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ASYMPTOTIC BEHAVIOUR OF A DISCRETE DYNAMICAL SYSTEM GENERATED BY A SIMPLE EVOLUTIONARY PROCESS

IWONA KARCZ-DULĘBA*

* Wrocław University of Technology
Institute of Engineering Cybernetics
ul. Wybrzeże Wyspiańskiego 27, 50–370 Wrocław, Poland
e-mail: `kdiwona@ict.pwr.wroc.pl`

A simple model of phenotypic evolution is introduced and analysed in a space of population states. The expected values of the population states generate a discrete dynamical system. The asymptotic behaviour of the system is studied with the use of classical tools of dynamical systems. The number, location and stability of fixed points of the system depend on parameters of a fitness function and the parameters of the evolutionary process itself. The influence of evolutionary process parameters on the stability of the fixed points is discussed. For large values of the standard deviation of mutation, fixed points become unstable and periodical orbits arise. An analysis of the periodical orbits is presented.

Keywords: phenotypic evolution, population dynamics, small populations, asymptotic behaviour, space of population states

1. Introduction

Although evolutionary algorithms have been recently widely applied and examined, there is still a need for new methods to describe and analyse complex population dynamics. The main reason behind searching for new methods is the fact that standard analytic tools are usually useless, especially in dealing with models of small populations. Equipped with a mathematical model of an evolving population, we are interested in a long-term behaviour of the population. An asymptotic analysis for a class of Genetic Algorithms was proposed with the use of stochastic Markov chain theory (Vose, 1999). In this paper we take advantage of a special representation of populations in the space of population states, where every point describes the whole population instead of particular individuals (Galar and Karcz-Dulęba, 1994; Dulęba and Karcz-Dulęba, 1996). The state-space approach facilitates the analysis of the population as a whole. The population's evolution is naturally represented as a trajectory in the state space. Consequently, classical tools of the theory of dynamical systems can be used to answer important questions concerning the number of fixed points, their stability and other phenomena. Obviously, when translated into the evolutionary domain, the answers admit to discover many interesting characteristics of the population. For example: How fast and where does the population tend to? Is the population stable after an infinite period of time? Can the population exhibit a periodic behaviour?

In this paper the model of asexual phenotypic evolution, based on the Darwinian theory of evolution, is considered. Populations of individuals evolve according to the rules of proportional selection and normally distributed mutation. The simplest case of evolution is analysed when a two-element population evolves in a one-dimensional search space. These assumptions allow us to characterize the populations analytically and to visualize population states easily. The aim of keeping the number of parameters as small as possible is to present phenomena of evolution in the simplest and the most transparent way, without a mess caused by increasing their number. Despite the simplicity of the evolutionary process presented, its behaviour seems to be quite complicated. Evolutionary processes are usually governed by indeterministic (stochastic) rules. The analysis of each particular exemplification of the process is not informative enough as it strongly depends on an initial distribution of the population and other parameters. Therefore, two alternatives can be proposed. The first one is to perform a large number of experiments and to draw statistically reliable conclusions. The second alternative explained in this paper is to observe the population as a whole rather than each of its elements separately. Thus a macro behaviour of the population can be investigated. The most important characteristic describing a population is its expected position in the consecutive iterations of the evolutionary process. It appears that the expected position of the population can be uniquely described by an appropriate discrete dynam-

ical system. Its equations (coordinates) are derived from the model of the evolutionary process. The asymptotic behaviour of the evolutionary process is determined by fixed points of the dynamical system. The fixed points are located in the vicinity of optima of the fitness functions and saddles between the optima. The stability of the fixed points depends primarily on a parameter of the evolutionary process — the standard deviation of mutation. Fixed points could be stable, unstable or may form periodical orbits. Both the theoretical and simulation analysis of the population evolution in the space of population states shed a light on the dynamics of reaching an equilibrium state by the population. If the number of fixed points is greater than one, the state space is naturally decomposed into separable components. Each component (set) is composed of those points that are attracted by a given fixed point. Those sets are called the basins later on. The structure of the basins of attraction is also presented.

The outline of the paper is as follows: in Section 2 a description of the general model of phenotypic evolution is given and made concrete for two-element populations. The discrete dynamical system generated by the expected values of the population states is given in Section 3. In Section 4, an analysis of fixed points of the system is provided. The basins of attraction of fixed points are studied in Section 5. The stability analysis of fixed points and periodic orbits is presented in Section 6. Conclusions are drawn in Section 7.

2. Model of Phenotypic Evolution

2.1. Model of Evolution

The model of phenotypic evolution called the *evolutionary search with soft selection* is considered (Galar, 1985). A population consists of m individuals $\boldsymbol{x} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m\}$. Each individual is described by an n-dimensional vector of traits (*type*) $\boldsymbol{x}_k \in \mathbb{R}^n$ ($k = 1, \ldots, m$) and the non-negative quality index $q(\boldsymbol{x}_k)$ (*fitness*). The successive generation is based on the current generation and created with the use of two operators only: selection and mutation. The simple form of selection, proportional to the individual's fitness, is applied. Descendants inherit parental traits mutated with independent random variables, normally distributed with identical standard deviation σ. The rules of proportional selection and normally distributed mutation define the conditional distribution $f_{\boldsymbol{x}}^{i+1}(\boldsymbol{x}|\boldsymbol{x}^i)$ of a new individual's position in $\mathbb{R}^n$,

$$f_{\boldsymbol{x}}^{i+1}(\boldsymbol{x}|\boldsymbol{x}^i) = \sum_{k=1}^{m} \alpha(\boldsymbol{x}_k^i) g(\boldsymbol{x}, \boldsymbol{x}_k^i)$$

$$= \sum_{k=1}^{m} \frac{q(\boldsymbol{x}_k^i)}{\sum_{j=1}^{m} q(\boldsymbol{x}_j^i)} g(\boldsymbol{x}, \boldsymbol{x}_k^i), \qquad (1)$$

where $f_{\boldsymbol{x}}^{i+1}(\boldsymbol{x}|\boldsymbol{x}_i)$ is the conditional distribution in the $(i+1)$-st generation when the position of the population in the i-th generation is $\boldsymbol{x}^i$, $\boldsymbol{x}_k^i \in \mathbb{R}^n$ stands for the type of the k-th individual in the i-th generation, $\alpha(\boldsymbol{x}_k^i)$ denotes the probability of selecting individual $\boldsymbol{x}_k^i$, $q(\boldsymbol{x}_k^i)$ means the fitness of individual $\boldsymbol{x}_k^i$, $g(\boldsymbol{x}, \boldsymbol{x}_k^i)$ is the distribution of mutation of the k-th individual, $g(\boldsymbol{x}, \boldsymbol{x}_k^i) = \mathbb{N}(\boldsymbol{x}_k^i, \sigma)$.

The distribution (1) is defined in the *space of types* (*traits*) $T = \mathbb{R}^n$, where each coordinate depicts an individual's trait and a point represents a type of an individual. The whole population is described by a set of m points and evolution is described as a movement of the set through the space of types. In the landscape of a fitness function, evolution leads the population to areas of a higher quality.

Another approach deals with a whole population represented by a single point in the *space of population states* S (Galar and Karcz-Dulęba, 1994; Dulęba and Karcz-Dulęba, 1996; Karcz-Dulęba, 2000; Chorążyczewski *et al.*, 2000; Karcz-Dulęba, 2002a). In the space S each point (state) $\boldsymbol{s}^i = (x_{1,1}^i, x_{1,2}^i, \ldots, x_{1,n}^i, x_{2,1}^i, x_{2,2}^i, \ldots, x_{m,1}^i, \ldots, x_{m,n}^i)$ represents the whole population in the i-th generation. The dimension of the state space S is equal to the number of individuals multiplied by the number of traits $\dim(S) = nm$, but it is not just a product of type spaces and it is not identical with $\mathbb{R}^{nm}$. The dynamics of evolution are independent of the ordering of individuals within the population. For example, the population state $(x_{1,1}^i, x_{1,2}^i, \ldots, x_{1,n}^i, x_{2,1}^i, x_{2,2}^i, \ldots, x_{m,1}^i, \ldots, x_{m,n}^i)$ represents the same state as $(x_{2,1}^i, x_{2,2}^i, \ldots, x_{2,n}^i, x_{1,1}^i, x_{1,2}^i, \ldots, x_{m,1}^i, \ldots, x_{m,n}^i)$. Therefore, the equivalence relation U that identifies all the points corresponding to the permutations of individuals in the population has to be defined on S. The non-redundant space of population states S becomes the factor (quotient) space $S_U = \mathbb{R}^{nm}/U$. Although the structure of S is rather complex, this approach facilitates the characterization of the population's evolution through the analysis of the population's trajectories in the quotient space.

Since the distributions (1) for every trial are the same and independent of one another, the joint distribution of the population state is just a product of m distributions (1). In the quotient space S_U corresponding to the permuting equivalence relation U the distribution of the population state in the $(i + 1)$-st iteration is given by the density function

$$\tilde{f}_{S_U}^{i+1}(\boldsymbol{s}|\boldsymbol{s}^i) = m! \prod_{j=1}^{m} f_x^{i+1}(\boldsymbol{x}_j|\boldsymbol{s}^i)$$

$$= m! \prod_{j=1}^{m} \sum_{k=1}^{m} \alpha(\boldsymbol{x}_k^i) g(\boldsymbol{x}_j, \boldsymbol{x}_k^i). \qquad (2)$$

The probability of locating the population at a particular state $\boldsymbol{s}$ is determined by the current state $\boldsymbol{s}^i$,

the fitness function $\alpha(\boldsymbol{x}_k^i)$ and the mutation distribution $g(\boldsymbol{x}, \boldsymbol{x}_k^i)$. The distribution (2) can be multimodal, cf. Figs. 3(b) and (d).

2.2. Two-Element Population

In one-element populations evolution is not possible because selection requires at least two individuals. Therefore, the simplest case of evolution when a two-element population evolves in a one-dimensional space of types ($m = 2$, $n = 1$) is considered. Even in such a simple case, the analysis of evolution is far from trivial and it becomes analytically intractable as m and n are increased.

In order to define the space of population states for the population of two individuals $\boldsymbol{x} = (x_1, x_2)$, the equivalence relation is introduced:

$$\mathbb{R}^2 \to S_U : \quad (x_1^i, x_2^i) \to \begin{cases} (x_1^i, x_2^i) & \text{for} \quad x_1^i \geq x_2^i, \\ (x_2^i, x_1^i) & \text{for} \quad x_1^i < x_2^i. \end{cases}$$

The mapping shrinks the space by identifying points symmetric with respect to the line $X_1 = X_2$, called the *identity axis*, and identifying the right half-plane with the factor space S_U (Fig. 1). In the space S_U, the distribution (2) of the two-individual population is given by

$$\tilde{f}_{S_U}^{i+1}(x_1, x_2 | \boldsymbol{s}^i)$$

$$= 2 f_x^{i+1}(x_1 | \boldsymbol{s}^i) f_x^{i+1}(x_2 | \boldsymbol{s}^i)$$

$$= 2 \big(\alpha(x_1^i) g(x_1, x_1^i) + \alpha(x_2^i) g(x_1, x_2^i) \big)$$

$$\times \big(\alpha(x_1^i) g(x_2, x_1^i) + \alpha(x_2^i) g(x_2, x_2^i) \big), \quad (3)$$

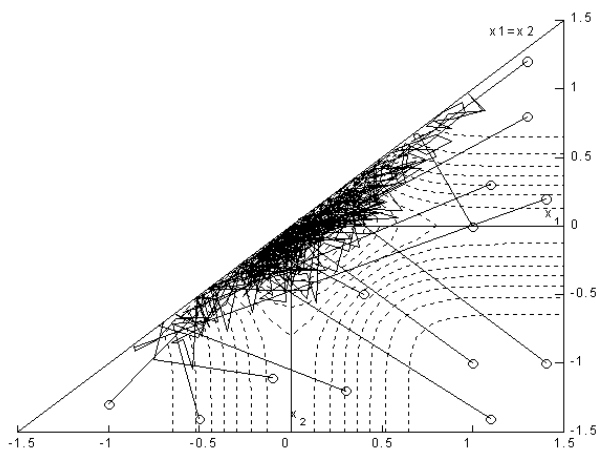where $\alpha(x_k^i) = q(x_k^i) / (q(x_1^i) + q(x_2^i))$, $k = 1, 2$.

Fig. 1. Evolution of two-element populations in the coordinate frames $X_1 X_2$ (the unimodal fitness function (4), $a = 5$, $\sigma = 0.1$; 50 generations are presented for initial states marked with open circles).

The presentation in the quotient space S_U becomes more convenient after the rotation of the coordinate frame $X_1 X_2$ with the angle $\gamma = \pi/4$. The new coordinate frame WZ is given by the transformations $w^i = (x_1^i - x_2^i)/\sqrt{2}$ and $z^i = (x_1^i + x_2^i)/\sqrt{2}$. Therefore, the population state depicted by traits $\boldsymbol{s}^i = (x_1^i, x_2^i)$ is mapped into the state $\boldsymbol{s}^i = (w^i, z^i)$ and the quotient space S_U is transformed to the right half-plane ($w \geq 0$) (Fig. 2). The new coordinates (w, z) admit a nice interpretation. The coordinate w describes a distance of the population state from the identity axis and it may be considered as a measure of the population's diversity. The coordinate z situates a state along the identity axis. In the new frame the identity axis coincides with the Z-axis.
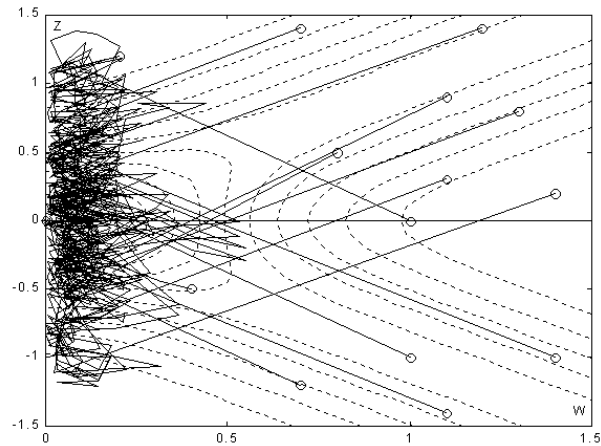

Fig. 2. Evolution of two-element populations in the coordinate frames WZ (the unimodal fitness function (4), $a = 5$, $\sigma = 0.1$; 50 generations are presented for initial states marked with open circles).

In order to illustrate evolution dynamics, simulations are performed in landscapes of uni- and multimodal one-dimensional fitness functions. Unimodal fitness functions allow us to check if and how fast the optimum is reached. Multimodal fitness functions are selected to evaluate the dynamics of crossing saddles between optima. As an example of the unimodal symmetrical fitness function, the following bell-shaped Gaussian function with the maximum located at zero is chosen:

$$q(x) = \exp(-ax^2). \quad (4)$$

Multimodal fitness functions are represented by sums of Gaussian functions

$$q(x) = \sum_k h_k \exp\big(-a(x - d_k)^2\big), \quad (5)$$

where h_k, d_k are real numbers and $h_k > 0$.

evolution trajectories   trajectories of the mean

(a) $s^0 = (1.3, 1.5)$, $G_s = 905$

(b) $s^0 = (0.6, 0.1)$, $G_s = 895$

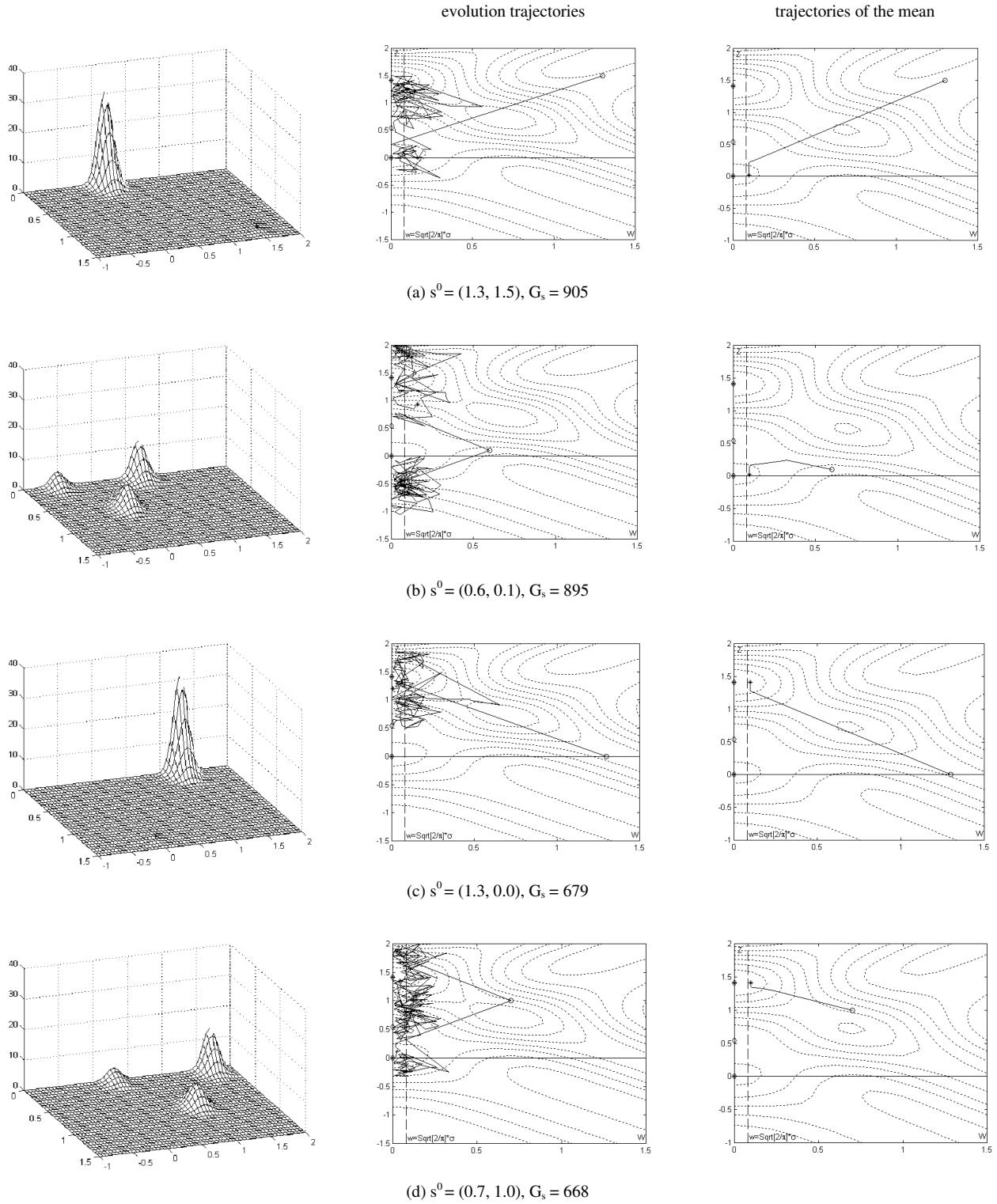(c) $s^0 = (1.3, 0.0)$, $G_s = 679$

(d) $s^0 = (0.7, 1.0)$, $G_s = 668$

Fig. 3. Population dynamics in the landscape of a bimodal fitness function (5) for four different initial states s^0. Distributions of the population states in the first generation (3) are presented in the left panels. The middle panels display examples of evolution trajectories in 100 generations. In (c) and (d) two trajectories show possible jumps of states to distinct optima. Trajectories of the expected states are presented in the right panels. G_s indicates the generation reaching a fixed point marked with an asterisk ($k = 2$, $a = 5$, $h_1 = 1$, $h_2 = 2$, $d_1 = 0$, $d_2 = 1$, $\sigma = 0.1$).

Simulations of evolution in the quotient space discover some characteristic properties of evolutionary dynamics (Figs. 2 and 3). The population quickly moves (almost jumps) toward states located near the identity axis. Then it wanders slowly toward the optimum of the fitness function. Finally, it maintains its stochastic equilibrium. The jump in the vicinity of the identity axis indicates unification of the (initially diversified) population and it is caused by the selection. The motion of an almost homogeneous population towards an optimum is effected by the mutation. The trajectory may stay around local optima for a relatively long period of time (for many iterations). This phase of evolution can be called the *quasi-equilibrium phase*. It can be abandoned if some individuals find a new, higher peak of an optimum.

3. Discrete Dynamical System Generated by Evolution

In the coordinate frame WZ, the expected values of the population state $\boldsymbol{s}^i = (w^i, z^i)$ in the next generation can be calculated analytically using the density function (3). The expected values of coordinates w and z, $E_{i+1}[w|\boldsymbol{s}^i]$ and $E_{i+1}[z|\boldsymbol{s}^i]$, in the $(i+1)$-st generation are equal to

$$E_{i+1}[w|\boldsymbol{s}^i] = \sqrt{\frac{2}{\pi}}\sigma + \left(1 - \Psi^{i^2}\right)$$
$$\times \sigma \left[\varphi_0\left(\frac{w^i}{\sigma}\right) + \frac{w^i}{\sigma}\Phi_0\left(\frac{w^i}{\sigma}\right)\right], \quad (6)$$

$$E_{i+1}[z|\boldsymbol{s}^i] = z^i + \Psi^i w^i, \quad (7)$$

where

$$\Psi^i(w,z) = \frac{q_1 - q_2}{q_1 + q_2}, \quad q_1 = q(x_1^i) = q\left(\frac{w^i + z^i}{\sqrt{2}}\right),$$

$$q_2 = q(x_2^i) = q\left(\frac{z^i - w^i}{\sqrt{2}}\right),$$

$$\varphi_0(x) = \frac{1}{\sqrt{2\pi}}\left[\exp\left(-\frac{x^2}{2}\right) - 1\right],$$

$$\Phi_0(x) = \frac{1}{\sqrt{2\pi}}\int_0^x \exp\left(-\frac{t^2}{2}\right)\mathrm{d}t,$$

q is a fitness function, and φ_0 and Φ_0 conform to the normal distribution of mutation.

Equation (6) consists of two components. The first depends only on the standard deviation of mutation σ. The second depends also on the fitness function (the coefficient Ψ) and on the current value of coordinate w. Since the fitness function $q(x)$ is non-negative, the coefficient $\Psi \subset [-1, 1]$, so $(1 - \Psi^2) \in [0, 1]$ and $\varphi_0 \geq 0$, $\Phi_0 \geq 0$, therefore $E_{i+1}[w|\boldsymbol{s}^i] \geq \sqrt{2/\pi}\sigma$. The expected value (6) is equal to $\sqrt{2/\pi}\sigma$ for $1 - \Psi^2 = 0$, i.e. when the differences in the individuals' fitness values are significant. The second component influences $E_{i+1}[w|\boldsymbol{s}^i]$ when $1 - \Psi^2 > 0$, which happens for states in the vicinity of axes W and Z. The expected value of coordinate z in (7) depends on the sign of the coefficient Ψ. The value of coordinate z increases or decreases depending on the ratio of the individuals' qualities $q(x_2)/q(x_1)$.

The expected values (6) and (7) generate a discrete dynamical system in the space S_U described by the following equations:

$$\begin{cases} w^{i+1} = \sqrt{\dfrac{2}{\pi}}\sigma + (1 - \Psi^{i^2}) \\ \qquad \times \sigma\left[\varphi_0\left(\dfrac{w^i}{\sigma}\right) + \dfrac{w^i}{\sigma}\Phi_0\left(\dfrac{w^i}{\sigma}\right)\right], \\ z^{i+1} = z^i + \Psi^i w^i. \end{cases} \quad (8)$$

Equation (6) defines two scalar mappings

$$(w, z) \to F(w, z) = \begin{bmatrix} F_1(w, z) \\ F_2(w, z) \end{bmatrix}. \quad (9)$$

Exemplary trajectories of the deterministic dynamical system (9) derived from the (stochastic) probability distribution (3) are depicted in Fig. 3. The trajectories exhibit some characteristic properties. At first, a jump to a state located in the vicinity of the identity axis is observed. Then, the current state slowly moves toward the optimum. Finally, the population maintains its equilibrium state.

4. Fixed Points of the Discrete Dynamical System

Fixed points are important in the study of dynamical systems, since they represent stationary or repeatable behaviuor. Evolution directs a population to the state of a selection-mutation equilibrium near the fixed point. Reaching an equilibrium state by the population does not necessarily mean that its elements are immobile. On the contrary, the positions of each individual in the population fluctuate but macroscopic behaviour of the population (the mean value) does not change. The coordinates of the fixed points of the dynamical system (9) are obtained from the equations

$$\begin{cases} w = F_1(w, z), \\ z = F_2(w, z). \end{cases} \quad (10)$$

For any quality function, the equilibrium states $\omega = (w^s, z^s)$ are characterized by the conditions

$$\begin{cases} w^s \cong 0.97\sigma, & (11) \\ \Psi(w^s, z^s) = 0 \Rightarrow q_1 = q_2. & (12) \end{cases}$$

The w-coordinate of the fixed points (11) does not depend on the fitness function. Since the w-coordinate indicates the diversity of the population, from (11) it follows that in an equilibrium state individuals do not have equal types but their types differ with respect to the standard deviation of mutation σ. The z-coordinate of an equilibrium point depends on the fitness function and it satisfies the condition

$$q\left(\frac{z^s + w^s}{\sqrt{2}}\right) = q\left(\frac{z^s - w^s}{\sqrt{2}}\right). \qquad (13)$$

The solution of (13) is determined by the points of intersection of two fitness functions q_1 and q_2 shifted by $\Delta_q = \sqrt{2}w^s \cong 1.37\sigma$, cf. Figs. 4(a) and 5(a). The number of intersection points depends on the number of the optima of the fitness function and on the shift Δ_q, i.e. on the standard deviation of mutation. When the value of σ increases, the number of fixed points may decrease. Fixed points are located in the neighborhood of optima and the saddles between the optima.

In the landscape of any unimodal fitness function, the dynamical system (9) has at most one fixed point (Karcz-Dulęba, 2000). For symmetric fitness functions with the optimum placed at the origin (e.g. (8)), a fixed point of the system lies on the axis W ($z^s = 0$) at the distance of 0.97σ from the optimum. The fixed point $\omega = (0.97\sigma, 0)$ can be called the optimum fixed point. The population in its equilibrium state consists of two individuals with equal fitness values, located symmetrically on the opposite hillsides of the fitness function: $\boldsymbol{x} = (x^s, -x^s)$ with $x^s \cong 0.69\sigma$. The evolution does not lead a population of two individuals exactly to the optimum but to its neighbourhood. However, the distance from the equilibrium point to the optimum decreases as the value of σ decreases. In this case the speed of reaching the equilibrium slows down as well. The asymmetry in the unimodal fitness function influences the value of the z-coordinate of the fixed point leaving its w-coordinate unchanged (Karcz-Dulęba, 2002a).

In the case of a bimodal fitness function, the dynamical system (9) has one or three fixed points (Fig. 4(b)), depending on the value of the standard deviation of mutation. As an example of a symmetrical fitness function, a sum of two bell-shaped functions with optima at the same distance from the origin of the coordinate system
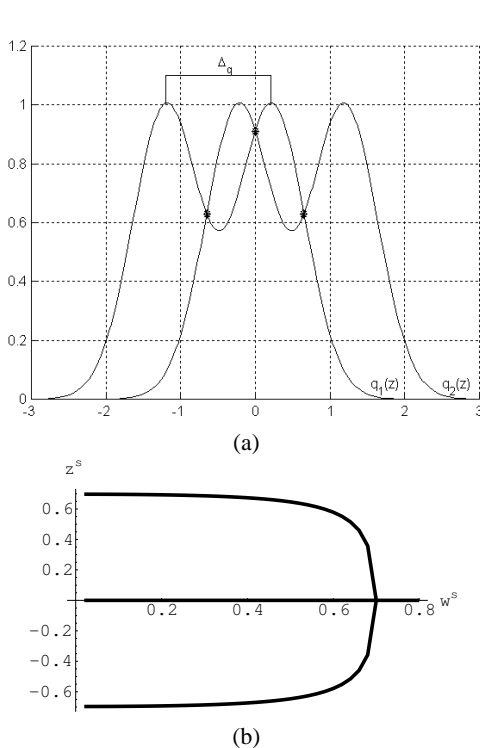

(a)

(b)

Fig. 4. Dynamical system (9) in the landscape of a symmetric bimodal fitness function (5) ($a = 5.0$, $d_1 = -1.0$, $d_2 = 1.0$, $h_1 = h_2 = 1.0$, $\sigma = 0.5$): (a) the z coordinates of fixed points obtained as intersection points of two shifted functions q_1, q_2, (b) location of fixed points as a function of $w^s (\approx \sigma)$.
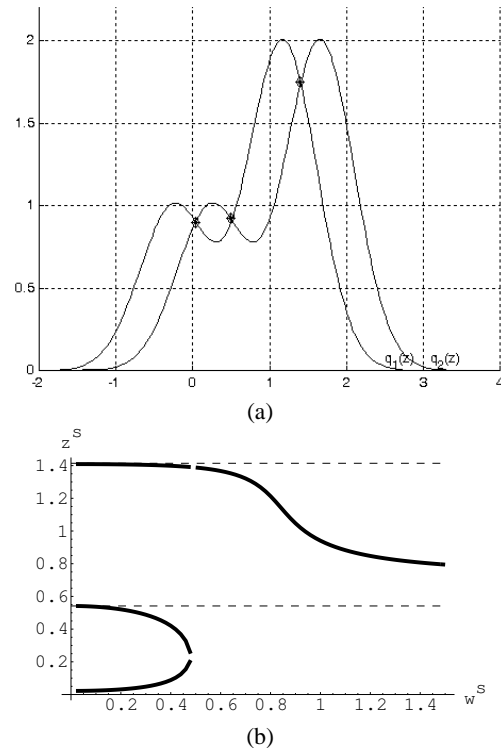

(a)

(b)

Fig. 5. Dynamical system (9) in the landscape of an asymmetric bimodal fitness function (5) ($a = 5.0$, $d_1 = 0.0$, $d_2 = 1.0$, $h_1 = 1.0$, $h_2 = 2.0$, $\sigma = 0.25$): (a) the z coordinates z of fixed points obtained as intersection points of two shifted functions q_1, q_2, (b) location of fixed points as a function of $w^s (\approx \sigma)$.

and equal hill hights (e.g. the function (5) with $k = 2$, $d_1 = -d_2$ and $h_1 = h_2 = 1$) were used. In this case, two symmetrical optima and one saddle fixed points appear. When the standard deviation of mutation σ surpasses its critical value σ_g, two fixed points disappear and one fixed point remains. For an asymmetrical bimodal fitness function (e.g. the function (5) with $d_1 = 0$, $d_2 = 1$, $h_1 = 1$, $h_2 = 2$), one fixed point is located near the local optimum, the second one lying close to the saddle and the third near the global optimum. As the value of the standard deviation σ increases, the fixed points that correspond to the local optimum and the saddle get closer and closer and finally disappear leaving only one global fixed point (Fig. 5(b)). When the standard deviation σ is comparable with the distance between optima, individuals may jump from one hill to another. In this case the global fixed point approaches the saddle.

Generally, for a fitness function with k optima, the number of fixed points may vary from one to $2k + 1$. They are situated near the optima and saddles of the fitness function. For a symmetric fitness function, one of the points is always located on the symmetry axis while others are symmetrically paired (Karcz-Dulęba, 2000). When the value of σ is increased, fixed points around local optima disappear and the global optimum fixed point remains.

5. Basins of Attraction

The basin of attraction $B(x)$ of a given fixed point x is a set composed of those initial points of the evolutionary process at hand that converge (when iterated) to the fixed point or, mathematically,

$$x = \begin{pmatrix} w \\ z \end{pmatrix}, \quad B\big(x | F(x) = x\big) = \Big\{ x_0 | F^i(x_0) \underset{i \to \infty}{\longrightarrow} x \Big\}.$$

Now, basins of attraction of the dynamical system (9) in landscapes of multimodal fitness functions will be considered. The shape of a basin depends on the parameters of a fitness function (the slope of the hillside described by the parameter a, differences of heights between optima) and the parameters of the dynamical system (the standard deviation of mutation σ) (Karcz-Dulęba, 2002b).

Basins of attraction and landscapes of the mean fitness are presented in Fig. 6 for sums of two and three Gaussian functions (5). Interesting facts can be observed in both cases. There exist states initiating trajectories that are attracted by a lower optimum although the states are located closer to a higher optimum. This phenomenon can be explained by looking at the contribution of separate optima characterized by parameters (h_k, d_k) of the multimodal fitness function (5) and the probability distribution of the population state (3). Let us define the *impact*

area of a given optimum. The area is composed of the states where the contribution to the mean value of the fitness resulting from a particular optimum is higher than the contribution of any other optimum considered separately, cf. Fig. 7. For the bimodal fitness function depicted in Fig. 7(a) the impact area of the lower hill is composed of the states (w, z) characterized by the condition

$$h_1 \exp\big(- a(x_1 - d_1)^2 \big) + h_1 \exp\big(- a(x_2 - d_1)^2 \big)$$
$$> h_2 \exp\big(- a(x_1 - d_2)^2 \big)$$
$$+ h_2 \exp\big(- a(x_2 - d_2)^2 \big),$$

where $x_1 = (z + w)/\sqrt{2}$, $x_2 = (z - w)/\sqrt{2}$.

In the next generation, the distribution given by (3) shifts the population towards the local optimum (Fig. 3(a)). The initial states presented in Fig. 3 belong to different basins of attraction. The initial states in Figs. 3(a) and (b) are attracted by a local optimum, while those in Figs. 3(c) and (d) by the global optimum. For some initial states the probability distributions (3) are multimodal with similar heights of optima (Figs. 3(b) and 3(d)). Although the states belong to the basin of attraction of a local or a global optimum, evolution from these states may direct the trajectory to a local or a global optimum (both the cases are illustrated in the figures). The trajectories of the mean value started at these states (thus being attractors of the states) are likely to be determined by the highest peak of the distribution (3). This phenomenon shows the difference between the stochastic behaviour of the evolving population and the deterministic behaviour of its mean.

In the case of equally high optima, one may expect that the space of states is split into two basins of attraction separated by a saddle. In fact, the basins have more complicated structures and they were located along separate optima ridges (Fig. 8(a)). The foregoing arguments confirmed earlier observations (Chorążyczewski *et al.*, 2000) that evolution is attracted mainly by ridges of fitness functions and proceeds along ridges.

Varying the values of other parameters of the evolutionary process (the standard deviation of mutation σ) and fitness functions (the slope a and the height h) influence basins of attractions. The slope a affects the size of basins rather than their shape. When the parameter a is increased (the selection intensity), the number of generations G_s needed to achieve a fixed point decreases and the precision of the optimum localization increases (Karcz-Dulęba, 2002b). When the difference between the heights of the optima decreases, the basin of attraction of the lower hill is enlarged, Fig. 8. While increasing the standard deviation of mutation σ, the basin of attraction of the local optimum is shrunk and disappears when there remains only one global fixed point (Fig. 5(b)).
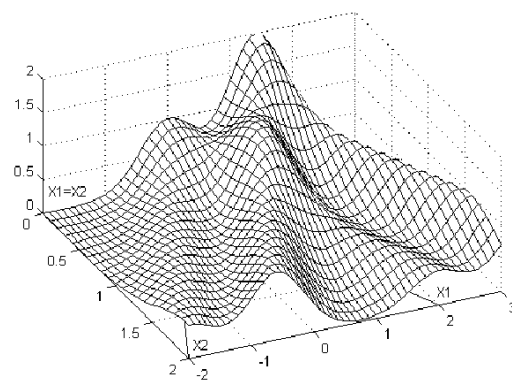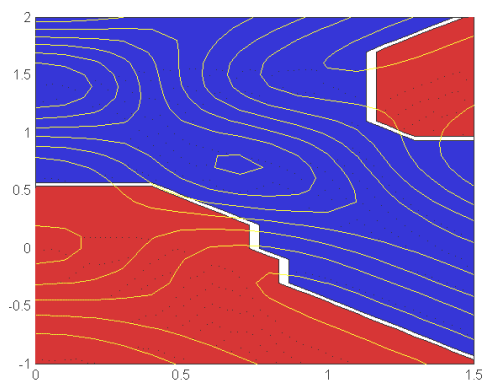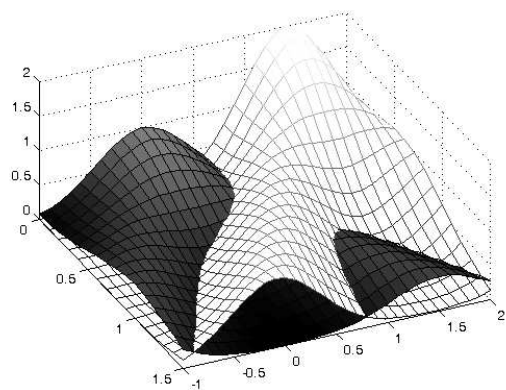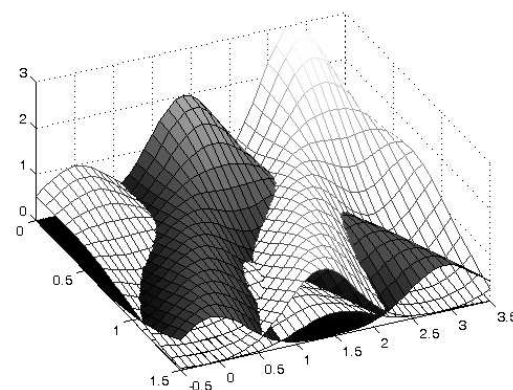
(a)
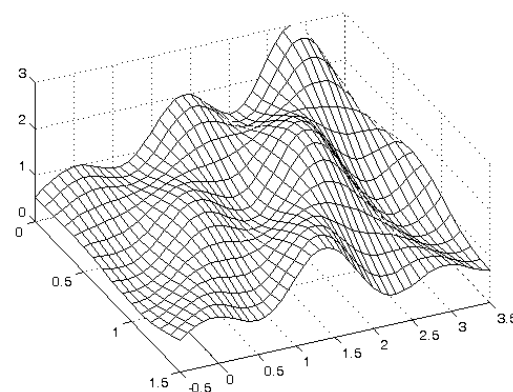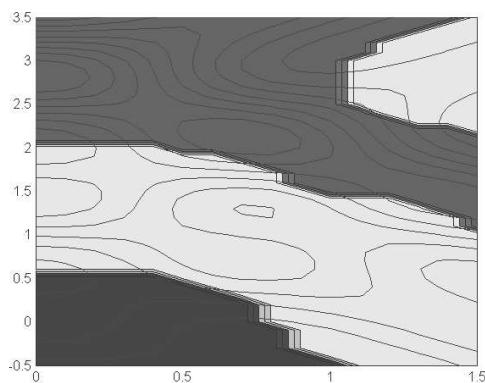
(b)

Fig. 6. Basins of attraction (left panel) and landscapes of the mean fitness (right panel) for multimodal fitness functions ($a = 5.0$, $\sigma = 0.1$): (a) a bimodal function (5), $d_1 = 0.0$, $d_2 = 1.0$, $h_1 = 1.0$, $h_2 = 2.0$, (b) function (5) with three optima, $d_1 = 0.0$, $d_2 = 1.0$, $d_3 = 2.0$, $h_1 = 1.0$, $h_2 = 2.0$, $h_3 = 3.0$.

(a)

(b)

Fig. 7. Impact area of optima for: (a) bimodal fitness function (5), $a = 5.0$, $d_1 = 0.0$, $d_2 = 1.0$, $h_1 = 1.0$, $h_2 = 2.0$, (b) function (5) with three optima, $a = 5.0$, $d_1 = 0.0$, $d_2 = 1.0$, $d_3 = 2.0$, $h_1 = 1.0$, $h_2 = 2.0$, $h_3 = 3.0$.

(a) $h_1 = 1$, $h_2 = 1$, $a = 5$

(b) $h_1 = 1$, $h_2 = 1.5$, $a = 5$

(c) $h_1 = 1$, $h_2 = 2$, $a = 10$
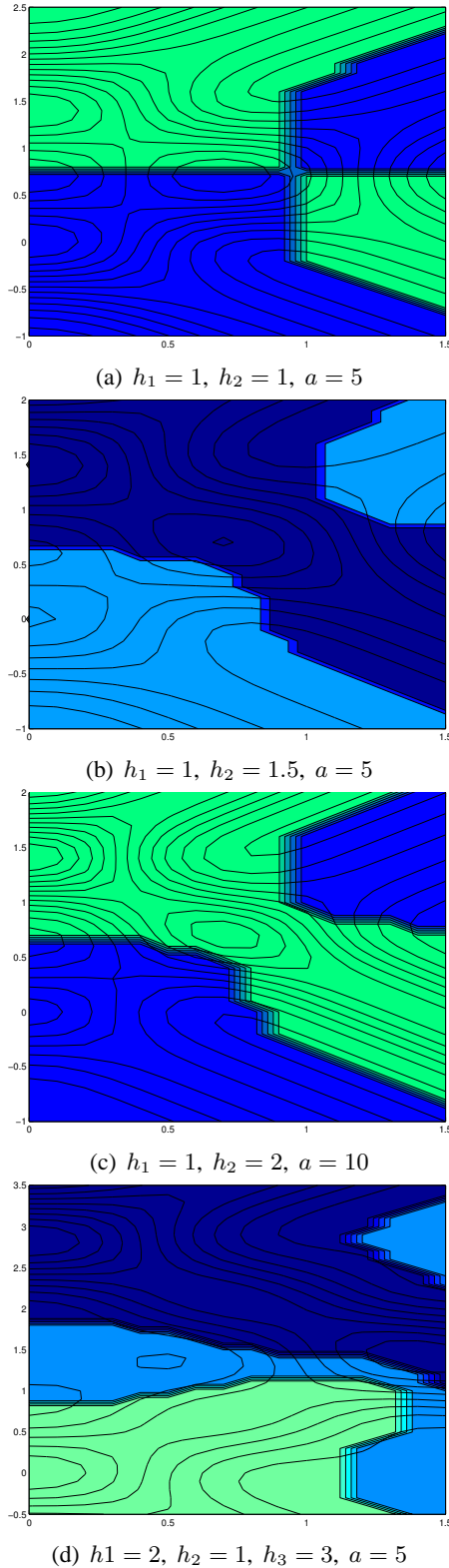

(d) $h1 = 2$, $h_2 = 1$, $h_3 = 3$, $a = 5$

Fig. 8. Basins of attraction for multimodal fitness functions as a function of the optimum height and the hillside slope a: (a)–(c) bimodal function (5), $d_1 = 0.0$, $d_2 = 1.0$, (d) function (5) with three optima, $d_1 = 0.0$, $d_2 = 1.0$, $d_3 = 2.0$, ($\sigma = 0.1$).

6. Stability of Fixed Points and Periodical Orbits

The asymptotic behaviour of the dynamical system (9) depends on the fitness function. The matrix of the linear approximation (the Jacobi matrix of partial derivatives) of the system (9) at a fixed point is diagonal and its eigenvalues are equal to $\lambda_1 = \Phi_0(w^s)$ and $\lambda_2 = w^s \partial \Psi / \partial z + 1$. We have $|\lambda_1| < 1$ because $w^s > 0$. The value of λ_2 depends on the fitness function (coefficient Ψ) and the standard deviation of mutation $\sigma(w^s \cong 0.97\sigma)$. Since at fixed points Eqn. (13) is fulfilled, in order to determine stability the following inequality has to be solved:

$$-2 < w^s \frac{\partial \Psi}{\partial z}\bigg|_{\substack{w^s \\ z^s}} < 0. \tag{14}$$

It appears that the saddle fixed points are always unstable whereas the stability of other fixed points depends on the parameters of the fitness function and the evolutionary process (the standard deviation of mutation σ). Given the parameters of the fitness function, the stability of the system (9) depends on the standard deviation of mutation σ only. When σ is increased, fixed points near local optima disappear and only the fixed point close to the global optimum remains.

Simulations and analytical studies of system stability in the vicinity of fixed points show that for small values of σ fixed points are stable, for larger values of σ fixed points become unstable and a periodic orbit emerges (Fig. 9). A general theoretical analysis of the stability of fixed points is not possible because of system complexity. Analytical results regarding the stability of fixed points and a study of periodic orbits are presented for a simple unimodal fitness function (Fig. 9(a)).

For a symmetric unimodal fitness function (4), the stability of the unique fixed point $(0.97\sigma, 0)$ depends on the slope parameter a and the standard deviation of mutation σ. There exists either a single stable equilibrium point, or a stable orbit with a period of two (Dulęba and Karcz-Dulęba, 1996). The inequality (14) takes the form $-2 < -aw^{s^2} < 0$, for which the right-hand side is always true. Fixing the value of the parameter a makes stability dependent on σ. For small values of σ the fixed point is stable and while increasing σ it looses stability and become unstable. The second eigenvalue is $\lambda_2 = -1$ for the critical value of the standard deviation $\sigma_c \approx \sqrt{2/a}$. Mutation with a critical value σ_c may move offspring of the parent with optimal fitness to the area with very low values of the fitness function (practically equal to zero). Therefore, differences in the fitness of individuals are insignificant, the selection intensity is much smaller and the mutation (with large σ) has a dramatic impact on the population.
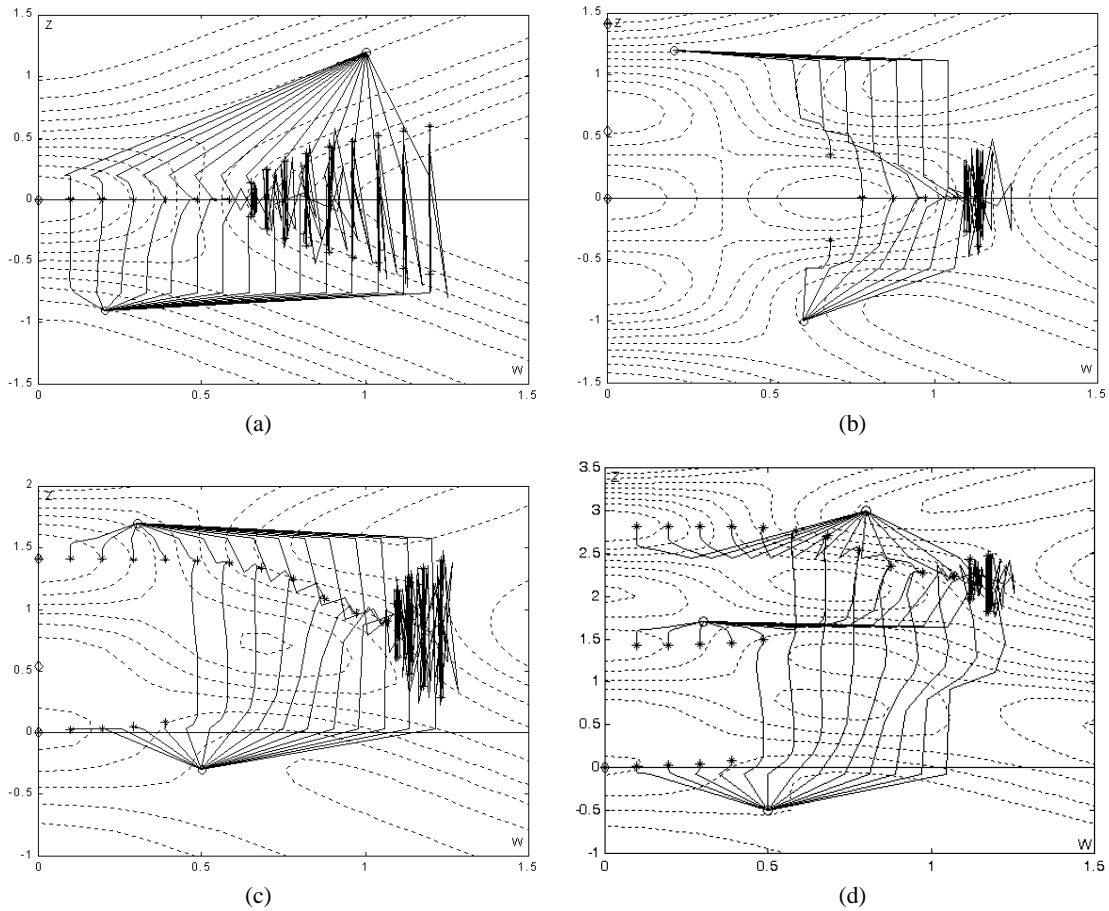
Fig. 9. Trajectories of the expectations in the landscape of various fitness functions. While varying the standard deviation of mutation, stable fixed points and periodic orbits are observed, $\sigma \in [0.1, 1.4]$ with a step of 0.1. Initial states, chosen from different basins of attraction, are denoted by open circles and fixed points are denoted by asterisks: (a) symmetric unimodal fitness function (4) ($a = 5$), trajectories initialized at states $(0.2, -0.9)$ and $(1.0, 1.2)$, (b) symmetric bimodal function (5) ($a = 5$, $d_1 = -0.5$, $d_2 = 0.5$, $h_1 = h_2 = 1.0$), trajectories started at $(0.6, -1.0)$ and $(0.2, 1.2)$, (c) bimodal function (5), $d_1 = 0.0$, $d_2 = 1.0$, $h_1 = 1.0$, $h_2 = 2.0$, trajectories from two initial states $(0.5, -0.3)$ and $(0.3, 1.7)$, (d) function (5) with three optima, $d_1 = 0.0$, $d_2 = 1.0$, $d_3 = 2.0$, $h_1 = 1.0$, $h_2 = 2.0$, $h_3 = 3.0$, trajectories initialized at states $(0.5, -0.5)$, $(0.3, 1.7)$ and $(0.8, 3.0)$.

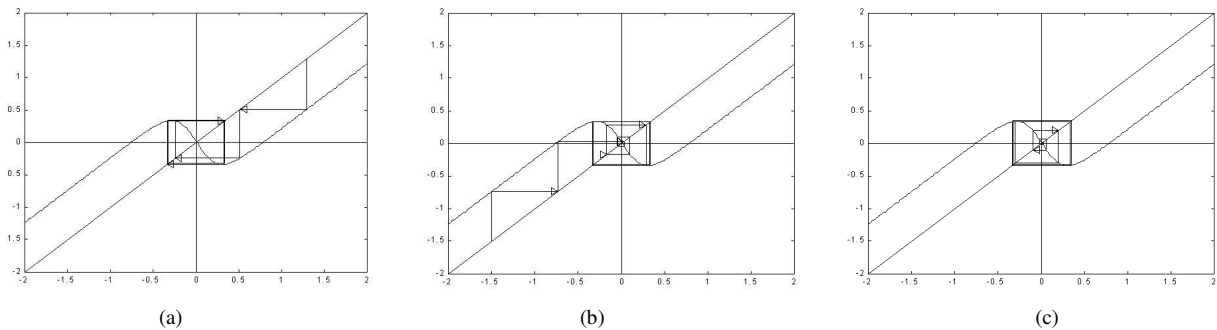
Fig. 10. Graphical analysis of iterating $F_2(w^s, z)$ starting at different initial points ($\sigma = 0.8$): (a) $z_0 = 1.3$, (b) $z_0 = -1.5$, (c) $z_0 = 0.01$.

Above the value of σ_c the pitchfork (period-doubling) bifurcation is observed: a fixed point becomes unstable and a stable periodic orbit of period 2 appears (Fig. 11). Figure 10 shows what happens when $F_2(w^s, z)$ is iterated starting from different values of z. In all cases, trajectories finally alternate between two values of the periodic orbit. It can be noticed that a periodic orbit appears when the standard deviation σ reaches the value where two ridges emerge in the surface of the average fitness (Fig. 9(a)) and the orbit points are located on internal slopes of the ridges. The ridges correspond to the states when one of the individuals is almost optimal.
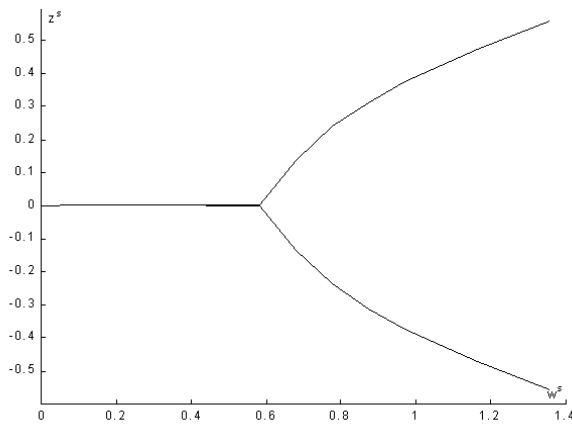

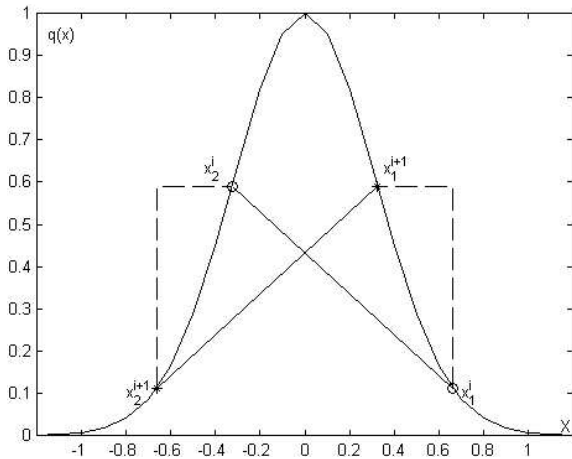Fig. 11. Bifurcation diagram for $F_2(w^s, z)$ showing the pitchfork bifurcation at $\sigma_c = 0.63$.

Fig. 12. Two populations (one denoted by open circles and the other by asterisks) which formed the periodic orbit shown in the landscape of the fitness function (4), $\sigma = 0.8$. The individuals of one population are connected by a solid line, the parents and the offspring are connected by a dotted line.

The periodic orbit consists of two states located symmetrically on both the sides of the fitness function (Fig. 12). In each state the population contains of two individuals: one with a higher fitness value and the other with a much smaller fitness value located on the other side of the hill. The distance between the individuals is slightly greater than σ. In the next generation, the offspring "exchange" coordinates, i.e. $x_1^{i+1} = -x_2^i$ and $x_2^{i+1} = -x_1^i$. For example, for $\sigma = 0.8$ the periodic orbit is composed of two populations $(x_1^i, x_2^i) = (0.66, -0.33)$ and $(x_1^{i+1}, x_2^{i+1}) = (0.33, -0.66)$. The Euclidean distance between the parent and the offspring is about $\sigma/2$.

Simulations showed that for a bimodal fitness function the global optimum fixed point becomes unstable for values of σ greater than the distance between optima. The critical value of the standard deviation practically does not depend on the difference in the hill heights.

7. Conclusion

In this paper the asymptotic behaviour of the discrete dynamical system generated by the evolutionary model was presented. Despite the simplicity of the analysed process, its behaviour is surprisingly complex. Two phases in the trajectories of the expected states of the populations were observed. The first phase was a fast jump towards the vicinity of the identity axis (the concentration of a population) while the second phase relied on a slow drift, of a nearly homogenous population, towards the optimum of the fitness function. The system converges to fixed points. Their number depended on the modality of the fitness function. The fixed points are usually located in a neighbourhood of the optima and saddles of the fitness function. The stability of the fixed points depends on the standard deviation of mutation. Fixed points of the optima are asymptotically stable for small σ. For larger values of σ the evolution process does not converge to a fixed point but forms a periodical orbit. The stability analysis of fixed points indicates a considerable influence of the standard deviation of mutation on the efficiency of the evolutionary process. Larger values of σ accelerate finding optima and crossing saddles, but too large a value of σ may cause some instability of fixed points disturbing the convergence to optima. Thus, the standard deviation of mutation can be regarded as the most important parameter affecting the evolutionary process.

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