aligned}$$

where c is the constant given by $c = \int_0^\delta e^{-\pi^2\sigma} \, \mathrm{d}\sigma$.

Hence $\mathrm{Im} H_N \subset E(e_1)$. Conversely, if $x \in E(e_1)$, there exists $\beta \in \mathbb{R}$ such that $x = \beta e_1$. Choose $u = (u(0), \ldots, u(N-1))$ such that $u(0) = \cdots = u(N-2) = 0$ and $u(N-1) = \beta / c$. Then $H_N u = x$ and

$$\mathrm{Im} H_N = E(e_1). \tag{35}$$

Consequently,

$$F_0 = \overline{\mathrm{Im} H_N} = E(e_1),$$
$$F_N = E(e_1). \quad \blacksquare$$

Now, for every integer $N \geq 1$ we have $x_d - A^N x_0 \in \mathrm{Im} H_N$, since $x_0 = 0$ and $x_d \in \mathrm{Im} H_N$. Hence from Remark 1 and Proposition 1 it follows that every integer $N \geq 1$ is admissible. In order to solve the equation $\Lambda_N f = x_d$, we first determine the adjoint operators B^* and H_N^*. By simple calculations we establish that for every $x \in L^2(\Omega)$, we have

$$B^* x = c \langle e_1, x \rangle, \tag{36}$$

$$\begin{aligned} H_N^* x &= \big((H_N^* x)_0, \ldots, (H_N^* x)_{N-1}\big), \\ (H_N^* x)_i &= R^{-1} B^* A^{N-1-i} x \\ &= \frac{c}{R} e^{-\pi^2 (N-1-i)\delta} \langle x, e_1 \rangle. \end{aligned} \tag{37}$$

Let $f \in F_N (= E(e_1))$ be such that

$$\Lambda_N f = x_d \ \text{ in } \ F_N'.$$

Then

$$\langle \Lambda_N f, x \rangle = \langle x_d, x \rangle, \quad \forall\, x \in F_0. \tag{38}$$

Since $f = a_N e_1$ for some $a_N \in \mathbb{R}$, (38) implies

$$a_N \langle \Lambda_N e_1, \beta e_1 \rangle = \langle x_d, \beta e_1 \rangle, \quad \forall\, \beta \in \mathbb{R}$$

or, equivalently,

$$a_N \langle H_N^* e_1, H_N^* e_1 \rangle_R = \langle x_d, e_1 \rangle = \alpha.$$

Thus

$$a_N = \frac{\alpha}{\|H_N^* e_1\|^2} = \frac{\alpha}{\|e_1\|_{F_N}^2}. \tag{39}$$

Consequently, the optimal cost corresponding to u^N is

$$\begin{aligned} J(N, u^N) &= N^2 + \|f\|_{F_N}^2 = N^2 + a_N^2 \|e_1\|_{F_N}^2 \\ &= N^2 + \frac{\alpha^2}{\|e_1\|_{F_N}^2}. \end{aligned} \tag{40}$$

Using (37), we establish

$$\|e_1\|_{F_N}^2 = \frac{c^2 (e^{-2\pi^2 (N-1)\delta} - e^{2\pi^2\delta})}{R(1 - e^{2\pi^2\delta})}.$$

For numerical simulation we take $\alpha = 10$, $\delta = 0.1$, $R = 1$, $N_0 = 7$. Then we apply Proposition 3 to deduce that the minimum time N^* exists in the interval $[1, 147]$ and is equal to 4. The optimal control is $u^* = H_{N^*} f$, where $f = (2132.4) e_1$. The evolution of $J(N, u^N)$ with respect to N is given in Fig. 1.

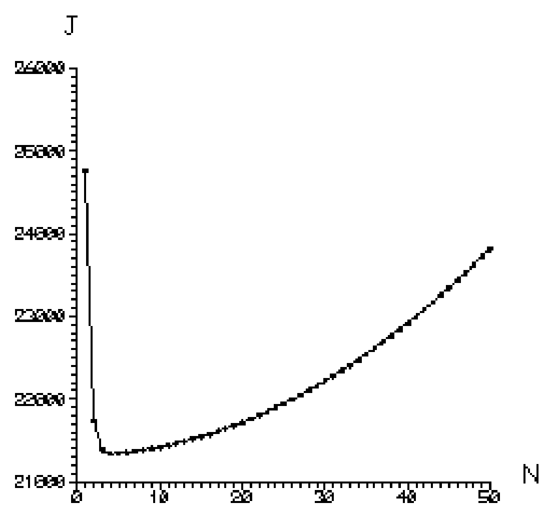

Fig. 1. The evolution of $J(N, u^N)$ with respect to N.

3. The Case of a Free End Point

In this case, we consider a cost functional $J(N,u)$ which includes time, energy and precision terms, i.e.

$$J(N,u) = \varphi(N) + \sum_{i=0}^{N-1} \left[\langle u(i), Ru(i)\rangle + \langle x(i), Mx(i)\rangle\right] + \langle x(N), Gx(N)\rangle, \tag{41}$$

where $M \in \mathcal{L}(X), G \in \mathcal{L}(X)$ are self-adjoint positive operators and $R \in \mathcal{L}(U)$ is a self-adjoint positive definite operator.

Then we investigate the optimal sequence (N^*, u^*) where N^* is as small as possible and

$$J(N^*, u^*) = \min_{(N,u)\in\mathbb{N}\times U^N} J(N,u). \tag{42}$$

To show that this problem has a unique solution (N^*, u^*), we proceed in two steps: In the first one, for any fixed integer N, we determine the optimal control u^N which minimizes the cost $J(N,u)$ over all controls $u \in U^N$. In the second step, we minimize $J(N,u^N)$ over all integers N. By convention, we set

$$J(0,u) = \varphi(0) + \langle x_0, Gx_0\rangle, \quad \forall N \in \mathbb{N}, \quad \forall u \in U^N. \tag{43}$$

For a fixed $N \in \mathbb{N}^*$, if we denote by $u^N \in U^N$ the optimal control which satisfies

$$J(N,u^N) = \min_{u\in U^N} J(N,u), \tag{44}$$

then u^N is unique and given by the following proposition:

Proposition 4. *Let $N \in \mathbb{N}^*$ and K_i: $X \to X$, $i = 0,\dots,N-1$ be a family of operators given by*

$$\begin{cases} K_{i+1} = A^*K_i(I + BR^{-1}B^*K_i)^{-1}A + M, \\ \qquad\qquad i = 0,\dots,N-1, \\ K_0 = G. \end{cases}$$

Given an initial condition $x_0 \in X$, the optimal control u^N is given in feedback form by

$$u^N(i) = -R^{-1}B^*K_{N-1-i}(I + BR^{-1}B^*K_{N-1-i})^{-1} \times Ax(i), \quad i = 0,\dots,N-1.$$

The corresponding cost is

$$J(N,u^N) = \langle K_N x_0, x_0\rangle.$$

Proof. For the proof, see (Zabczyk, 1974). ■

Finally, the optimal pair (N^*, u^*) being a solution of (42) is determined by the following result:

Proposition 5. *Consider $(N_0, M) \in \mathbb{N}^2$ such that $\varphi(M) > J(N_0, u^{N_0})$. Then the minimum time N^* is the smallest integer that minimizes $J(N,u^N)$ over the interval $[0,M]$. Moreover, we have $u^* = u^{N^*}$.*

Proof. The proof is similar to the one of Proposition 3. ■

4. Discrete Time Delay Systems

Consider the discrete time delay system described by

$$\begin{cases} x(i+1) = \displaystyle\sum_{j=0}^{m} A_j x(i-j) \\ \qquad\qquad + \displaystyle\sum_{j=0}^{q} B_j u(i-j), \quad i = 0,\dots,N-1, \\ x(0) = x_0, \\ x(r) = \alpha_r, \qquad -m \le r \le -1, \\ u(r) = \mu_r, \qquad -q \le r \le -1, \end{cases} \tag{45}$$

where $x(i) \in X, u(i) \in U, X$ and U are Hilbert spaces, $A_j \in \mathcal{L}(X)$, $j = 0,\dots,m$ and $B_j \in \mathcal{L}(U,X)$, $j = 0,\dots,q$. Furthermore, $(\alpha_r)_r$ and $(\mu_r)_r$ are fixed initial conditions. Here $m \ge 0$ and $q \ge 1$ are given integers.

Given the cost functional

$$J(N,u) = \varphi(N) + \sum_{i=0}^{N-1} \langle u(i), Ru(i)\rangle \tag{46}$$

and a desired final state x_d, we investigate the optimal pair (N^*, u^*) which steers the system from the initial state $(x_0, (\alpha_r)_{-m\le r\le -1})$ to x_d with a minimal cost. We recall that φ: $\mathbb{N} \to \mathbb{R}_+$ is a positive increasing map satisfying (6) and $R \in \mathcal{L}(U)$ is a self adjoint positive definite operator. Similarly to the case of discrete systems without delays, the determination of the optimal pair (N^*, u^*) follows from solving the following optimization problems:

Find $u^N \in U^N$ such that

$$J(N,u^N) = \min_{u\in U^N} J(N,u), \tag{47}$$

and $x(N) = x_d$, where N is an admissible integer.

The determination of N^* is then performed by minimizing $J(N,u^N)$ over an appropriate bounded subset of $\mathbb{N}$.

First, we establish some results which are useful for the sequel. Define a new state variable $e(i) \in X^{m+1} \times U^q$ by

$$e(i) = \big(x(i), x(i-1),\dots,x(i-m), u(i-1), \dots, u(i-q)\big)^T. \tag{48}$$

Then $e(\cdot)$ satisfies the difference equation

$$\begin{cases} e(i+1) = \Phi e(i) + \bar{B}u(i), & i = 0, \dots, N-1, \\ \quad e(0) = e_0, \end{cases} \tag{49}$$

where

$$\Phi = \begin{pmatrix} A_0 & A_1 & \dots & A_m & B_1 & \dots & B_q \\ I & 0 & & 0 & 0 & & 0 \\ & \ddots & \ddots & & & & \vdots \\ 0 & & I & 0 & 0 & & 0 \\ 0 & & & 0 & 0 & \dots & 0 \\ \vdots & & & & I & \ddots & \vdots \\ & & & & & \ddots & \\ 0 & & \dots & & & I & 0 \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} B_0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ I \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{50}$$

and $e_0 = (x_0, \alpha_{-1}, \dots, \alpha_{-m}, \mu_{-1}, \dots, \mu_{-q})^T$.

Let $P \in \mathcal{L}(X^{m+1} \times U^q, X)$ be the projection operator defined by

$$P\colon \quad \begin{aligned} X^{m+1} \times U^q &\to X, \\ (y_1, \dots, y_{m+1}, v_1, \dots, v_q) &\hookrightarrow y_1. \end{aligned} \tag{51}$$

Then from (49) it follows that

$$x(N) = Pe(N) = P\Phi^N e_0 + P\bar{H}_N u, \tag{52}$$

where $\bar{H}_N$ is the operator

$$\bar{H}_N\colon \quad \begin{aligned} U^N &\to X, \\ (u(0), \dots, u(N-1)) &\hookrightarrow \sum_{i=0}^{N-1} \Phi^{N-1-i} \bar{B}u(i). \end{aligned} \tag{53}$$

Let $K_N = P\bar{H}_N$ and $G_0 = \overline{\mathrm{Im} K_N}$. Then consider the semi-norm $\|\cdot\|_{G_N}$ defined on X by

$$\|x\|_{G_N} = \|K_N^* x\|_R, \quad \forall\, x \in X, \tag{54}$$

where K_N^* is the adjoint operator of K_N defined with respect to the inner products $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_R$. Since $G_0 = \overline{\mathrm{Im} K_N} = (\mathrm{Ker} K_N^*)^\perp$, we deduce that the functional $\|\cdot\|_{G_N}$ is a norm on G_0. Denote by G_N the completion of G_0 under the norm $\|\cdot\|_{G_N}$ and consider the operator L_N given by

$$L_N\colon \quad \begin{aligned} G_0 &\to G_0, \\ x &\hookrightarrow K_N K_N^* x. \end{aligned} \tag{55}$$

Clearly, L_N defines a bounded operator on G_0 endowed with the norm $\|\cdot\|_{G_N}$. By standard results (Lions, 1988), the operator L_N may be extended to an isomorphism denoted also by L_N and defined from G_N to its dual $G_N^{'}$.

Proposition 6. *An integer $N \geq 1$ is admissible if and only if $x_d - P\Phi^N e_0 \in G_N^{'}$.*

Proof. If N is admissible, then there exists a control sequence $u \in U^N$ such that $x(N) = x_d$, which implies $Pe(N) = P\Phi^N e_0 + K_N u = x_d$, or $x_d - P\Phi^N e_0 = K_N u$. Since $\mathrm{Im} K_N \subset G_N^{'}$, we deduce that $x_d - P\Phi^N e_0 \in G_N^{'}$. Conversely, suppose that $x_d - P\Phi^N e_0 \in G_N^{'}$. Then there exists $y \in G_N$ such that $L_N y = x_d - P\Phi^N e_0$. Hence $x_d = P\Phi^N e_0 + K_N K_N^* y$ or $x_d = x(N)$, where $u = K_N^* y$. Thus N is admissible. ■

Proposition 7. *For each admissible integer N, the control u^N exists, is unique and given by $u^N = K_N^* g$, where $g \in G_N$ is the unique solution of the algebraic equation*

$$L_N g = x_d - P\Phi^N e_0.$$

Moreover, the optimal cost is $J(N, u^N) = \varphi(N) + \|g\|_{G_N}^2$.

Proposition 8. *Let $\mathcal{A}$ be the set of all admissible integers. If $\mathcal{A}$ is bounded, then N^* is the smallest integer that minimizes $J(N, u^N)$ over $\mathcal{A}$. Otherwise, consider $N_0 \in \mathcal{A}$ and $M \in \mathcal{A}$ such that $\varphi(M) > J(N_0, u^{N_0})$. Then N^* is the smallest integer that minimizes $J(N, u^N)$ over the interval $[1, M]$.*

Remark 4. By obvious modifications, Remark 2 remains also valid.

4.1. The Case of a Free End Point

In this case, the cost functional $J(N, u)$ depends on time, energy, state and also delays in the states, i.e.

$$\begin{aligned} J(N,u) = \varphi(N) &+ \sum_{i=0}^{N-1} \langle u(i), Ru(i) \rangle \\ &+ \sum_{i=0}^{N-1} \Big\langle \sum_{j=0}^{m_1} M_j x(i-j), M \sum_{j=0}^{m_1} M_j x(i-j) \Big\rangle \\ &+ \langle x(N), Gx(N) \rangle, \end{aligned} \tag{56}$$

where $M_j \in \mathcal{L}(X)$, $j = 0, \dots, m_1$ and $m_1 \in \mathbb{N}$ is such that $m_1 \leq m$.

The problem is to determine the optimal pair (N^*, u^*) which satisfies

$$J(N^*, u^*) = \min_{(N,u) \in \mathbb{N} \times U^N} J(N, u) \tag{57}$$

such that N^* is as small as possible. To determine the unique solution (N^*, u^*), we proceed in two steps: In the first one, for any fixed integer N, we determine the optimal control u^N which minimizes the cost $J(N,u)$ over all controls $u \in U^N$. In the second step, we minimize $J(N,u^N)$ over all integers N. By convention, we set

$$J(0,u) = \varphi(0) + \Big\langle \sum_{j=0}^{m_1} M_j \alpha_{-j}, M \sum_{j=0}^{m_1} M_j \alpha_{-j} \Big\rangle + \langle x_0, Gx_0 \rangle, \quad \forall\, u \in U^N. \tag{58}$$

To settle this problem, we rewrite the cost functional in terms of the state variable $e(\cdot)$ given in (49). Indeed, since

$$\sum_{j=0}^{m_1} M_j x(i-j) = \bar{M} e(i), \quad i = 0,\dots,N-1,$$

$$x(N) = Pe(N), \tag{59}$$

where $\bar{M} = [M_0,\dots,M_{m_1}, \underbrace{0,\dots,0}_{m-m_1+q}]$, we deduce that

$$J(N,u) = \varphi(N) + \sum_{i=0}^{N-1} \langle u(i), Ru(i) \rangle + \sum_{i=0}^{N-1} \langle e(i), \bar{M}^T M \bar{M} e(i) \rangle + \langle e(N), P^T G P e(N) \rangle. \tag{60}$$

Consequently, in order to settle the problem (57), we consider the cost functional defined by (60), where $e(\cdot)$ is the solution of the system without delay given by (49). Then we apply the results of Section 3 to obtain the following propositions:

Proposition 9. *Let* $N \in \mathbb{N}^*$ *and* $K_i : X^{m+1} \times U^q \to X^{m+1} \times U^q$, $i = 0,\dots,N-1$ *be a family of operators given by*

$$\begin{cases} K_{i+1} = \Phi^* K_i (I + \bar{B} R^{-1} \bar{B}^* K_i)^{-1} \Phi + \bar{M}^T M \bar{M}, \\ \qquad\qquad i = 0,\dots,N-1, \\ K_0 = P^T G P. \end{cases}$$

Given initial conditions $x_0, (\alpha_r)_r$, *the optimal control* u^N *is given in feedback form by*

$$u^N(i) = -R^{-1} \bar{B}^* K_{N-1-i} (I + \bar{B} R^{-1} \bar{B}^* K_{N-1-i})^{-1} \times \Phi e(i), \quad i = 0,\dots,N-1$$

and the corresponding cost is

$$J(N,u^N) = \langle K_N e_0, e_0 \rangle.$$

Proposition 10. *Consider* $(N_0, M) \in \mathbb{N}^2$ *such that* $\varphi(M) > J(N_0, u^{N_0})$. *Then the minimum time* N^* *is the smallest integer that minimizes* $J(N,u^N)$ *over the interval* $[0,M]$. *Moreover,* $u^* = u^{N^*}$.

5. Conclusion

We have solved the linear quadratic minimum-time problem for discrete distributed systems and discrete distributed time delay systems. On certain assumptions, we can prove the existence and uniqueness of the solution. We consider the problem in two variants, with fixed and free end point. In the first variant, we establish that the optimal pair can be determined by solving a finite sequence of algebraic equations and by minimizing a time functional over a finite sub-interval of $\mathbb{N}$. In the second variant, we use a similar technique and some results of (Zabczyk, 1974). For discrete distributed time delay systems, in order to solve the problem we have defined a new state variable which is the solution of a discrete system without delay.

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