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A BOUNDARY–VALUE PROBLEM FOR LINEAR PDAEs

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We analyze a boundary-value problem for linear partial differential algebraic equations, or PDAEs, by using the method of the separation of variables. The analysis is based on the Kronecker-Weierstrass form of the matrix pencil $[A, -\lambda_n B]$. A new theorem is proved and two illustrative examples are given.

Keywords: differential algebraic equations, boundary-value problems, linear multivariable systems

1. Introduction

Recently there has been great interest in the analysis of coupled systems of differential and algebraic equations. Initially, most physical plants are modeled by systems of ordinary and/or partial differential equations coupled with algebraic constraints. Such systems are usually called partial differential algebraic equations, or PDAEs, and have found numerous applications as models of electrical, mechanical (constrained), and chemical engineering problems (Brenan *et al.*, 1996; Campbell, 1982; Griepentrog and März, 1986; Lewis, 1986; Pipilis, 1990).

In (Campbell and Marszałek, 1999; Marszałek and Trzaska, 1995) linear PDAEs were analyzed using the method of the separation of variables (modal analysis), and in (Campbell and Marszałek, 1999) a detailed analysis of the *index* of PDAEs was given. Interesting applications of PDAEs to traveling wave solutions in magnetohydrodynamics were studied in (Campbell and Marszałek, 1997; Marszałek and Campbell, 1999). A numerical solution of boundary value problems (BVPs) for linear time-varying differential algebraic equations, or DAEs for short, was considered in (Clark and Petzold, 1989), and a necessary and sufficient condition for the existence of the solution was given in terms of the invertibility of the *shooting* matrix (Clark and Petzold, 1989, Thm. 3.1). In this paper we shall analytically solve a two-point BVP for linear time-invariant PDAEs using the modal analysis of (Campbell and Marszałek, 1996; Marszałek and Trzaska, 1995). This BVP for the PDAE problem differs in many ways from the conventional BVPs considered in the literature on DAEs. Also, the convergence analysis of the separable method we provide differs slightly from the conventional convergence analysis (Haberman, 1998; Strauss, 1992) as it fully utilizes the Kronecker-Weierstrass form of the matrix pencil.

2. Linear PDAEs

Consider the following system of linear PDAEs:

$$Au_t + Bu_{xx} = f(x,t) \tag{1}$$

for $0 \le x \le L$, with

$$M_1 u(x,t)_{|x=0} = 0, \quad M_2 u(x,t)_{|x=L} = 0, \tag{2}$$

where linear operators M_i, $i = 1, 2$ specify the boundary conditions. The initial and final conditions are related through

$$Q_1 u(x,0) + Q_2 u(x,T) = \beta(x), \tag{3}$$

where Q_1 and Q_2 denote the appropriate matrices.

We allow A to be singular, and assume that the boundary conditions (2) yield a set of real eigenvalues λ_n and the corresponding orthogonal eigenfunctions $\phi_n(x)$, $n = 0, 1, 2, \dots$. In most applications, the matrix B in (1) is positive definite, and $\beta(x) = \sum_{n=0}^{\infty} \beta_n \phi_n(x)$, $\beta_n \in \mathbb{R}^{r_n}$ with r_n being the core-rank of $A_{n,s}$, where $A_{n,s} = (sA - \lambda_n B)^{-1} A$ (Clark and Petzold, 1989). We also assume that the rank conditions of matrices Q_1 and Q_2 are satisfied (Clark and Petzold, 1989).

The modal analysis of (1) and (2) yields the following DAEs:

$$Au_n' - \lambda_n B u_n = f_n, \tag{4}$$

where $u_n(t)$ and $f_n(t)$ result from the modal series of $u(x,t)$ and $f(x,t)$, respectively. Our further analysis of the problem is based on the analysis of the series of matrix pencils $[A, -\lambda_n B]$, $n = 0, 1, 2, \ldots$.

A similar BVP for DAEs was considered in (Clark and Petzold, 1989), where the *shooting* theory and a numerical approach were used. If $f(x,t) \equiv 0$, $\beta(x) \equiv 0$, and (2) yields the zero eigenvalue, say $\lambda_0 = 0$, then the BVP for (1) has a non-unique solution, as is shown by the following example.

Example 1. Consider (1) with $f(x,t) \equiv 0$, $\beta(x) \equiv 0$, $T = 1$ and

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \frac{2}{3\pi}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{5}$$

$$Q_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

with $M_1 = \partial/\partial x$, $M_2 = I$, for $0 \le x \le 1$.

Note that the matrices A and B are already in Kronecker-Weierstrass form (see Section 3 for details about this form). Also, $\lambda_0 = 0$ is one of the eigenvalues of the spatial problem. It yields the non-uniqueness of the solution of the nilpotent part of the DAEs (see (11) below). Additionally, some of the non-zero eigenvalues $\lambda_n \neq 0$ yield the non-uniqueness of the solution of the regular part of the DAE. The analysis is as follows:

Let

$$u(x,t) = \sum_{n=0}^{\infty} u_n(t)\phi_n(x),$$

and

$$u_n(t) \equiv \begin{bmatrix} u_{n,\mathrm{reg}}(t) \\ u_{n,\mathrm{nil}}(t) \end{bmatrix},$$

i.e. $u_n(t)$ is partitioned into the regular and nilpotent parts. The boundary condition (2) gives $\lambda_0 = 0$, $\lambda_n = (\pi^2/4)(2n-1)^2$, $n = 1, 2, \ldots$. The eigenvalues λ_n with $n = 2, 5, 8, \ldots$ are responsible for the non-uniqueness of the solution of the regular part of the DAEs. The remaining eigenvalues do not cause any problems. This is easily seen if we consider the regular part of the DAEs (5):

$$u_{n,\mathrm{reg}}'(t) - \frac{2}{3\pi}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\frac{\pi^2}{4}(2n-1)^2 u_{n,\mathrm{reg}}(t) = 0, \tag{6}$$

where $u_{n,\mathrm{reg}}(t) \in \mathbb{R}^2$, $n = 2, 5, 8, \ldots$.

If we assume that $u_{n,\mathrm{nil}}(T) = 0$ (this is true, e.g., if $f(x,t) \equiv 0$), then the first two columns of Q_1 and Q_2 together with (5) yield

$$\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\exp\left(\frac{2}{3\pi}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\frac{\pi^2}{4}(2n-1)^2\right)\right) \times u_{n,\mathrm{reg}}(0) = 0, \tag{7}$$

which gives

$$\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} u_{n,\mathrm{reg}}(0) = 0, \tag{8}$$

for $n = 2, 5, 8, \ldots$. Therefore the eigenvalues λ_n, $n = 2, 5, 8, \ldots$ of the spatial spectrum allow the first component of $u_{n,\mathrm{reg}}(0) \in \mathbb{R}^2$ to be any non-zero numbers. This gives the non-uniqueness of the solution of (6) and, as a consequence, the non-uniqueness of $u_2(x,t)$, one of the components of the vector u in (1). One can easily check that

$$u_{\mathrm{reg}}(x,t) = \begin{bmatrix} a_2\cos\left(\frac{3\pi}{2}t\right)\cos\left(\frac{3\pi}{2}x\right) + a_5\cos\left(\frac{27\pi}{2}t\right)\cos\left(\frac{9\pi}{2}x\right) + \cdots \\ \hdashline -a_2\sin\left(\frac{3\pi}{2}t\right)\cos\left(\frac{3\pi}{2}x\right) - a_5\sin\left(\frac{27\pi}{2}t\right)\cos\left(\frac{9\pi}{2}x\right) + \cdots \end{bmatrix} \tag{9}$$

is a solution of the above two-point BVP for any real numbers $a_2, a_5, \ldots$. ♦

3. Analytical Solution of the BVP for PDAEs

In what follows we shall assume that each pencil $[A, -\lambda_n B]$ is regular and the initial-value problem for each DAE with the pencil $[A, -\lambda_n B]$ is impulse-free.

Recall that a DAE is of the Kronecker-Weierstrass form if

$$[A, -\lambda_n B] = \left(\begin{bmatrix} I & 0 \\ 0 & J_0 \end{bmatrix}, \begin{bmatrix} J\lambda_n & 0 \\ 0 & I\lambda_n \end{bmatrix}\right), \tag{10}$$

where $J \in \mathbb{R}^{n_1}$ is of the Jordan form with finite eigenvalues, $J_0 \in \mathbb{R}^{n_2}$ is nilpotent with *index* n_{0n}, and I is the identity matrix.

The impulse-free response of such a system is

$$u_{n,\mathrm{reg}}(t) = e^{-J\lambda_n t} u_{n,\mathrm{reg}}(0) + \int_0^t e^{-J\lambda_n (t-s)} f_{n,\mathrm{reg}}(s)\,\mathrm{d}s, \tag{11}$$

$$u_{n,\mathrm{nil}}(t) = \sum_{i=1}^{n_{0n}} (-1)^{i+1} J_0^{i-1} \lambda_n^{-i} f_{n,\mathrm{nil}}^{(i-1)}(t),$$

where $f_{n,\mathrm{reg}}(t)$ and $f_{n,\mathrm{nil}}(t)$ are the components of $f_n(t)$ in $f(x,t) = \sum_n f_n(t)\phi_n(x)$ corresponding to the partition of the vector $u_n(t)$ into the regular and nilpotent parts.

Let Q_1 and Q_2 in (3) be partitioned according to the partition of the vector $u_n(t)$ into the regular and nilpotent parts, i.e. $Q_1 = [Q_1^{n_1} Q_1^{n_2}]$ and $Q_2 = [Q_2^{n_1} Q_2^{n_2}]$, where n_1 and n_2 are the dimensions of the regular and nilpotent parts, respectively.

Theorem 1. *The BVP for the PDAE problem (1) with*

$$u_n(t) = \begin{bmatrix} u_{n,\mathrm{reg}}(t) \\ u_{n,\mathrm{nil}}(t) \end{bmatrix}$$

and $\det\left[Q_1^{n_1} + Q_2^{n_1} e^{-J\lambda_n T}\right] \neq 0$ *has the solution* $u(x,t) = \sum_n u_n(t)\phi_n(x)$*, where*

$$\begin{aligned}
&\left[Q_1^{n_1} + Q_2^{n_1} e^{-J\lambda_n T}\right] e^{J\lambda_n t} u_{n,\mathrm{reg}}(t) \\
&\quad = \sum_{i=1}^{n_{0n}} (-1)^i \lambda_n^{-i} \Big[Q_1^{n_2} J_0^{i-1} f_{n,\mathrm{nil}}^{(i-1)}(0) \\
&\qquad + Q_2^{n_2} J_0^{i-1} f_{n,\mathrm{nil}}^{(i-1)}(T) \Big] \\
&\qquad + Q_1^{n_1} \int_0^t e^{J\lambda_n s} f_{n,\mathrm{reg}}(s)\,\mathrm{d}s \\
&\qquad - Q_2^{n_1} e^{-J\lambda_n T} \int_t^T e^{J\lambda_n s} f_{n,\mathrm{reg}}(s)\,\mathrm{d}s + \beta_n
\end{aligned} \tag{12}$$

and $u_{n,\mathrm{nil}}(t)$ *is given by (11).*

Proof. From (3) with partitioned matrices Q_1 and Q_2, we obtain

$$\begin{aligned}
&Q_1^{n_1} u_{n,\mathrm{reg}}(0) + Q_1^{n_2} u_{n,\mathrm{nil}}(0) \\
&\quad + Q_2^{n_1} u_{n,\mathrm{reg}}(T) + Q_2^{n_2} u_{n,\mathrm{nil}}(T) = \beta_n.
\end{aligned} \tag{13}$$

Here $u_{n,\mathrm{nil}}(0)$, $u_{n,\mathrm{nil}}(T)$ and $u_{n,\mathrm{reg}}(T)$ can easily be computed from (11) and substituted into (13). We get exactly the same formula as we would get from (12) for $t = 0$. In addition, differentiating (12) yields (note that the expression in the square brackets on the right-hand side of (12) is constant):

$$\begin{aligned}
&\left[Q_1^{n_1} + Q_2^{n_1} e^{-J\lambda_n T}\right] J\lambda_n e^{J\lambda_n t} u_{n,\mathrm{reg}}(t) \\
&\quad + \left[Q_1^{n_1} + Q_2^{n_1} e^{-J\lambda_n T}\right] e^{J\lambda_n t} u'_{n,\mathrm{reg}}(t) \\
&= Q_1^{n_1} e^{J\lambda_n t} f_{n,\mathrm{reg}}(t) \\
&\quad - Q_2^{n_1} e^{-J\lambda_n T} \left[-e^{J\lambda_n t} f_{n,\mathrm{reg}} f(t)\right].
\end{aligned} \tag{14}$$

Dividing both sides of (14) by common factors, we obtain

$$u'_{n,\mathrm{reg}}(t) + J\lambda_n u_{n,\mathrm{reg}}(t) = f_{n,\mathrm{reg}}(t), \tag{15}$$

which is the regular part for each DAE of the Kronecker-Weierstrass form. It is an easy exercise to show that $u_{n,\mathrm{nil}}(t)$ in (11) satisfies the equation

$$J_0 u'_{n,\mathrm{nil}}(t) + \lambda_n u_{n,\mathrm{nil}}(t) = f_{n,\mathrm{nil}}(t). \tag{16}$$

This completes the proof. ■

Corollary 1. If $f(x,t) \equiv 0$ and $\beta(x) \equiv 0$, then the BVP has a non-zero solution if and only if $\left[Q_1^{n_1} + Q_2^{n_1} e^{-J\lambda_n T}\right]$ is singular for at least one λ_n, $n = 0, 1, 2, \ldots$. The solution is not unique.

Proof. It follows immediately from (12) with the zero right-hand side. ■

Remark 1. Equation (12) depends only on the initial and final values of $f_{n,\mathrm{nil}}(t)$. The above analysis can also be extended to the case when (1) is $Au_t + Bu_{xx} + Cu = f(x,t)$. Then $-\lambda_n B$ in (4) is replaced with $-\lambda_n B + C$, as was shown in (Campbell and Marszałek, 1996; Marszałek and Trzaska, 1995).

4. Convergence Properties and Error Analysis

We shall assume that no zero eigenvalue appears among the eigenvalues generated by the conditions (2) and that every matrix $\left[Q_1^{n_1} + Q_2^{n_1} e^{-\lambda_n JT}\right]$ is non-singular for $n = 0, 1, \ldots$. Also, we assume that the orthogonal series expansions $\sum_{n=0}^{\infty} \beta_n \phi_n(x)$ and the input $\sum_{n=0}^{\infty} f_n(t)\phi_n(x)$ are convergent in the L^2 sense to $\beta(x)$, $0 \le x \le L$ and $f(x,t)$, $0 \le x \le L, 0 \le t \le T$, respectively.

The convergence properties of the above method can be analyzed by noticing that the Kronecker-Weierstrass decomposition of matrices

$$A \equiv \begin{bmatrix} I & 0 \\ 0 & J_0 \end{bmatrix} \quad \text{and} \quad B \equiv \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}$$

effectively splits the problem into two linear equations:

$$\begin{aligned} \frac{\partial u_{\rm reg}}{\partial t} + J\frac{\partial^2 u_{\rm reg}}{\partial x^2} &= f_{\rm reg}(x,t), \\ J_0\frac{\partial u_{\rm nil}}{\partial t} + \frac{\partial^2 u_{\rm nil}}{\partial x^2} &= f_{\rm nil}(x,t), \end{aligned} \tag{17}$$

with J and J_0 described in the previous section, and where $f_{\rm reg}(x,t)$ and $f_{\rm nil}(x,t)$ follow from the transformation of $f(x,t)$ into two vectors corresponding to the Kronecker-Weierstrass decomposition of $u(x,t)$ into the *regular* and *nilpotent* vectors. Also, we can use the partition of

$$u(x,t) = \begin{bmatrix} u_{\rm reg}(x,t) \\ u_{\rm nil}(x,t) \end{bmatrix}$$

to rewrite (3) as

$$\begin{aligned} &Q_1^{n_1} u_{\rm reg}(x,0) + Q_1^{n_2} u_{\rm nil}(x,0) \\ &\quad + Q_2^{n_1} u_{\rm reg}(x,T) + Q_2^{n_2} u_{\rm nil}(x,T) = \beta(x). \end{aligned} \tag{18}$$

Thus the vectors $u_{\rm reg}(x,t)$ and $u_{\rm nil}(x,t)$, which would normally be separated (see (17)), are in fact linked together through (18).

Let $u_{\rm reg}(x,t)$ and $u_{\rm nil}(x,t)$ be the exact solutions of (17) and suppose that the method of the separation of variables yields the solutions

$$\bar{u}_{\rm reg}(x,t) = \sum_{n=0}^{\infty} u_{n,\rm reg}(t)\phi_n(x)$$

and

$$\bar{u}_{\rm nil}(x,t) = \sum_{n=0}^{\infty} u_{n,\rm nil}(t)\phi_n(x)$$

for the *regular* and *nilpotent* parts, respectively.

Consider the mean-square convergence (also known as the L^2 convergence (Watkins, 1991)) of the solution as measured by

$$\int_\Omega \left\| \begin{bmatrix} u_{\rm reg}(x,t) \\ u_{\rm nil}(x,t) \end{bmatrix} - \begin{bmatrix} \bar{u}_{\rm reg}(x,t) \\ \bar{u}_{\rm nil}(x,t) \end{bmatrix} \right\|^2 {\rm d}x\,{\rm d}t,$$

where $\Omega \equiv \{(x,t) \mid 0 \le x \le L, 0 \le t \le T\}$ and for any vector $a \in \mathbb{R}^q$ we have $\|a\| = \sqrt{\sum_{i=1}^q |a_i|^2}$.

Define the errors $y_{\rm reg}(x,t) \equiv u_{\rm reg}(x,t) - \bar{u}_{\rm reg}(x,t)$ and $y_{\rm nil}(x,t) \equiv u_{\rm nil}(x,t) - \bar{u}_{\rm nil}(x,t)$. We shall show that both the errors are equal to zero in Ω. Substituting $u_{\rm reg}(x,t)$, $u_{\rm nil}(x,t)$ and the respective series of $f_{\rm reg}(x,t)$ and $f_{\rm nil}(x,t)$ into (17), we obtain that the errors $y_{\rm reg}(x,t)$ and $y_{\rm nil}(x,t)$ satisfy the equations

$$\begin{aligned} \frac{\partial y_{\rm reg}}{\partial t} + J\frac{\partial^2 y_{\rm reg}}{\partial x^2} &= 0, \\ J_0\frac{\partial y_{\rm nil}}{\partial t} + \frac{\partial^2 y_{\rm nil}}{\partial x^2} &= 0. \end{aligned} \tag{19}$$

Moreover, the condition (18) for the errors $y_{\rm reg}(x,t)$ and $y_{\rm nil}(x,t)$ is

$$\begin{aligned} &Q_1^{n_1} y_{\rm reg}(x,0) + Q_1^{n_2} y_{\rm nil}(x,0) \\ &\quad + Q_2^{n_1} y_{\rm reg}(x,T) + Q_2^{n_2} y_{\rm nil}(x,T) = 0. \end{aligned} \tag{20}$$

The second of the two equations (19), with the nilpotent matrix J_0, has only the trivial solution $y_{\rm nil}(x,t) = 0$. The reason is as follows: Because of the zero right-hand side of that equation and the block structure of the nilpotent matrix J_0, we have here many subsystems of equations of various sizes, but a typical block, of size, say $s \times s$, has the form

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \frac{\partial y_{\rm nil}(x,t)}{\partial t} + \frac{\partial^2 y_{\rm nil}(x,t)}{\partial x^2} = 0. \tag{21}$$

If we assume that $\bar{y}_{\rm nil}(x,t) = \sum_{n=0}^{\infty} y_{n,\rm nil}(t)\phi_n(x)$, then $y_{n,\rm nil}(t)$ is the solution of

$$J_0 y'_{n,\rm nil}(t) + \lambda_n y_{n,\rm nil}(t) = 0, \tag{22}$$

where J_0 is the nilpotent $s \times s$ matrix of the left-hand side of (21). Since the boundary conditions yield no zero eigenvalues λ_n, (22) has only the trivial solution $y_{n,\rm nil}(t) = 0$ for $n = 0, 1, \dots$.

Since $y_{n,\rm nil}(t) = 0$ yields $y_{\rm nil}(x,t) = 0$, (20) reduces now to $Q_1^{n_1} y_{\rm reg}(x,0) + Q_2^{n_1} y_{\rm reg}(x,T) = 0$. Also, if M_i, $i = 1, 2$ in (2) are partitioned as

$$M_i = \begin{bmatrix} M_{i,\rm reg} \\ M_{i,\rm nil} \end{bmatrix},$$

then

$$M_{1,\rm reg} y_{\rm reg}(x,t)_{|x=0} = 0$$

and

$$M_{2,\rm reg} y_{\rm reg}(x,t)_{|x=L} = 0.$$

Note that the initial and boundary conditions stated above along with the first equation in (19) yield $y_{\rm reg}(x,t) = \sum_{n=0}^{\infty} T_n(t)\phi_n(x)$, where $T_n(t) = e^{-\lambda_n J t} A_n$, A_n being constant vectors, $n = 0, 1, \dots$, and

$$\sum_{n=0}^{\infty} \left(Q_1^{n_1} + Q_2^{n_1} e^{-\lambda_n J T} \right) A_n \phi_n(x) = 0. \tag{23}$$

Since every matrix $[Q_1^{n_1} + Q_2^{n_1} e^{-\lambda_n J T}]$ was assumed to be non-singular, (23) is satisfied if and only if $A_n = 0$ for all $n = 0, 1, \dots$. This gives $y_{\rm reg}(x,t) = 0$. This fact combined with $y_{\rm nil}(x,t) = 0$ implies $u_{\rm reg}(x,t) - \bar{u}_{\rm reg}(x,t) = 0$ and $u_{\rm nil}(x,t) - \bar{u}_{\rm nil}(x,t) = 0$, which completes the convergence analysis.

Example 2. Consider (1) with $T = 1,\ f(x,t) \equiv 0$, zero Dirichlet conditions at $x = 0$ and $x = 1$, and

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\beta(x) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \sin(2\pi x). \tag{24}$$

Since $\lambda_n = (n+1)^2\pi^2,\ n = 0, 1, 2, \dots$ and

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

we get that

$$S_n \equiv \left[Q_1^{n_1} + Q_2^{n_1} e^{-J\lambda_n T}\right]$$

is non-singular for all n. However, since $\beta_2 = [\ 1 \quad 2\ ]'$, $\beta_i = 0,\ i \neq 2$, we have $u_{2,\mathrm{reg}}(t) = e^{-4\pi^2 Jt} S_2^{-1} \beta_2$, $u_{i,\mathrm{reg}}(t) = 0,\ i \neq 2$, and

$$u(x,t) = \begin{bmatrix} u_{\mathrm{reg}}(x,t) \\ u_{\mathrm{nil}}(x,t) \end{bmatrix} \in \mathbb{R}^3,$$

$$u_{\mathrm{reg}}(x,t) = u_{2,\mathrm{reg}}(t) \sin(2\pi x), \quad u_{\mathrm{nil}}(x,t) = 0. \qquad \blacklozenge$$

5. Conclusion

This note determines a general solution to a linear BVP for the PDAE problem. The analytical solution may not be unique if $[Q_1^{n_1} + Q_2^{n_1} e^{-J\lambda_n T}]$ is singular. This matrix is equivalent to the *shooting* matrix of the BVP for the DAE problem considered in (Clark and Petzold, 1989). The converegence properties of the modal method for this linear BVP for the PDAE problem have also been established.

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