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AN ELASTIC MEMBRANE WITH AN ATTACHED NON-LINEAR THERMOELASTIC ROD

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We study a thermo-mechanical system consisting of an elastic membrane to which a shape-memory rod is glued. The slow movements of the membrane are controlled by the motions of the attached rods. A quasi-static model is used. We include the elastic feedback of the membrane on the rods. This results in investigating an elliptic boundary value problem in a domain $\Omega \subset \mathbb{R}^2$ with a cut, coupled with non-linear equations for the vertical motions of the rod and the temperature on the rod. We prove the existence of a unique global weak solution to this problem using a fixed point argument.

Keywords: thermoelastic materials, non-linear partial differential equations

1. Introduction

In this note we investigate a model where one or more thin linear rods of shape memory alloy are attached to an elastic membrane. Heating these rods can be used to change their shape and, in turn, to deform the membrane. We assume that the reference domain for the membrane is a bounded domain $\Omega \subset \mathbb{R}^2$. The motions of the membrane are governed by a linear wave equation. However, we will assume that the motions of the rod and the membrane are slow compared with the vibrations of the membrane, and will therefore restrict ourselves to a quasi-static model, i.e. we will model the membrane statically and the actuating rod dynamically. The membrane itself will act on the rod via the elastic force. The feedback of the structure on the actuator will change the system dynamics. A precursor of this model was previously introduced by Horn and Sokołowski (2000), where a proof of a local existence theorem was sketched. In the present paper we will conduct a more thorough investigation of the analytic properties of this model.

The rod could be either attached to the boundary or in the interior of the membrane. The boundary case is less complicated and its mathematical properties follow directly from the case when the actuating rod is attached to the interior. The case of a finite number of rods is analogous to the case when there is only one rod. We will also assume that the membrane does not conduct heat.

Section 2 of this paper contains a comprehensive description of the model at hand. We will introduce the necessary terminology and state the major results, a local existence and uniqueness theorem, as well as a global version of the same. Since this situation involves the solution of an elliptic boundary value problem in a domain containing a cut, the solutions will have less regularity than the results known for one-dimensional models of shape memory alloys (Brokate and Sprekels, 1996; Bubner and Sprekels, 1998; Sprekels and Zheng, 1989). It will therefore be necessary to consider a weak formulation of the problem.

In Section 3 we will prove the local existence and uniqueness of weak solutions. This was outlined for a somewhat simpler model by Horn and Sokołowski (2000). We will adapt this outline to the analysed situation and provide the necessary details. In Section 4 we will prove uniform *a-priori* estimates to extend the local solutions to the global ones.

Throughout the article we will use the same model for shape memory alloys as in the papers cited above. We refer the reader to those papers and the works cited therein for a derivation of this model and its properties. The basic techniques of the present paper are also following the techniques of these earlier papers, but in the present situation we have to perform many of the steps with significantly less regularity because of the low regularity of the feedback term.

Several authors have studied related questions. A one-dimensional model of an adaptive structure was introduced in Bubner *et al.* (2001). Two-dimensional models can be found in (Hoffmann and Żochowski, 1992; Pawłow and Żochowski, 2000; Żochowski, 1992). However, these models differ from the present one in that they use visco-elastic shape memory materials, and that they do not consider the feedback of the structure (in our case the membrane) on the actuator.

2. Model Description

In this section we will discuss the situation with an actuator in the interior of the domain. We assume that $\Omega \subset \mathbb{R}^2$ is a bounded connected domain with a C^1 boundary Γ. Let Q be a line segment which lies in the interior of Ω. We will assume that $Q = \{\mathbf{x} \in \Omega\colon\ 0 \le x \le 1, y = 0\}$. Furthermore, we assume that we can extend Q to a smooth non-self-intersecting curve γ which intersects Γ transversally at two points and divides Ω into two subdomains Ω_+ and Ω_- with Lipschitz boundaries as indicated in Fig. 1. This will allow us to apply Green's formula type arguments. The vertical displacement U of the

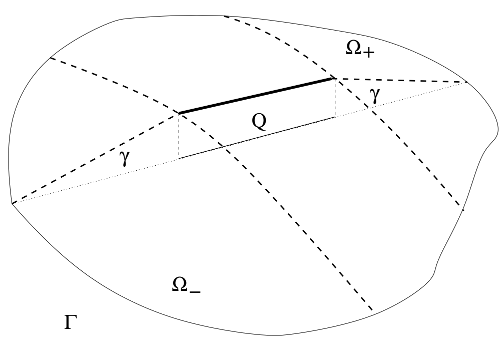

Fig. 1. Domain Ω.

membrane satisfies

$$-\Delta U = f(\mathbf{x},t) \quad \text{for} \quad \mathbf{x} = (x,y) \in \Omega \setminus Q, \tag{1}$$

$$U(\mathbf{x},t)|_Q = v(x,t), \tag{2}$$

$$U(\mathbf{x},t)|_\Gamma = w(\mathbf{x},t), \tag{3}$$

for all $0 \le t \le T$. We will also assume that $f \in C^1([0,T]; H^1(\Omega))$ and $w \in C^1([0,T]; H^{\frac{5}{2}}(\Gamma))$.

The function $v(x,t)$ is the vertical displacement of the actuator. It satisfies the non-linear system for shape memory rods given in (Sprekels and Zheng, 1989):

$$v_{tt} - \big(\sigma(\theta, v_x)\big)_x + R v_{xxxx} = f_1, \tag{4}$$

$$\theta_t - \kappa\theta_{xx} - \theta\big(\sigma(\theta, v_x)\big)_\theta v_{xt} = g. \tag{5}$$

Here θ is the absolute temperature of the rod and the function σ is given by

$$\sigma(\theta, v_x) = \gamma\,(\theta_1 - \theta)\, v_x - \beta v_x^3 + \alpha v_x^5. \tag{6}$$

θ_1, α, β, γ, κ and R are positive constants. We refer the reader to (Sprekels and Zheng, 1989) for a detailed investigation of this model. The function v can be interpreted as either the tangential (as in the work cited above) or the normal displacement (as in (Żochowski, 1992), for example). We will interpret it here as the normal displacement. The function g represents an external heat source. As in the previous papers, we will assume that $g \in L^2(0,T; L^2(Q))$ and that $g(x,t) \ge 0$ on $Q \times [0,T]$. This positivity condition is necessary to apply the maximum principle to (5).

The function f_1 represents an external force on the rod. In this paper we assume that the only external force is the elastic force acting on the rod from the deformation of the membrane. Following Hooke's law, the elastic force is proportional to the normal derivative of the displacement

$$f_1 = c\left[\frac{\partial U}{\partial y}\right], \tag{7}$$

where

$$\left[\frac{\partial U}{\partial y}\right] = \left(-\lim_{y\to 0^+}\frac{\partial U}{\partial y} + \lim_{y \to 0^-}\frac{\partial U}{\partial y}\right). \tag{8}$$

To simplify the notation we will assume that $c = 1$.

The system of equations (4), (5) is augmented by the following set of initial and boundary conditions:

$$\theta_x(0,t) = \theta_x(1,t) = 0, \tag{9}$$

$$v(0,t) = v(1,t) = v_{xx}(0,t) = v_{xx}(1,t) = 0, \tag{10}$$

$$\theta(x,0) = \theta_0(x), \tag{11}$$

$$v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x). \tag{12}$$

These boundary conditions, especially $v(0,t) = v(1,t) = 0$, are rather restrictive. In order to get a more general model, we will assume that the actuator is glued to the membrane and moves with it even if it is not deformed. For this, the vertical displacement U of the membrane is composed of two parts: $U(\mathbf{x},t) = \hat{u}(\mathbf{x},t) + u(\mathbf{x},t)$, where the function $\hat{u}$ is smooth across Q and does not contribute to the force acting on the rod. This function will satisfy the system

$$-\Delta\hat{u} = 0 \quad \text{on} \quad \Omega \setminus Q, \tag{13}$$

$$\left[\frac{\partial \hat{u}}{\partial y}\right]_Q = 0, \tag{14}$$

$$\hat{u}|_\Gamma = w. \tag{15}$$

The function u, which is the second part of U, describes the part of the vertical displacement that acts on the rod. This part will be used to model the entire interaction between the membrane and the rod. The function $u(\mathbf{x},t)$ satisfies

$$-\Delta u = 0 \quad \text{on} \quad \Omega \setminus Q, \tag{16}$$

$$u|_Q = v, \tag{17}$$

$$u|_\Gamma = 0. \tag{18}$$

The system for $\hat{u}$ can be treated separately. Any function $\hat{u} \in H^2(\Omega)$ which satisfies

$$-\Delta \hat{u} = 0 \quad \text{on} \quad \Omega,$$

$$\hat{u}|_\Gamma = w,$$

will automatically satisfy the boundary condition on Q. The classical elliptic regularity theory guarantees the existence of a unique solution $\hat{u}$ on the smooth domain Ω.

We will therefore only consider (16), (18) coupled with the non-linear system (4)–(5) via the force

$$f_1 = \left[\frac{\partial u}{\partial y}\right]. \tag{19}$$

For any $v \in H^1(Q)$ the solutions to (16)–(18) are only in $H^1(\Omega \setminus Q)$, and therefore $f_1 \in (H^{\frac{1}{2}}(Q))'$. It is thus necessary to define f_1 as a linear functional. We follow the approach used in (Hazounet and Joly, 1979). Then for any $\phi \in H_0^1(\Omega)$, define

$$\langle f_1, \phi\rangle = -\int_\Omega \nabla u \nabla \phi \,\mathrm{d}x, \tag{20}$$

where u satisfies (16)–(18). Formally, this is Green's formula, and

$$\langle f_1, \phi\rangle = \int_Q \left[\frac{\partial u}{\partial y}\right] \phi \,\mathrm{d}x.$$

The functional f_1 is supported on Q. We can therefore restrict it to functions $\hat{\phi} \in H^{\frac{1}{2}}(Q)$ as follows, by taking $\phi \in H_0^1(\Omega)$, an extension of $\hat{\phi}$ to Ω. We denote by

$$\langle f_1, \phi\rangle_{(H^{\frac{1}{2}}(Q))' \times H^{\frac{1}{2}}(Q)}$$

the action by f_1 on $H^{\frac{1}{2}}(Q)$. In order to write a weak formulation of (4), (5) we introduce the following spaces:

$$X_1(t) = C\big(0,t;H^3(Q)\big) \cap C^1\big(0,t;H^1(Q)\big), \tag{21}$$

$$X_2(t) = L^2\big(0,t;H^2(Q)\big) \cap C\big(0,t;H^1(Q)\big) \cap C^1\big(0,t;L^2(Q)\big). \tag{22}$$

These spaces are Banach spaces with the following norms:

$$\|u\|_{X_1(t)} = \max\left\{ \max_{0\le s\le t} \|u(s)\|_{H^3(Q)}, \max_{0\le s\le t} \|u_t(s)\|_{H^1(Q)} \right\},$$

$$\|u\|_{X_2(t)} = \max\left\{ \left(\int_0^t \|u(s)\|^2_{H^2(Q)} \,\mathrm{d}s\right)^{\frac{1}{2}}, \max_{0\le s\le t} \|u(s)\|_{H^1(Q)}, \max_{0\le s\le t} \|u_t(s)\| \right\}.$$

For notational simplicity, we denote by $\|\cdot\|$ without any subscripts the norm on $L(Q)$.

We say that a pair $(v,\theta) \in X_1(t) \times X_2(t)$ is a weak solution to (4), (5) together with the initial and boundary conditions if (v,θ) satisfies the initial conditions and

$$\int_0^t \big(\langle v_t, \phi_t\rangle + \langle \sigma, \phi_x\rangle - R\langle v_{xxx}, \phi_x\rangle\big) \,\mathrm{d}s = \langle v_1, \phi\rangle - \int_0^t \langle f_1, \phi\rangle_{(H^{\frac{1}{2}}(Q))' \times H^{\frac{1}{2}}(Q)} \,\mathrm{d}s, \tag{23}$$

$$\int_0^t (\langle \theta_t, \psi\rangle + \kappa\langle \theta_x, \psi_x\rangle - \langle \sigma_\theta v_{xt}\theta, \psi\rangle) \,\mathrm{d}s = \int_0^t \langle g, \psi\rangle \,\mathrm{d}s, \tag{24}$$

for every pair $(\phi,\psi) \in \hat{X}_1(t) \times \hat{X}_2(t)$. Here

$$\hat{X}_1(t) = \big\{\phi \in H^1\big(0,t;H_0^1(Q)\big) : \phi(x,0) = 0\big\}, \tag{25}$$

$$\hat{X}_2(t) = L^2\big(0,t;H^1(Q)\big). \tag{26}$$

We can now state the main results of this paper. In Section 3 we will prove the following local existence theorem:

Proposition 1. *For a sufficiently small $t > 0$ there exists a unique triple*

$$(u,v,\theta) \in C\big(0,t;H^1(\Omega)\big) \times X_1(t) \times X_2(t)$$

such that u solves (16)–(18), and (v,θ) satisfies the initial conditions and (23), (24).

In Section 4 we will prove uniform *a-priori* estimates for the weak solutions which will give the following global existence result:

Proposition 2. *For any given $T > 0$, there exists a unique triple*

$$(u,v,\theta) \in C\big(0,T;H^1(\Omega)\big) \times X_1(T) \times X_2(T),$$

such that u solves (16)–(18), and (v,θ) satisfies the initial conditions and (23)–(24).

Remark 1. We could also investigate this situation by using weighted Sobolev space methods as described in (Kondrat'ev, 1967; Kondrat'ev and Oleinik, 1983; Kozlov *et al.*, 1997; Kozlov and Maz'ya, 1999). In particular, the solution u to (16)–(18) satisfies

$$u \in V_\beta^{l+1}(\Omega \setminus Q)$$

for $|\beta - l| < 1/2$. Its trace on the boundary $(\partial\Omega) \cup Q$ satisfies

$$u \in V_\beta^{l+\frac{1}{2}}(\partial\Omega \cup Q).$$

In this situation the weighted Sobolev spaces $V_\beta^{l+1}(\Omega \setminus Q)$ are endowed with the following norm:

$$\|u\|^2_{V_\beta^l(\Omega\setminus Q)} = \|(1-\zeta_1-\zeta_2)u\|^2_{H^l(\Omega\setminus Q)} + \sum_{j=1}^{2}\int_\Omega \sum_{|\alpha|\le l} |x-x_j|^{2(\beta-l+|\alpha|)} \times |D_x^\alpha(\zeta_j u)|^2 \, \mathrm{d}x,$$

where x_1 and x_2 denote the endpoints of the rod Q and ζ_j are C^∞-functions equal to one in a neighborhood of x_j and vanish outside a neighborhood of x_j. Here $\alpha = (\alpha_1, \alpha_2)$ is a multi-index.

However, this approach would require re-establishing the existence and uniqueness theorems for the evolutionary system (4), (5) in the setting of weighted Sobolev spaces. This might be worth undertaking in its own right, but it would be beyond the framework of this paper.

For more on the theory of weighted Sobolev spaces and its application to elliptic problems in non-smooth domains, we refer the reader to the cited works.

3. Proof of Proposition 1

Before proving this result we need to get some preliminary results. To do this, define the following linear pseudo-differential operator associated with the elliptic system (16)–(18):

$$\mathcal{F} : v \mapsto f_1 = \left[\frac{\partial u}{\partial y}\right]. \tag{27}$$

This operator involves the solution of (16)–(18). The next lemma gives a result about the regularity of $\mathcal{F}$. For this, let $(H^{\frac{1}{2}}(Q))'$ denote the usual dual space of $H^{\frac{1}{2}}(Q)$.

Lemma 1. *The operator*

$$\mathcal{F} : C^1\big(0,t; H^{\frac{1}{2}}(Q)\big) \to C^1\left(0,t; \big(H^{\frac{1}{2}}(Q)\big)'\right)$$

is bounded, i.e. there are constants C_1, C_2 which depend only on Ω and Q such that

$$\|\mathcal{F}(v)\|_{(H^{\frac{1}{2}}(Q))'} \le C_1 \|v\|_{H^{\frac{1}{2}}(Q)}, \tag{28}$$

$$\big\|\big(\mathcal{F}(v)\big)_t\big\|_{(H^{\frac{1}{2}}(Q))'} \le C_2 \|v_t\|_{H^{\frac{1}{2}}(Q)}. \tag{29}$$

Proof. Elliptic regularity theory implies (Lions and Magenes, 1972; Nazarov and Plamenevsky, 1994):

$$\|u\|_{H^1(\Omega\setminus Q)} \le C \|v\|_{H^{\frac{1}{2}}(Q)}. \tag{30}$$

We combine this estimate with the trace theorem and the compact inclusion $H^1(Q) \subset H^{\frac{1}{2}}(Q)$, and get the first estimate of the lemma.

Next observe that u_t satisfies

$$-\Delta u_t = 0 \quad \text{on} \quad \Omega \setminus Q, \tag{31}$$

$$u_t(\mathbf{x},t)|_Q = v_t(x,t), \tag{32}$$

$$u_t(\mathbf{x},t)|_\Gamma = 0. \tag{33}$$

We apply the same reasoning as above to this elliptic equation to get the second inequality in the lemma. ■

Proof of Proposition 1. It suffices to show that (4), (5) admit a weak solution in the sense (23), (24) for any right-hand side $\mathcal{F}(v) \in C^1(0,t;(H^{\frac{1}{2}}(Q))')$. We start with two observations: First, for σ given by (6) and $(\hat{v},\hat{\theta}) \in X_1(t) \times X_2(t)$, we have $\sigma \in L^2(0,t;H^2(Q)) \cap C(0,t;H^1(Q))$. Second, if $f_1 \in C^1(0,t;(H^{\frac{1}{2}}(Q))')$ and $g \in C(0,t;H^1(Q))$, we can define the linear form $\mathcal{L}(g)$ as follows:

$$\begin{aligned}\mathcal{L}(g) &= \langle f_1(t), g(t)\rangle_{(H^{\frac{1}{2}}(Q))'\times H^{\frac{1}{2}}(Q)} - \langle f_1(0), g(0)\rangle_{(H^{\frac{1}{2}}(Q))'\times H^{\frac{1}{2}}(Q)} - \int_0^t \langle f_{1t}, g\rangle_{(H^{\frac{1}{2}}(Q))'\times H^{\frac{1}{2}}(Q)} \,\mathrm{d}s \\ &= \int_0^t \langle f_1, g_t\rangle_{(H^{\frac{1}{2}}(Q))'\times H^{\frac{1}{2}}(Q)} \,\mathrm{d}s.\end{aligned} \tag{34}$$

Observe that the definition of $\mathcal{L}(g)$ involves only f_1 and f_{1t} acting on g, but not g_t. Formally, this is equivalent to integration by parts in the variable t.

To prove the proposition, we consider the following linear problem:

$$v_{tt} + Rv_{xxxx} = f_1 + \big(\sigma(\hat{\theta},\hat{v}_x)\big)_x, \tag{35}$$

$$\theta_t - \kappa\theta_{xx} = \hat{\theta}\big(\sigma(\hat{\theta},\hat{v}_x)\big)_{\hat{\theta}}\hat{v}_{xt} - g, \tag{36}$$

where

$$f_1 = \mathcal{F}(\hat{v}). \tag{37}$$

These equations are augmented by the initial and boundary conditions. Since the right-hand side of (35) contains $f_1 \in C^1(0,t;(H^{\frac{1}{2}}(Q))')$, the solutions to this equations are not classical solutions, and this equation must be understood in the weak sense as (23).

The linear system (35)–(37) defines a map

$$\mathcal{G} : (\hat{v}, \hat{\theta}) \mapsto (v, \theta). \tag{38}$$

To continue, for positive constants M_0 and M_1 define the following subset of $X_1(h) \times X_2(h)$:

$$\mathbf{B} = \left\{ \|v\|_{X_1(h)} \le M_0, \|\theta\|_{X_2(h)} \le M_1, \theta > 0 \right\}. \tag{39}$$

We will show that for a sufficiently small h the map $\mathcal{G}$ is a contraction

$$\mathcal{G} : \mathcal{B} \to \mathcal{B}. \tag{40}$$

We do this in several steps.

Step 1. We multiply (35) by v_t, integrate over $Q \times (0,t)$ and obtain after integration by parts

$$\begin{aligned} &\frac{1}{2}\left(\|v_t(t)\|^2 + R\,\|v_{xx}(t)\|^2\right) \\ &\quad \le \frac{1}{2}\left(\|v_1\|^2 + R\,\|v_{0xx}\|^2\right) + \int_0^t \left\|\sigma(\hat{\theta},\hat{v})_x\right\| \|v_t\| \,\mathrm{d}s \\ &\qquad + \int_0^t \langle f_1, v_t\rangle_{(H^{\frac{1}{2}}(Q))' \times H^{\frac{1}{2}}(Q)} \,\mathrm{d}s. \end{aligned} \tag{41}$$

We apply (34) with Hölder's and Young's inequalities to the last term on the right-hand side to get

$$\begin{aligned} &\frac{1}{2}\left(\|v_t(t)\|^2 + R\,\|v_{xx}(t)\|^2\right) \\ &\quad \le \frac{1}{2}\left(\|v_1\|^2 + R\,\|v_{0xx}\|^2\right) + C_1 t + \frac{1}{2}\int_0^t \|v_t\|^2 \,\mathrm{d}s \\ &\qquad + \frac{1}{2}\int_0^t \|\mathcal{F}(\hat{v}_t)\|^2_{H^{-1}(Q)} \,\mathrm{d}s + \frac{1}{2}\int_0^t \|v\|^2_{H^1(Q)} \,\mathrm{d}s \\ &\qquad + \langle \mathcal{F}(\hat{v}(t)), v(t)\rangle_{(H^{\frac{1}{2}}(Q))' \times H^{\frac{1}{2}}(Q)} \\ &\qquad - \langle \mathcal{F}(\hat{v}_0), v_0\rangle_{(H^{\frac{1}{2}}(Q))' \times H^{\frac{1}{2}}(Q)}, \end{aligned} \tag{42}$$

for an appropriate constant C_1. The second to last term can be treated via Hölder's and Young's inequalities again. The last term is bounded by $C_2\,\|v_0\|^2_{H^1(Q)}$. Since $v(0,s) = v(1,s) = 0$, for each $s \in [0,t]$ there is a $\xi \in Q$ such that $v_x(\xi, s) = 0$. We can therefore apply Poincaré's inequality to both v and v_x to obtain

$$\|v\|_{H^1(Q)} \le C_3\,\|v_{xx}\|, \tag{43}$$

for an appropriate C_3 (see (Bubner, 1995) for details). Combining these results, we get

$$\begin{aligned} &\frac{1}{2}\left(\|v_t(t)\|^2 + \hat{R}\,\|v_{xx}(t)\|^2\right) \\ &\quad \le \frac{1}{2}\left(\|v_1\|^2 + R\,\|v_{0xx}\|^2\right) \\ &\qquad + C_4 t + \int_0^t \left(\|v_t\|^2 + \|v_{xx}\|^2\right) \mathrm{d}s, \end{aligned}$$

for an appropriate suitable positive constant C_4 which depends only on the initial data and $(\hat{\theta}, \hat{v}) \in \mathcal{B}$. Applying Gronwall's inequality, we get

$$\begin{aligned} &\|v_t(t)\|^2 + \hat{R}\,\|v_{xx}(t)\|^2 \\ &\quad \le e^t\left(\|v_1\|^2 + R\,\|v_{0xx}\|^2 + C_5 t\right), \end{aligned} \tag{44}$$

where C_5 depends only on M_0 and M_1.

Step 2. We multiply (36) by θ to get, after integration by parts,

$$\begin{aligned} &\frac{1}{2}\,\|\theta(t)\|^2 + \int_0^t \|\theta_x\|^2 \,\mathrm{d}s \\ &\quad \le \frac{1}{2}\,\|\theta_0\| + C_6 + \int_0^t \|\theta\|^2 \,\mathrm{d}s, \end{aligned} \tag{45}$$

for an appropriate constant C_6. Next we multiply (36) by θ_t and obtain by applying integration by parts and Young's inequality

$$\frac{1}{2}\,\|\theta_x(t)\|^2 + \int_0^t \|\theta_t\|^2 \,\mathrm{d}s \le \frac{1}{2}\,\|\theta_{0x}\| + C_7, \tag{46}$$

for an appropriate constant C_7. We combine these last two results and apply Gronwall's lemma to get

$$\|\theta(t)\|^2_{H^1(Q)} \le e^t\left(\|\theta_0\|^2_{H^1(Q)} + tC_3\right), \tag{47}$$

where C_3 again depends only on M_0 and M_1.

Step 3. In this step we multiply (35) by $-v_{xxt}$, integrate the result over $Q \times (0,t)$ and integrate it by parts. The right-hand side of the resulting equation contains the term

$$\int_0^t \langle f_1, v_{xxt}\rangle_{(H^{\frac{1}{2}}(Q))' \times H^{\frac{1}{2}}(Q)} \,\mathrm{d}s,$$

which is again treated using (34). Using a similar argument to that of Step 1, we get

$$\begin{aligned} &\|v_{xt}(t)\|^2 + \tilde{R}\,\|v_{xxx}\|^2 \\ &\quad \le e^t\left(\|v_{1x}(t)\|^2 + \tilde{R}\,\|v_{0xxx}\|^2 + tC_8\right), \end{aligned} \tag{48}$$

where C_8 depends only on M_0 and M_1.

Step 4. We combine the first three steps to get the following inequalities:

$$\|v\|^2_{X_1(t)} \leq e^t \left(\|v_0\|^2_{H^3(Q)} + \|v_1\|^2_{H^1(Q)} + tK_1\right), \quad (49)$$

$$\|\theta\|^2_{X_2(t)} \leq e^t \left(\|\theta_0\|^2_{H^1(Q)} + tK_2\right), \quad (50)$$

where K_1 and K_2 depend on M_0 and M_1. Furthermore, since (36) satisfies a maximum principle, we have $\theta > 0$. We can now pick M_0, M_1 and h such that the map $\mathcal{G}$ satisfies

$$\mathcal{G} : \mathbf{B} \to \mathbf{B}. \quad (51)$$

Step 5. It remains to be shown that the map $\mathcal{G}$ is actually a contraction. To do this, observe that $\mathcal{F}$ is linear and therefore we have

$$\left\|\mathcal{F}(v^1 - v^2)\right\|_{(H^{\frac{1}{2}}(Q))'} \leq C_1 \left\|v^1 - v^2\right\|_{H^1(Q)}, \quad (52)$$

$$\left\|\left(\mathcal{F}(v^1 - v^2)\right)_t\right\|_{(H^{\frac{1}{2}}(Q))'} \leq C_3 \left\|v^1_t - v^2_t\right\|_{H^1(Q)} \quad (53)$$

for any functions v^1 and v^2 in $C^1(0,t;H^1(Q))$. To prove that $\mathcal{G}$ is a contraction, we will use similar *a-priori* estimates for $\mathcal{G}(v^1,\theta^1) - \mathcal{G}(v^2,\theta^2)$ as in the previous steps. The feedback term can be treated using (52), (53) combined with the techniques of the previous steps. If necessary, we can use smaller values for h, M_0 and M_1 in order to show that $\mathcal{G}$ is a contraction.

We can now apply the Banach Fixed-Point Theorem to obtain the existence of a unique pair $(v,\theta) \in X_1(t) \times X_2(t)$ which solves (23) and (24). To get Proposition 1, we solve (16) for the given v. ■

4. Uniform *A-Priori* Estimates

In this section we will prove some uniform *a-priori* estimates which will then imply Proposition 2. In general, these estimates follow the same lines as the estimates in (Sprekels and Zheng, 1989). However, the authors of that paper require the inhomogeneity f_1 to be in $H^1(0,T;H^1(Q))$. In the present situation this function is in $C^1(0,T;(H^{\frac{1}{2}}(Q))')$. In other words, we have slightly more regularity in time, but significantly less regularity in space. We will therefore need to modify the approach. We start with the following preliminary lemma. We will, however, only use the third assertion of this lemma, but we will state and prove the others for the sake of completeness.

Lemma 2. *Let u satisfy (16)–(18). Define the bi-linear form*

$$\mathcal{B} : H^{\frac{1}{2}}(Q) \times H^{\frac{1}{2}}(Q) \to \mathbb{R}$$

as follows:

$$\mathcal{B}(\phi,\psi) = \langle \mathcal{F}(\phi), \psi \rangle_{(H^{\frac{1}{2}}(Q))' \times H^{\frac{1}{2}}(Q)}.$$

Then the following estimates hold:

$$\mathcal{B}(v,v) \leq 0, \quad (54)$$

$$|\mathcal{B}(v,v)| \leq C\,\|v\|^2_{H^1(Q)}, \quad (55)$$

$$\int_0^t \mathcal{B}(v,v_t)\,\mathrm{d}s \leq \frac{1}{2}\,\|u(\cdot,0)\|^2_{H^1(\Omega\setminus Q)} < \hat{C}\,\|v_0\|^2_{H^1(Q)}, \quad (56)$$

$$\left|\int_0^t \mathcal{B}(v,v_t)\,\mathrm{d}s\right| \leq \tilde{C}\max\left\{\|v(t)\|^2_{H^1(Q)},\ \|v_0\|^2_{H^1(Q)}\right\}, \quad (57)$$

where the constants C, $\hat{C}$ and $\tilde{C}$ depend only on the data.

Proof. For the first two assertions, observe that, by the definition of $\mathcal{B}$ and (20), we have

$$\mathcal{B}(v,v) = \langle \mathcal{F}(v), v \rangle_{(H^{\frac{1}{2}}(Q))' \times H^{\frac{1}{2}}(Q)} = -\int_\Omega |\nabla u|^2\,\mathrm{d}x,$$

since u is an extension of v to $H^1_0(D)$. The result follows immediately.

For the third and fourth assertions we have

$$\mathcal{B}(v,v_t) = \langle \mathcal{F}(v), v_t \rangle_{(H^{\frac{1}{2}}(Q))' \times H^{\frac{1}{2}}(Q)} = -\int_\Omega \nabla u \nabla u_t\,\mathrm{d}x.$$

The result follows from integration over $(0,t)$.

We can now proceed analogously to (Sprekels and Zheng, 1989; Zheng, 1995). We will state the estimates. However, we will not give the proofs unless there is a significant difference. The only differences are due to the terms involving inhomogeneity $f_1 = \mathcal{F}(v)$.

Lemma 3. *There exists a constant C which depends only on the initial data and g such that*

$$\sup_{0<t<T}\Big(\|v_t(t)\|^2 + \|v(t)\|^2_{H^2(Q)} + \|v_x(t)\|^6_{L^6(Q)} + \|v_x(t)\|^2_{L^\infty(Q)} + \|\theta\|_{L^1(Q)}\Big) \leq C. \quad (58)$$

Proof. We start by multiplying (4) by v_t and integrating the result over Q to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|v_t\|^2+\frac{c_1}{2}\|v_x\|^2+\frac{\alpha}{6}\|v_x\|^6_{L^6(Q)}+\frac{R}{2}\|v_{xx}\|^2\right)$$
$$=\mathcal{B}(v,v_t)+\frac{\beta}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|v_x\|^4_{L^4(Q)}-\int_Q\gamma\theta v_x v_{xt}\,\mathrm{d}x.$$

Next we take (5) and integrate it over Q to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|=\int_Q g\,\mathrm{d}x+\int_Q\gamma\theta v_x v_{xt}\,\mathrm{d}x.$$

By adding this equation to the equation above, the coupling term

$$\int_Q\gamma\theta v_x v_{xt}\,\mathrm{d}x$$

is cancelled. To continue, we integrate the result over $(0,t)$ and apply (56) to estimate the term $\int_0^t\mathcal{B}(v,v_t)\,\mathrm{d}s$. The L^4 term on the right-hand side can be estimated against the L^6 term on the left. The same argument as in Step 1 of the preceding section is used to estimate the H^2 norm with $\|v_{xx}\|$. The result then follows by taking the supremum over $(0,T)$ and applying the Sobolev Imbedding Theorem to v_x in order to get an estimate for $\|v_x\|_{L^\infty(Q)}$.

The next estimate is concerned only with the energy balance (5). Thus its proof is identical to the proof in the previous papers. We state the result for completeness.

Lemma 4. *There exists a constant* C *which depends only on the initial data and the inhomogeneity* g *such that*

$$\sup_{0<t<T}\|\theta(t)\|^2+\int_0^T\left(\|\theta_x(s)\|^2+\|\theta(s)\|^2_{L^\infty(Q)}\right)\,\mathrm{d}s\le C. \tag{59}$$

We continue as in the proof of Lemma 2.6 of (Sprekels and Zheng, 1989) by multiplying (4) by $-v_{xxt}$ and (5) by θ_t. Only the term

$$-\int_0^t\mathcal{B}(v,v_{xxt})\,\mathrm{d}s$$

requires a difference from the treatment. For this term we use (34) to get

$$-\int_0^t\mathcal{B}(v,v_{xxt})\,\mathrm{d}s=\int_0^t\mathcal{B}(v_t,v_{xx})\,\mathrm{d}s$$
$$-\mathcal{B}\big(v(t),v_{xx}(t)\big)+\mathcal{B}(v_0,v_{0xx}).$$

The last two terms on the right-hand side are bounded by virtue of Lemma 3. For the first term on the right-hand side observe that

$$\left|\int_0^t\mathcal{B}(v_t,v_{xx})\,\mathrm{d}s\right|$$
$$\le\frac{1}{2}\int_0^t\left(\|v_t(s)\|^2_{H^1(Q)}+\|v_{xx}(s)\|^2_{H^1(Q)}\right)\,\mathrm{d}s.$$

This term will be estimated by the application of Gronwall's inequality using the terms

$$\frac{1}{2}\left(\|v_{xt}(t)\|^2+\|v_{xxx}(t)\|^2\right),$$

which appear on the left. Continuing as in the previous works we arrive at the following result:

Lemma 5. *There exists a constant* C *which depends only on the initial data and* g *such that*

$$\sup_{0<t<T}\left(\|v_{xt}(t)\|^2+\|v_{xxx}(t)\|^2+\|\theta_x(t)\|^2\right)$$
$$+\int_0^T\left(\|\theta_t(s)\|^2+\|\theta_{xx}(s)\|^2\right)\,\mathrm{d}s\le C. \tag{60}$$

Finally, we can combine all the previous estimates to deduce that

$$\sup_{0<t<T}\|v_{tt}(t)\|^2_{H^{-1}(Q)}\le C, \tag{61}$$

for a constant C that depends only on the initial data and g.

Proposition 2 follows immediately from these estimates.

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