NEURAL NETWORK-BASED MRAC CONTROL OF DYNAMIC NONLINEAR SYSTEMS

GHANIA DEBBACHE[*], ABDELHAK BENNIA[**]
NOUREDDINE GOLÉA[*]

[*] Electrical Engineering Institute, Oum El-Bouaghi University
04000 Oum El-Bouaghi, Algeria
e-mail: {gdebbache,nour_golea}@yahoo.fr

[**] Electronic Department, Constantine University
25000 Constantine, Algeria
e-mail: abdelhak.bennia@laposte.net

This paper presents direct model reference adaptive control for a class of nonlinear systems with unknown nonlinearities. The model following conditions are assured by using adaptive neural networks as the nonlinear state feedback controller. Both full state information and observer-based schemes are investigated. All the signals in the closed loop are guaranteed to be bounded and the system state is proven to converge to a small neighborhood of the reference model state. It is also shown that stability conditions can be formulated as linear matrix inequalities (LMI) that can be solved using efficient software algorithms. The control performance of the closed-loop system is guaranteed by suitably choosing the design parameters. Simulation results are presented to show the effectiveness of the approach.

Keywords: neural networks, reference model, nonlinear systems, adaptive control, observer, stability, LMI

1. Introduction

Nonlinear control design can be viewed as a nonlinear function approximation problem (see, e.g., (Billings *et al.*, 1992; Narendra and Parthasarathy, 1990)). On the other hand, neural networks have been proved to be universal approximators for nonlinear systems (Cybenko, 1989; Cotter, 1990; Ferrari and Stengel, 2005; Park and Sandberg, 1991). Neural networks are capable of learning and reconstructing complex nonlinear mappings and have been widely studied by control engineers in the identification analysis and design of control systems. A large number of control structures have been proposed, including direct inverse control (Plett, 2003), model reference control (Narendra and Parthasarathy, 1990; Patino and Liu, 2000), sliding mode control (Zhihong *et al.*, 1998), internal model control (Rivals and Personnaz, 2000), feedback linearization (Yesildirek and Lewis, 1995), backstepping (Zhang *et al.*, 2000), indirect adaptive control (Chang and Yen, 2005; Narendra and Parthasarathy, 1990; Neidhoefer *et al.*, 2003; Poznyak *et al.*, 1999; Patino and Liu, 2000; Seshagiri and Khalil, 2000; Yu and Annaswamy, 1997; Zhang *et al.*, 2000), and direct adaptive control (Ge *et al.*, 1999; Huang *et al.*, 2006; Sanner and Slotine, 1992; Spooner and Passino, 1996; Zhihong *et al.*, 1998). The principal types of neural networks used for control problems are the multilayer perceptron (MLP), neural networks with sigmoidal units, and radial basis function (RBF) neural networks.

Model reference adaptive control (Landau, 1979) is a technique well established in the framework of linear systems. In the direct model reference control approach, the parameters of the linear controller are adapted directly either by gradient- or stability-based approaches to drive the plant output to follow a desired reference model. This structure can be extended by utilizing the nonlinear function approximation capability of feedforward neural networks, such as the multilayer perceptron (MLP), to form a more general, flexible control law (Narendra and Parthasarathy, 1990; Patino and Liu, 2000; Zhihong *et al.*, 1998). Such a network could be trained on-line within the MRAC structure, to facilitate the generation of a nonlinear control law that may be necessary to drive the output of the plant to follow the desired linear reference model.

In this paper, we propose a new neural-network-based model reference adaptive control scheme for a class of nonlinear systems. Unlike in the cited works, neural networks are not directly used to learn the system nonlinearities, but they are used to adaptively realize, like in linear MRAC, the model following conditions. The idea behind this approach is that activation functions play the role of a continuous scheduler of local linear feedback, such that at each operating point there exists a controller that achieves the model following objective. Both full state information and observer-based approaches are developed. The stability in both cases is established using Lyapunov tools. It is shown that this approach is robust against external disturbances, and the tracking and observation errors are bounded. Compared with the previously developed direct adaptive approaches, this adaptive scheme requires few assumptions, and its implementation is much simpler, i.e., neither supervisory nor switching control is needed to ensure stability, and no upper bound on the high-frequency gain derivative is required. Simulation results for an unstable inverted pendulum show the capabilities of the proposed approach for both tracking a reference model and estimating the system states.

The rest of the paper is organized as follows: Section 2 introduces the neural networks used and states the nonlinear systems direct model reference adaptive control problem. Section 3 develops the full state information approach, based on neural network state feedback. Section 4 develops the observer-based approach using an adaptive observer to estimate the system states. Section 5 presents simulation results, and Section 6 concludes the paper.

2. Preliminaries

The following notation will be used throughout the paper: The symbol $|\underline{y}|$ denotes the usual Euclidean norm of a vector $\underline{y}$. In case y is a scalar, $|y|$ denotes its absolute value. If A is a matrix, then $\|A\|$ denotes the Frobenius matrix norm defined as

$$\|A\|^2 = \sum_{ij} a_{ij}^2 = \mathrm{tr}\left[A^T A\right],$$

where $\mathrm{tr}[\cdot]$ denotes the trace of the matrix.

2.1. Neural Networks. The general function of a one hidden layer feedforward network can be described as a weighted combination of N activation functions

$$y = \sum_{i=1}^{N} \varphi_i\left(\underline{x}, \underline{\theta}_i\right) w_i. \tag{1}$$

Here the input vector $\underline{x}$ and $\varphi_i(\cdot)$ represent the i-th activation function (with its parameter vector $\underline{\theta}_i$) connected to the output by the weight w_i. The number of the input and output layers coincides with the dimension of the input vector and the output information number, respectively. The activation functions are chosen following the neural network at hand. For example, the MLP uses generally the sigmoid function defined as

$$\varphi_i\left(\underline{x}, \underline{\theta}_i\right) = \frac{1}{1+\exp(-\underline{a}_i \underline{x})}, \tag{2}$$

or the logistic function defined by

$$\varphi_i\left(\underline{x}, \underline{\theta}_i\right) = \frac{1-\exp(-\underline{a}_i \underline{x})}{1+\exp(-\underline{a}_i \underline{x})}. \tag{3}$$

On the other hand, RBF networks use Gaussian functions defined as

$$\varphi_i\left(\underline{x}, \underline{\theta}_i\right) = \exp\left(-\frac{|\underline{x}-\underline{c}_i|^2}{2\sigma_i^2}\right). \tag{4}$$

Since the above neural networks will be trained on-line to realise a control task, in order to reduce computation load, we will assume that the parameters of the activation functions $\underline{\theta}_i$ are fixed, i.e., their number and shapes are fixed *a priori*. The only adjustable parameters are the weights w_i. Then (1) is rewritten in the compact form

$$y = \underline{\phi}\left(\underline{x}\right) \underline{w}, \tag{5}$$

where

$$\underline{\phi}\left(\underline{x}\right) = \begin{bmatrix} \varphi_1\left(\underline{x}\right) & \cdots & \varphi_N\left(\underline{x}\right) \end{bmatrix}$$

and

$$\underline{w}^T = \begin{bmatrix} w_1 & \cdots & w_N \end{bmatrix}.$$

From approximation theory it is known that the modeling error can be reduced arbitrarily by increasing the number N, i.e., the number of linear independent activation functions in the network. That is, a smooth function $f\left(\underline{x}\right), \underline{x} \in \Omega_{\underline{x}} \subset \mathbb{R}^n$ can be written as

$$f\left(\underline{x}\right) = \underline{\phi}\left(\underline{x}\right) \underline{w}^* + \epsilon\left(\underline{x}\right), \tag{6}$$

where $\epsilon\left(\underline{x}\right)$ is the approximation error inherent in the network and $\underline{w}^*$ is an optimal weight vector. Following the universal approximation results, for the neural network (5) there exist a finite set of $\underline{w}^*$ and a constant ϵ^* such that (6) holds with $|\epsilon\left(\underline{x}\right)| < \epsilon^*$.

2.2. Problem Statement. Consider the class of continuous-time nonlinear systems given by

$$\dot{\underline{x}} = A\underline{x} + \underline{b}\left[f(\underline{x}) + g(\underline{x})u\right], \tag{7}$$

$$y = \underline{c}^T \underline{x}, \tag{8}$$

where $\underline{x}^T = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \in \mathbb{R}^n$ is the state vector, $u, y \in \mathbb{R}$ are the system input and output, respectively, and $f\left(\underline{x}\right)$, $g\left(\underline{x}\right)$ are smooth nonlinear unknown functions. Furthermore,

$$A = \begin{bmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ \multicolumn{2}{c}{0_{1\times n}} \end{bmatrix}_{n\times n},$$

$$\underline{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{n\times 1}, \quad \underline{c} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1},$$

where I_n is the $n \times n$ identity matrix.

The nonlinear system (7)–(8) satisfies the following assumptions:

P1. The high-frequency gain $g(\underline{x})$ is strictly positive and bounded by $0 < g(\underline{x}) < g_0$, for some known constant g_0, in the relevant control space $\Omega_{\underline{x}}$. The negative case can also be handled.

P2. The nonlinear function $f(\underline{x})$ can be decomposed as $f(\underline{x}) = \underline{a}(\underline{x})\underline{x} + a_0(\underline{x})$ with $\underline{a}(\underline{x}) = [\, a_1(\underline{x}) \;\; a_2(\underline{x}) \;\; \cdots \;\; a_n(\underline{x}) \,] \in \mathbb{R}^n$ as an unknown smooth nonlinear vector function, and $a_0(\underline{x})$ as a bounded nonlinear function.

Note that P1 is a usual assumption in adaptive control, and P2 specifies the discussed class of nonlinear systems. Note that, from a practical point of view, a large class of real plants satisfies Assumption P2, e.g., robot manipulators, electrical drives, hydraulic rigs, power plants and turbofan engines, etc.

The reference model is defined by the following stable LTI state equation:

$$\dot{\underline{x}}_m = A_m \underline{x}_m + \underline{b} b_m r, \tag{9}$$

$$y_m = \underline{c}^T x_m, \tag{10}$$

where $\underline{x}_m \in \mathbb{R}^{n\times 1}$ is the state vector, y_m is the reference model output, r is a bounded reference input, A_m is given by

$$A_m = \begin{bmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ & -\underline{a}_m \end{bmatrix}_{n\times n},$$

with $\underline{a}_m \in \mathbb{R}^{1\times n}$ and $b_m > 0$.

Substracting (7) and (8) from (9) and (10), respectively, and using P2 yield the following tracking error dynamic:

$$\begin{aligned} \dot{\underline{e}} = A_m \underline{e} + \underline{b}\big[&-(\underline{a}_m + \underline{a}(\underline{x}))\,\underline{x} + b_m r \\ &- g(\underline{x})u - a_0(\underline{x}) \big], \end{aligned} \tag{11}$$

$$e_y = \underline{c}^T \underline{e}, \tag{12}$$

where $\underline{e} = \underline{x}_m - \underline{x}$ and $e_y = y_m - y$ are the state and output tracking errors, respectively.

The control problem can be stated as follows: Design the control input u such that the reference model following is achieved, under the condition that all involved signals in the closed loop remain bounded.

3. Neuro-State Feedback Design

The neural adaptive controller to be designed consists of three neural components (see Fig. 1) such that

$$u = \underline{\phi}_1(\underline{x}) K_1 \underline{x} + \underline{\phi}_2(\underline{x}) \underline{k}_2 + \underline{\phi}_3(\underline{x}) \underline{k}_3 r, \tag{13}$$

where $\underline{\phi}_1(\underline{x}) \in \mathbb{R}^{1\times N_1}$, $\underline{\phi}_2(\underline{x}) \in \mathbb{R}^{1\times N_2}$ and $\underline{\phi}_3(\underline{x}) \in \mathbb{R}^{1\times N_3}$ are the vectors of activation functions in the three networks of dimensions N_1, N_2 and N_3, respectively. $K_1 \in \mathbb{R}^{N_1\times n}$, $\underline{k}_2 \in \mathbb{R}^{N_2\times 1}$ and $\underline{k}_3 \in \mathbb{R}^{N_3\times 1}$ are adjustable parameters of the three networks. The first and third terms in (13) are a nonlinear replica of the standard reference model feedback control. The second term is added to account for the nonlinearity $a_0(\underline{x})$. As can be seen from Fig. 1, the first term in (13) is realized by a single multi-output (n outputs) network, and the others are formed by single output networks.

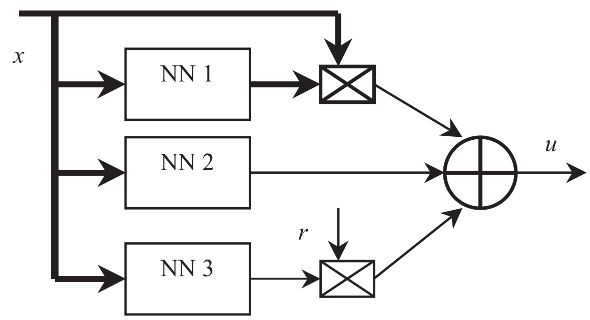


Fig. 1. Neural state feedback structure.

Now, introducing (13) in (11) yields

$$\begin{aligned} \dot{\underline{e}} = A_m \underline{e} + \underline{b}\Big[&- \Big(\underline{a}_m + \underline{a}(\underline{x}) + g(\underline{x})\underline{\phi}_1(\underline{x}) K_1\Big)\underline{x} \\ &- \Big(a_0(\underline{x}) + g(\underline{x})\underline{\phi}_2(\underline{x})\underline{k}_2\Big) \\ &- \Big(g(\underline{x})\underline{\phi}_3(\underline{x})\underline{k}_3 - b_m\Big) r\Big]. \end{aligned} \tag{14}$$

According to the universal approximation results, there exist optimal parameters K_1^*, $\underline{k}_2^*$ and $\underline{k}_3^*$ defined as

$$K_1^* = \arg\min_{K_1\in\Omega_1} \left\{ \sup_{\underline{x}\in\Omega_{\underline{x}}} \left| \underline{a}_m + \underline{a}(\underline{x}) + g(\underline{x})\underline{\phi}_1(\underline{x})K_1^* \right| \right\},$$

$$\underline{k}_2^* = \arg\min_{\underline{k}_2\in\Omega_2} \left\{ \sup_{\underline{x}\in\Omega_{\underline{x}}} \left| a_0(\underline{x}) + g(\underline{x})\underline{\phi}_2(\underline{x})\underline{k}_2^* \right| \right\},$$

$$\underline{k}_3^* = \arg\min_{\underline{k}_3\in\Omega_3} \left\{ \sup_{\underline{x}\in\Omega_{\underline{x}}} \left| g(\underline{x})\underline{\phi}_3(\underline{x})\underline{k}_3^* - b_m \right| \right\},$$

where Ω_1, Ω_2, Ω_3 and $\Omega_{\underline{x}}$ are constraint sets for K_1, $\underline{k}_2$, $\underline{k}_3$ and $\underline{x}$, respectively. Define the minimum approximation errors

$$\underline{a}_m + \underline{a}(\underline{x}) + g(\underline{x})\underline{\phi}_1(\underline{x})K_1^* = \underline{\varepsilon}_1(\underline{x}), \tag{15}$$

$$a_0(\underline{x}) + g(\underline{x})\underline{\phi}_2(\underline{x})\underline{k}_2^* = \varepsilon_2(\underline{x}), \tag{16}$$

$$g(\underline{x})\underline{\phi}_3(\underline{x})\underline{k}_3^* - b_m = \varepsilon_3(\underline{x}), \tag{17}$$

where $\underline{\varepsilon}_1(\underline{x}) \in \mathbb{R}^{1\times n}$ and $\varepsilon_2(\underline{x}), \varepsilon_3(\underline{x}) \in \mathbb{R}$ are the minimum approximation errors achieved by the three neural networks in (13) with the optimal parameters K_1^*, $\underline{k}_2^*$ and $\underline{k}_3^*$.

The properties (15)–(17) can be viewed as a nonlinear version of the linear model following conditions, where the activation functions play the role of scheduling parameters, i.e., for every sample $\underline{x}(t)$, the activation functions $\underline{\phi}_1(\underline{x})$ to $\underline{\phi}_3(\underline{x})$ quantify the contribution of the linear feedback to the controller output.

The adaptive neural controller (13) satisfies the following assumptions:

C1. The activation functions in $\underline{\phi}_3(\underline{x})$ are such that $|\dot{\underline{\phi}}_3(\underline{x})| < \varphi_0$, where φ_0 is a known positive constant, and $\underline{\phi}_3(\underline{x})\underline{k}_3^* > 0$.

C2. Suitable upper bounds $\Omega_1 = \{K_1^* \mid \|K_1^*\| < \kappa_1\}$, $\Omega_2 = \{\underline{k}_2^* \mid |\underline{k}_2^*| < \kappa_2\}$ and $\Omega_3 = \{\underline{k}_3^* \mid |\underline{k}_3^*| < \kappa_3\}$, with $\kappa_1, \kappa_2, \kappa_3 > 0$, are given.

C3. The approximation errors are upper bounded by $|\underline{\varepsilon}_1(\underline{x})| \le \epsilon_1$, $|\varepsilon_2(\underline{x})| \le \epsilon_2$ and $|\varepsilon_3(\underline{x})| \le \epsilon_3$, for some positive constants ϵ_1, ϵ_1 and ϵ_3.

Assumption C1 indicates that the activation functions in $\underline{\phi}_3(\underline{x})$ should be chosen as smooth functions with a bounded rate of change. Note that $\dot{\underline{\phi}}_3(\underline{x})$ may not necessarily be globally bounded, but it will have a constant bound within $\Omega_{\underline{x}}$ due to the continuity of $\underline{\phi}_3(\underline{x})$. Assumption C2 is made to prevent any drift of the neural controller parameters. Assumption C3 follows from the universal approximation property of neural networks.

Hence, using (15)–(17) in (14), we obtain

$$\begin{aligned}\dot{\underline{e}} = A_m\underline{e} + \underline{b}\Big[&g(\underline{x})\underline{\phi}_1(\underline{x})\widetilde{K}_1\underline{x} + g(\underline{x})\underline{\phi}_2(\underline{x})\widetilde{\underline{k}}_2\\ &+ g(\underline{x})\underline{\phi}_3(\underline{x})\widetilde{\underline{k}}_3 r - \epsilon_1(\underline{x})\underline{x} - \epsilon_2(\underline{x})\\ &- \epsilon_3(\underline{x})r\Big],\end{aligned} \tag{18}$$

where $\widetilde{K}_1 = K_1^* - K_1$, $\widetilde{\underline{k}}_2 = \underline{k}_2^* - \underline{k}_2$ and $\widetilde{\underline{k}}_3 = \underline{k}_3^* - \underline{k}_3$ are the estimation errors of the neural networks parameters.

Further exploiting the property (17), (18) can be rewritten as

$$\begin{aligned}\dot{\underline{e}} &= A_m\underline{e}\\ &+ \frac{b_m}{\underline{\phi}_3(\underline{x})\underline{k}_3^*}\underline{b}\left[\underline{\phi}_1(\underline{x})\widetilde{K}_1\underline{x} + \underline{\phi}_2(\underline{x})\widetilde{\underline{k}}_2 + \underline{\phi}_3(\underline{x})\widetilde{\underline{k}}_3 r\right]\\ &+ \frac{1}{\underline{\phi}_3(\underline{x})\underline{k}_3^*}\underline{b}\left[\underline{\alpha}_1\underline{e} + \xi\right],\end{aligned} \tag{19}$$

with

$$\begin{aligned}\xi &= -\underline{\alpha}_1\underline{x}_m + \alpha_2 + \alpha_3 r,\\ \underline{\alpha}_1 &= \underline{\varepsilon}_1(\underline{x})\underline{\phi}_3(\underline{x})\underline{k}_3^* - \varepsilon_3(\underline{x})\underline{\phi}_1(\underline{x})\widetilde{K}_1,\\ \alpha_2 &= \varepsilon_3(\underline{x})\underline{\phi}_2(\underline{x})\widetilde{\underline{k}}_2 - \varepsilon_2(\underline{x})\underline{\phi}_3(\underline{x})\underline{k}_3^*,\\ \alpha_3 &= \varepsilon_3(\underline{x})\underline{\phi}_3(\underline{x})\underline{k}_3.\end{aligned}$$

Combining (12) and (19) yields the following transfer function:

$$\begin{aligned}e_y &= G(s)\frac{b_m}{\underline{\phi}_3(\underline{x})\underline{k}_3^*}\\ &\quad\times\left[\underline{\phi}_1(\underline{x})\widetilde{K}_1\underline{x} + \underline{\phi}_2(\underline{x})\widetilde{\underline{k}}_2 + \underline{\phi}_3(\underline{x})\widetilde{\underline{k}}_3 r\right]\\ &\quad+ \frac{G(s)}{\underline{\phi}_3(\underline{x})\underline{k}_3^*}\left[\underline{\alpha}_1\underline{e} + \xi\right],\end{aligned} \tag{20}$$

where $G(s) = \underline{c}^T[sI_n - A_m]^{-1}\underline{b}$, and s is the Laplace operator.

Define the polynomial $H(s) = s^{n-1} + \beta_2 s^{n-2} + \dots + \beta_{n-2}s + \beta_{n-1}$ such that $H^{-1}(s)$ is a proper stable transfer function. Then, dividing and multiplying (20) by $H(s)$, we get

$$\begin{aligned}e_y &= G_0(s)\frac{b_m}{\underline{\phi}_3(\underline{x})\underline{k}_3^*}\\ &\quad\times\left[\underline{\phi}_{1f}(\underline{x})\widetilde{K}_1\underline{x} + \underline{\phi}_{2f}(\underline{x})\widetilde{\underline{k}}_2 + \underline{\phi}_{3f}(\underline{x})\widetilde{\underline{k}}_3 r\right]\\ &\quad+ \frac{G_0(s)}{\underline{\phi}_3(\underline{x})\underline{k}_3^*}\left[\underline{\alpha}_{1f}\underline{e} + \xi_f\right],\end{aligned} \tag{21}$$

with $G_0(s) = H(s)G(s)$, $\underline{\alpha}_{1f} = H^{-1}(s)\underline{\alpha}_1$, $\xi_f = H^{-1}(s)\xi$ and $\underline{\phi}_{if}(\underline{x}) = H^{-1}(s)\underline{\phi}_i(\underline{x})$ for $i = 1, \dots, 3$.

Hence, (21) can be rewritten in the following state space form:

$$\begin{aligned}\dot{\underline{e}} &= A_m\underline{e} + \frac{b_m}{\underline{\phi}_3(\underline{x})\underline{k}_3^*}\underline{b}_c\\ &\quad\times\left[\underline{\phi}_{1f}(\underline{x})\widetilde{K}_1\underline{x} + \underline{\phi}_{2f}(\underline{x})\widetilde{\underline{k}}_2 + \underline{\phi}_{3f}(\underline{x})\widetilde{\underline{k}}_3 r\right]\\ &\quad+ \frac{1}{\underline{\phi}_3(\underline{x})\underline{k}_3^*}\underline{b}_c\left[\underline{\alpha}_{1f}\underline{e} + \xi_f\right]\end{aligned} \tag{22}$$

$$e_y = \underline{c}^T\underline{e}, \tag{23}$$

where $\underline{b}_c^T = [\,0 \ \cdots \ 0 \ 1 \ \beta_2 \ \cdots \ \beta_{n-1}\,] \in \mathbb{R}^n$.

Since $G(s)$ is Hurwitz, and the pairs $\{A_m, \underline{b}\}$ and $\{A_m, \underline{c}\}$ are controllable and observable, respectively, $H(s)$ can always be chosen such that $G_0(s)$ is strictly positive real (SPR). It follows that, for a given $Q > 0$, there exists $P = P^T > 0$ such that the following equations are satisfied (Landau, 1979):

$$A_m^T P + P A_m = -Q, \tag{24}$$

$$\underline{b}_c^T P = \underline{c}^T. \tag{25}$$

The parameters of the neural controller are updated using the following projection algorithm (Sastry and Bodson, 1989):

$$\begin{aligned}\dot{K}_1 &= \gamma_1 b_m e_y \underline{\phi}_{1f}^T(\underline{x})\underline{x}^T \\ &\quad - I_1\gamma_1 b_m e_y \operatorname{tr}\Big[K_1^T \underline{\phi}_{1f}^T(\underline{x})\,\underline{x}^T\Big]\Big(\frac{1+\|K_1\|}{\kappa_1}\Big)^2 K_1,\end{aligned} \tag{26}$$

$$\dot{\underline{k}}_2 = \gamma_2 b_m e_y \underline{\phi}_{2f}^T(\underline{x}) - I_2\gamma_2 b_m e_y \frac{\underline{k}_2\underline{k}_2^T\underline{\phi}_{2f}^T(\underline{x})}{|\underline{k}_2|^2}, \tag{27}$$

$$\dot{\underline{k}}_3 = \gamma_3 b_m e_y \underline{\phi}_{3f}^T(\underline{x}) r - I_3\gamma_3 b_m e_y \frac{\underline{k}_3\underline{k}_3^T\underline{\phi}_{3f}^T(\underline{x})\, r}{|\underline{k}_3|^2}, \tag{28}$$

where

$$I_1 = \begin{cases} 0 & \text{if } \begin{cases} \|K_1\| < \kappa_1 \text{ or} \\ \|K_1\| = \kappa_1 \\ \quad \text{and } e_y \operatorname{tr}\Big[K_1^T\underline{\phi}_{1f}^T(\underline{x})\,\underline{x}^T\Big] \le 0, \end{cases} \\ 1 & \text{if } \|K_1\| = \kappa_1 \text{ and } e_y \operatorname{tr}\Big[K_1^T\underline{\phi}_{1f}^T(\underline{x})\,\underline{x}^T\Big] > 0, \end{cases}$$

$$I_2 = \begin{cases} 0 & \text{if } \begin{cases} |\underline{k}_2| < \kappa_2 \text{ or} \\ |\underline{k}_2| = \kappa_2 \text{ and } e_y\underline{\phi}_{2f}(\underline{x})\,\underline{k}_2 \le 0, \end{cases} \\ 1 & \text{if } |\underline{k}_2| = \kappa_2 \text{ and } e_y\underline{\phi}_{2f}(\underline{x})\,\underline{k}_2 > 0, \end{cases}$$

$$I_3 = \begin{cases} 0 & \text{if } \begin{cases} |\underline{k}_3| < \kappa_3 \text{ or} \\ |\underline{k}_3| = \kappa_3 \text{ and } e_y\underline{\phi}_{3f}(\underline{x})\,\underline{k}_3 r \le 0, \end{cases} \\ 1 & \text{if } |\underline{k}_3| = \kappa_3 \text{ and } e_y\underline{\phi}_{3f}(\underline{x})\,\underline{k}_3 r > 0, \end{cases}$$

and $\gamma_1, \gamma_2, \gamma_3 > 0$ are design parameters.

The update laws (26)–(28) ensure that $\|K_1\| \le \kappa_1$, $|\underline{k}_2| \le \kappa_2$ and $|\underline{k}_3| \le \kappa_3$, $\forall t \ge 0$, if the initial values of the parameters are set properly.

To prove that $\|K_1\| \le \kappa_1$, let $V_K = \frac{1}{2}\|K_1\|^2 = \frac{1}{2}\operatorname{tr}[K_1^T K_1]$. Then

$$\dot{V}_K = \frac{1}{2}\frac{\mathrm{d}\,\|K_1\|^2}{\mathrm{d}t} = \operatorname{tr}\Big[K_1^T \dot{K}_1\Big].$$

If $I_{K_1} = 0$, we have either $\|K_1\| < \kappa_1$ or $\|K_1\| = \kappa_1$, and

$$\dot{V}_K = \gamma_1 b_m e_y \operatorname{tr}\Big[K_1^T\underline{\phi}_{1f}^T(\underline{x})\underline{x}^T\Big] \le 0,$$

i.e., we always have $\|K_1\| \le \kappa_1$. If $I_{K_1} = 1$, we have $\|K_1\| = \kappa_1$ and

$$\begin{aligned}\dot{V}_K &= \gamma_1 b_m e_y \operatorname{tr}\Big[K_1^T\Big(\underline{\phi}_{1f}^T(\underline{x})\underline{x}^T - \operatorname{tr}\Big[K_1^T\underline{\phi}_{1f}^T(\underline{x})\,\underline{x}^T\Big] \\ &\qquad \times \Big(\frac{1+\|K_1\|}{\kappa_1}\Big)^2 K_1\Big)\Big] \\ &= \gamma_1 b_m e_y \Big(\operatorname{tr}\Big[K_1^T\underline{\phi}_{1f}^T(\underline{x})\underline{x}^T\Big] - \operatorname{tr}\Big[K_1^T\underline{\phi}_{1f}^T(\underline{x})\,\underline{x}^T\Big] \\ &\qquad \times \Big(\frac{1+\|K_1\|}{\kappa_1}\Big)^2 \operatorname{tr}\Big[K_1^T K_1\Big]\Big) \\ &= \gamma_1 b_m e_y \operatorname{tr}\Big[K_1^T\underline{\phi}_{1f}^T(\underline{x})\underline{x}^T\Big] \\ &\qquad \times \Big(1 - \Big(\frac{1+\|K_1\|}{\kappa_1}\Big)^2 \operatorname{tr}\Big[K_1^T K_1\Big]\Big).\end{aligned} \tag{29}$$

Since $e_y \operatorname{tr}[K_1^T\underline{\phi}_{1f}^T(\underline{x})\underline{x}^T] > 0$ and $\|K_1\| = \kappa_1$, we get $\dot{V}_K \le 0$, which implies that $\|K_1\| \le \kappa_1$. Therefore, we get $\|K_1\| \le \kappa_1$, $\forall t \ge 0$ a similar analysis can be used to show that $|\underline{k}_2| \le \kappa_2$ and $|\underline{k}_3| \le \kappa_3$ $\forall t \ge 0$.

Now, consider the Lyapunov function

$$\begin{aligned}V &= \frac{1}{2}\underline{\phi}_3(\underline{x})\,\underline{k}_3^*\underline{e}^T P\underline{e} + \frac{1}{2\gamma_1}\operatorname{tr}\Big[\widetilde{K}_1^T\widetilde{K}_1\Big] \\ &\quad + \frac{1}{2\gamma_2}\widetilde{\underline{k}}_2^T\widetilde{\underline{k}}_2 + \frac{1}{2\gamma_3}\widetilde{\underline{k}}_3^T\widetilde{\underline{k}}_3,\end{aligned} \tag{30}$$

where, for a given $Q > 0$, P is the solution of (24) and (25).

The differentiation of (30) along the trajectory of (22) yields

$$\begin{aligned}\dot{V} &= -\frac{1}{2}\underline{\phi}_3(\underline{x})\,\underline{k}_3^*\underline{e}^T Q\underline{e} + \frac{1}{2}\dot{\underline{\phi}}_3(\underline{x})\,\underline{k}_3^*\underline{e}^T P\underline{e} \\ &\quad + b_m\underline{e}^T P\underline{b}_c\Big[\underline{\phi}_{1f}(\underline{x})\,\widetilde{K}_1\underline{x} + \underline{\phi}_{2f}(\underline{x})\widetilde{\underline{k}}_2 + \underline{\phi}_{3f}(\underline{x})\widetilde{\underline{k}}_3 r\Big] \\ &\quad + \underline{e}^T P\underline{b}_c\underline{\alpha}_{1f}\underline{e} + \underline{e}^T P\underline{b}_c\xi_f \\ &\quad + \frac{1}{\gamma_1}\operatorname{tr}\Big[\widetilde{K}_1^T\dot{\widetilde{K}}_1\Big] + \frac{1}{\gamma_2}\widetilde{\underline{k}}_2^T\dot{\widetilde{\underline{k}}}_2 + \frac{1}{\gamma_3}\widetilde{\underline{k}}_3^T\dot{\widetilde{\underline{k}}}_3.\end{aligned} \tag{31}$$

Further, using the facts that $\dot{\widetilde{K}}_1 = -\dot{K}_1$, $\dot{\widetilde{\underline{k}}}_2 = -\dot{\underline{k}}_2$ and $\dot{\widetilde{\underline{k}}}_3 = -\dot{\underline{k}}_3$, we can arrange (31) as

$$\begin{aligned}\dot{V} &= -\frac{1}{2}\underline{\phi}_3(\underline{x})\underline{k}_3^*\underline{e}^T Q\underline{e} + \frac{1}{2}\dot{\underline{\phi}}_3(\underline{x})\underline{k}_3^*\underline{e}^T P\underline{e} \\ &\quad + \underline{e}^T P\underline{b}_c\underline{\alpha}_{1f}\underline{e} + \underline{e}^T P\underline{b}_c\xi_f\end{aligned}$$

$$+\frac{1}{\gamma_1}\operatorname{tr}\left[\widetilde{K}_1^T\left(\gamma_1 b_m e_y \underline{\phi}_{1f}^T(\underline{x})\underline{x}^T-\dot{K}_1\right)\right]$$

$$+\frac{1}{\gamma_2}\underline{\widetilde{k}}_2^T\left(\gamma_2 b_m e_y \underline{\phi}_{2f}^T(\underline{x})-\underline{\dot{k}}_2\right)$$

$$+\frac{1}{\gamma_3}\underline{\widetilde{k}}_3^T\left(\gamma_3 b_m e_y \underline{\phi}_{3f}^T(\underline{x})r-\underline{\dot{k}}_3\right). \quad (32)$$

Then, introducing the update laws (26)–(28) in (32) yields

$$\dot{V}=-\frac{1}{2}\underline{\phi}_3(\underline{x})\underline{k}_3^*\underline{e}^TQ\underline{e}+\frac{1}{2}\underline{\dot{\phi}}_3(\underline{x})\underline{k}_3^*\underline{e}^TP\underline{e}$$
$$+\underline{e}^TP\underline{b}_c\underline{\alpha}_{1f}\underline{e}+\underline{e}^TP\underline{b}_c\xi_f$$
$$+I_1\operatorname{tr}\left[\widetilde{K}_1^T b_m e_y \operatorname{tr}\left[K_1^T\underline{\phi}_{1f}^T(\underline{x})\,\underline{x}^T\right]\times\left(\frac{1+\|K_1\|}{\kappa_1}\right)^2K_1\right]$$
$$+I_2\underline{\widetilde{k}}_2^T\frac{b_m e_y\underline{k}_2\underline{k}_2^T\underline{\phi}_{2f}^T(\underline{x})}{|\underline{k}_2|^2}$$
$$+I_3\underline{\widetilde{k}}_3^T\frac{b_m e_y\underline{k}_3\underline{k}_3^T\underline{\phi}_{3f}^T(\underline{x})\,r}{|\underline{k}_3|^2}. \quad (33)$$

Further, (33) can be arranged as

$$\dot{V}=-\frac{1}{2}\underline{\phi}_3(\underline{x})\underline{k}_3^*\underline{e}^TQ\underline{e}+\frac{1}{2}\underline{\dot{\phi}}_3(\underline{x})\underline{k}_3^*\underline{e}^TP\underline{e}$$
$$+\underline{e}^TP\underline{b}_c\underline{\alpha}_{1f}\underline{e}+\underline{e}^TP\underline{b}_c\xi_f$$
$$+I_1 b_m e_y\operatorname{tr}\left[\widetilde{K}_1^TK_1\right]\operatorname{tr}\left[K_1^T\underline{\phi}_{1f}^T(\underline{x})\,\underline{x}^T\right]\times\left(\frac{1+\|K_1\|}{\kappa_1}\right)^2$$
$$+I_2\underline{\widetilde{k}}_2^T\underline{k}_2\frac{b_m e_y\underline{k}_2^T\underline{\phi}_{2f}^T(\underline{x})}{|\underline{k}_2|^2}$$
$$+I_3\underline{\widetilde{k}}_3^T\underline{k}_3\frac{b_m e_y\underline{k}_3^T\underline{\phi}_{3f}^T(\underline{x})}{|\underline{k}_3|^2}r. \quad (34)$$

We now prove that the last three terms in (34) are always nonpositive. If $I_1=I_2=I_3=0$, the result is trivial. If $I_1=1$, then $\|K_1\|=\kappa_1$ and $e_y\operatorname{tr}[K_1^T\underline{\phi}_{1f}^T(\underline{x})\,\underline{x}^T]>0$. On the other hand, we have

$$\operatorname{tr}\left[\widetilde{K}_1^TK_1\right]=\frac{1}{2}\operatorname{tr}\left[K_1^{*T}K_1^*\right]-\frac{1}{2}\operatorname{tr}\left[K_1^TK_1\right]-\frac{1}{2}\operatorname{tr}\left[\widetilde{K}_1^T\widetilde{K}_1\right]. \quad (35)$$

Since $\operatorname{tr}[K_1^{*T}K_1^*]\le\kappa_1^2$, $\operatorname{tr}[K_1^TK_1]=\kappa_1^2$ and $\operatorname{tr}[\widetilde{K}_1^T\widetilde{K}_1]\ge 0$, we get $\operatorname{tr}[\widetilde{K}_1^TK_1]\le 0$, which means that the fourth term in (34) is always nonpositive. The same analysis can be used to show that the last two terms in (34) are also nonpositive.

Using the above result, (34) can be rewritten as

$$\dot{V}\le-\frac{1}{2}\underline{\phi}_3(\underline{x})\underline{k}_3^*\underline{e}^TQ\underline{e}+\frac{1}{2}\underline{\dot{\phi}}_3(\underline{x})\underline{k}_3^*\underline{e}^TP\underline{e}+\underline{e}^TP\underline{b}_c\underline{\alpha}_{1f}\underline{e}+\underline{e}^TP\underline{b}_c\xi_f. \quad (36)$$

Further, using the fact that $\underline{e}^TP\underline{b}_c\xi_f\le\frac{1}{2}\underline{e}^TP\underline{b}_c\underline{b}_c^TP\underline{e}+\frac{1}{2}\xi_f^2$ and $\|\underline{b}_c\underline{\alpha}_{1f}\|\le\alpha_0$ (where α_0 is a given upper bound) in (36), we obtain

$$\dot{V}\le-\frac{1}{2}\underline{\phi}_3(\underline{x})\underline{k}_3^*\underline{e}^TQ\underline{e}+\frac{1}{2}\underline{\dot{\phi}}_3(\underline{x})\underline{k}_3^*\underline{e}^TP\underline{e}+\alpha_0\underline{e}^TP\underline{e}+\frac{1}{2}\underline{e}^TP\underline{b}_c\underline{b}_c^TP\underline{e}+\frac{1}{2}\xi_f^2, \quad (37)$$

which, using C1 and the fact that $|\underline{\phi}_3(\underline{x})|\le 1$, can be upper bounded by

$$\dot{V}\le-\frac{1}{2}\kappa_3\underline{e}^TQ\underline{e}+\frac{1}{2}\varphi_0\kappa_3\underline{e}^TP\underline{e}+\alpha_0\underline{e}^TP\underline{e}+\frac{1}{2}\underline{e}^TP\underline{b}_c\underline{b}_c^TP\underline{e}+\frac{1}{2}\xi_f^2. \quad (38)$$

Finally, (38) can be rearranged as

$$\dot{V}\le-\frac{1}{2}\kappa_3\underline{e}^T\left(Q-\left(\varphi_0+2\frac{\alpha_0}{\kappa_3}\right)P-\frac{1}{\kappa_3}P\underline{b}_c\underline{b}_c^TP\right)\underline{e}+\frac{1}{2}\xi_f^2. \quad (39)$$

If the matrices Q and P are chosen such that

$$Q-\left(\varphi_0+2\frac{\alpha_0}{\kappa_3}\right)P-\frac{1}{\kappa_3}P\underline{b}_c\underline{b}_c^TP\ge Q_1 \quad (40)$$

for some positive definite matrix Q_1, then (39) becomes

$$\dot{V}\le-\frac{1}{2}\kappa_3\underline{e}^TQ_1\underline{e}+\frac{1}{2}\xi_f^2, \quad (41)$$

which can be upper bounded by

$$\dot{V}\le-\frac{1}{2}\kappa_3\lambda_{Q_1}|\underline{e}|^2+\frac{1}{2}|\xi_f|^2, \quad (42)$$

where λ_{Q_1} is the smallest eigenvalue of Q_1.

Then $\dot{V}\le 0$ outside the bounded region defined by

$$|\underline{e}|\le\frac{1}{\sqrt{\kappa_3\lambda_{Q_1}}}|\xi_f|. \quad (43)$$

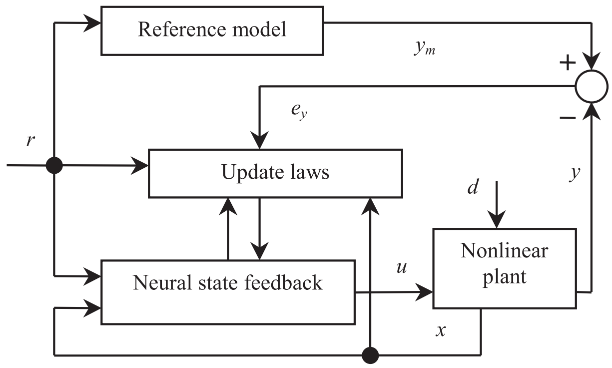


Fig. 2. Full state-based adaptive neural control structure.

The properties of the proposed adaptive neural state feedback approach are summarized by the following theorem:

Theorem 1. *The control system of Fig. 2, composed by the nonlinear system (7)–(8) satisfying P1–P2, the reference model (9)–(10), the neural controller (13) satisfying C1–C3, the update laws (26)–(28), and the matrices P, Q satisfying (24)–(25) and (40) guarantee the following:*

(i)

$$|\underline{x}| \leq |\underline{x}_m| + c_1 |\xi_f|, \tag{44}$$

(ii)

$$|u| \leq \kappa_1 |\underline{x}_m| + \kappa_3 |r| + c_2 |\xi_f|, \tag{45}$$

(iii)

$$\int_0^T |\underline{e}|^2 \,\mathrm{d}t \leq c_3 + c_4 \int_0^T |\xi_f|^2 \,\mathrm{d}t. \tag{46}$$

Proof. (i) Using (43) and the fact that $|\underline{x}| \leq |\underline{x}_m| + |\underline{e}|$, we have

$$|\underline{x}| \leq |\underline{x}_m| + \frac{1}{\sqrt{\kappa_3 \lambda_{Q_1}}} |\xi_f|. \tag{47}$$

Then (44) holds by setting $c_1 = 1/\sqrt{\kappa_3 \lambda_{Q_1}}$.

(ii) From (13) and the parameter bounds, the control input is bounded by

$$|u| \leq \kappa_1 |\underline{x}| + \kappa_2 + \kappa_3 |r|, \tag{48}$$

where the fact that $|\underline{\phi}_i(\underline{x})|_{i=1,\dots,3} \leq 1$ is used. Then, introducing (47) in (48), we get

$$|u| \leq \kappa_1 |\underline{x}_m| + \kappa_2 + \kappa_3 |r| + \kappa_1 c_1 |\xi_f|. \tag{49}$$

Therefore, (45) is proved with $c_2 = \kappa_1 c_1$.

(iii) From (43) we have that $\underline{e} \in L_\infty$, and since K_1, $\underline{k}_2, \underline{k}_3 \in L_\infty$, we obtain $V \in L_\infty$. Since all terms in (22) are bounded, $\dot{\underline{e}} \in L_\infty$ and, therefore, $\underline{e}$ is uniformly continuous. Integrating both sides of (42) yields

$$\int_0^T |\underline{e}|^2 \,\mathrm{d}t \leq \frac{2}{\kappa_3 \lambda_{Q_1}} \big[|V(0)| + |V(T)| \big] + \frac{1}{\kappa_3 \lambda_{Q_1}} \int_0^T |\xi_f|^2 \,\mathrm{d}t. \tag{50}$$

Hence, setting $c_3 = \frac{2}{\kappa_3 \lambda_{Q_1}} [|V(0)| + |V(T)|]$ and $c_4 = \frac{1}{\kappa_3 \lambda_{Q_1}}$ yields (46). ■

Remarks:

1. The control law (13) is a nonlinear form of the linear MRAC control structure. Various terms may be included, such as a term proportional to the tracking error, or a switching control term. Note that different control laws may be used in different control theory frameworks, such as feedback linearization or adaptive sliding mode control.
2. In the proposed approach, RBF or MLP networks may be used depending on the application context. RBF networks are simple to design and rapid in the training process, but they are not suitable for high dimensional systems since RBF suffers from the course of dimensionality. MLP networks are more suitable for highly nonlinear systems due to their parallelism and approximation capabilities.
3. The bounds κ_1, κ_2 and κ_3 are selected to force the realization of (15)–(17), i.e., they should be such that $\kappa_1 > \max |(\underline{a}_m + \underline{a}(\underline{x}))/g(\underline{x})|$, $\kappa_2 > \max |a_0(\underline{x})/g(\underline{x})|$ and $\kappa_3 > \max |b_m/g(\underline{x})|$. Since the system's nonlinearities are poorly known, these bounds should be large enough to ensure that the above limits are included.
4. It would be simpler to choose the same set of activation functions in (13), i.e., $\underline{\phi}_1(\underline{x}) = \underline{\phi}_2(\underline{x}) = \underline{\phi}_3(\underline{x}) = \underline{\phi}(\underline{x})$. But, in practice, $f(\underline{x})$ and $g(\underline{x})$ may depend on different state variables and, therefore, the proposed approach provides a mean to design the neural control input with a reduced number of activation functions and adjustable parameters.
5. Compared with the direct approach presented in (Spooner and Passino, 1996), this approach is simpler since only neural state feedback is used, and no supervisory control term is required to ensure the boundedness of variables.
6. Compared with (Ge *et al.*, 1999; Spooner and Passino, 1996), where an upper bound on $\dot{g}(\underline{x})$ is required to achieve the control objective, this scheme needs only an upper bound on $\dot{\underline{\phi}}_3(\underline{x})$, which is a user designed parameter and, therefore, is available for evaluation.

$$P > 0, \tag{51}$$

$$\begin{bmatrix} -A_m^T P - P A_m - \left(\varphi_0 + 2\frac{\alpha_0}{\kappa_3}\right) P - \frac{1}{\kappa_3} P \underline{b}_c \underline{b}_c^T P & P\underline{b}_c - \underline{c}^T \\ \underline{b}_c^T P - \underline{c} & 0 \end{bmatrix} > 0. \tag{52}$$

7. Using Barbalat's lemma (Sastry and Bodson, 1989), it can be shown that if $\xi_f \in L_2$, then the tracking error asymptotically converges to zero, i.e., $\lim_{t\to\infty} \underline{e}(t) = 0$.
8. The stability conditions encountered in (24)–(25) and (40) can be expressed in the LMI form (51) and (52), where the LMI (51) is introduced to force the positivity of P. Then, an LMI-based stability analysis is carried out to check whether the stability condition is satisfied. In the case where (51) and (52) are not satisfied, the reference model dynamics and/or the filter $H(s)$ should be modified so that the above LMIs are feasible.

4. Observer-Based Design

The above design was based on full state information. Since this is seldom the case in many practical situations, the problem will be solved using only the system's output as the available information.

To estimate the system states, define the following observer:

$$\dot{\underline{\widehat{e}}} = A\underline{\widehat{e}} + \underline{l}\left(e_y - \widehat{e}_y\right), \tag{53}$$

$$\widehat{e}_y = \underline{c}^T \underline{\widehat{e}}, \tag{54}$$

where $\underline{\widehat{e}} = \underline{x}_m - \underline{\widehat{x}}$ is the estimated tracking error, $\widehat{e}_y = y_m - \widehat{y}$ is the output estimated tracking error, $\underline{\widehat{x}}$, $\widehat{y}$ are the estimated states and output, respectively, and $\underline{l}^T = \begin{bmatrix} l_1 & l_2 & \cdots & l_n \end{bmatrix} \in \mathbb{R}^n$ is selected such that $\left[A - \underline{l}\underline{c}^T\right]$ is a Hurwitz matrix.

The observer (53)–(54) can be further arranged as

$$\dot{\underline{\widehat{e}}} = \left[A - \underline{l}\underline{c}^T\right]\underline{\widehat{e}} + \underline{l}\underline{c}^T \underline{e}, \tag{55}$$

$$\widehat{e}_y = \underline{c}^T \underline{\widehat{e}}. \tag{56}$$

The control input is now defined, using the estimated states, as

$$u = \underline{\phi}_1\left(\underline{\widehat{x}}\right) K_1 \underline{\widehat{x}} + \underline{\phi}_2\left(\underline{\widehat{x}}\right) \underline{k}_2 + \underline{\phi}_3\left(\underline{\widehat{x}}\right) \underline{k}_3 r. \tag{57}$$

Introducing the control input (57) in (11) and using (15)–(17) yields

$$\begin{aligned} \dot{\underline{e}} &= A_m \underline{e} \\ &\quad + \frac{b_m}{\underline{\phi}_3(\underline{x})\underline{k}_3^*}\underline{b}\left[\underline{\phi}_1(\underline{\widehat{x}})\widetilde{K}_1\underline{\widehat{x}} + \underline{\phi}_2(\underline{\widehat{x}})\underline{\widetilde{k}}_2 + \underline{\phi}_3(\underline{\widehat{x}})\underline{\widetilde{k}}_3 r\right] \\ &\quad + \frac{1}{\underline{\phi}_3(\underline{x})\underline{k}_3^*}\underline{b}\left[\underline{\alpha}_4 \underline{e} + \underline{\alpha}_5 \underline{\widehat{e}} + \zeta\right], \end{aligned} \tag{58}$$

with

$$\zeta = \underline{\alpha}_6 \underline{x}_m + \alpha_7 + \alpha_8 r,$$

$$\underline{\alpha}_4 = \underline{\phi}_3(\underline{x})\underline{k}_3^* g(\underline{x})\underline{\phi}_1(\underline{\widehat{x}})K_1^* - \varepsilon_3\left(\underline{x}\right)\underline{\phi}_1(\underline{\widehat{x}})\widetilde{K}_1,$$

$$\underline{\alpha}_5 = \underline{\phi}_3(\underline{x})\underline{k}_3^*\left(\underline{\varepsilon}_1\left(\underline{x}\right) - g(\underline{x})\left(\underline{\phi}_1(\underline{x})K_1^* + \underline{\phi}_1(\underline{x})\widetilde{K}_1\right)\right),$$

$$\underline{\alpha}_6 = -(\underline{\alpha}_4 + \underline{\alpha}_5),$$

$$\begin{aligned} \alpha_7 &= \varepsilon_3\left(\underline{x}\right)\underline{\phi}_2(\underline{\widehat{x}})\underline{\widetilde{k}}_2 \\ &\quad + \underline{\phi}_3(\underline{x})\underline{k}_3^*\left(g(\underline{x})\left(\underline{\widetilde{\phi}}_2(\underline{x})\underline{k}_2^* + \underline{\phi}_2(\underline{x})\underline{\widetilde{k}}_2\right) - \varepsilon_2\left(\underline{x}\right)\right), \end{aligned}$$

$$\begin{aligned} \alpha_8 &= \varepsilon_3\left(\underline{x}\right)\underline{\phi}_3(\underline{\widehat{x}})\underline{\widetilde{k}}_3 \\ &\quad + \underline{\phi}_3(\underline{x})\underline{k}_3^*\left(g(\underline{x})\left(\underline{\widetilde{\phi}}_3(\underline{x})\underline{k}_3^* + \underline{\phi}_3(\underline{x})\underline{\widetilde{k}}_3\right) - \varepsilon_3\left(\underline{x}\right)\right), \end{aligned}$$

where $\underline{\widetilde{\phi}}_i\left(\underline{x}\right) = \underline{\phi}_i\left(\underline{x}\right) - \underline{\phi}_i\left(\underline{\widehat{x}}\right)$, $i = 1, \ldots, 3$.

Then, dividing and multiplying (58) by $H(s)$ yields

$$\begin{aligned} \dot{\underline{e}} &= A_m \underline{e} \\ &\quad + \frac{b_m}{\underline{\phi}_3(\underline{x})\underline{k}_3^*}\underline{b}_c\left[\underline{\phi}_{1f}(\underline{\widehat{x}})\widetilde{K}_1\underline{\widehat{x}} + \underline{\phi}_{2f}(\underline{\widehat{x}})\underline{\widetilde{k}}_2 + \underline{\phi}_{3f}(\underline{\widehat{x}})\underline{\widetilde{k}}_3 r\right] \\ &\quad + \frac{1}{\underline{\phi}_3(\underline{x})\underline{k}_3^*}\underline{b}_c\left[\underline{\alpha}_{4f}\underline{\widehat{e}} + \underline{\alpha}_5 \underline{e} + \zeta_f\right], \end{aligned} \tag{59}$$

where $\zeta_f = H^{-1}(s)\zeta$, $\underline{\alpha}_{4f} = H^{-1}(s)\underline{\alpha}_4$ and $\underline{\alpha}_{5f} = H^{-1}(s)\underline{\alpha}_5$.

Using (12), (59) and (55)–(56), the augmented dynamics of the estimated error and tracking error is given by

$$\begin{aligned}\dot{\underline{v}} &= A_a\underline{v} \\ &\quad + \frac{b_m}{\underline{\phi}_3(\underline{x})k_3^*}\underline{b}_a\left[\underline{\phi}_{1f}(\widehat{\underline{x}})\widetilde{K}_1\widehat{\underline{x}} + \underline{\phi}_{2f}(\widehat{\underline{x}})\widetilde{K}_2 r\right] \\ &\quad + \frac{1}{\underline{\phi}_3(\underline{x})k_3^*}\underline{b}_a\left[\underline{\alpha}_f\underline{v} + \zeta_f\right], \qquad (60)\end{aligned}$$

$$e_y = \underline{c}_a^T\underline{v}, \qquad (61)$$

with $\underline{v}^T = [\ \widehat{\underline{e}}^T \quad \underline{e}^T\ ]$, $\underline{b}_a^T = [\ 0_{1\times n} \quad \underline{b}_c^T\ ]$, $\underline{c}_a = [\ 0_{1\times n} \quad \underline{c}^T\ ]$, $\underline{\alpha}_f = [\ \underline{\alpha}_{4f} \quad \underline{\alpha}_{5f}\ ]$ and

$$A_a = \begin{bmatrix} A - \underline{l}\underline{c}^T & \underline{l}\underline{c}^T \\ 0_{n\times n} & A_m \end{bmatrix}.$$

Note that for the state space realization (60)–(61), there exist two symmetric positive definite matrices P_a and Q_a such that

$$A_a^T P_a + P_a A_a = -Q_a, \qquad (62)$$

$$P_a\underline{b}_a = \underline{c}_a. \qquad (63)$$

This can be proved by observing that $\underline{c}_a^T\,[sI_{2n} - A_a]\,\underline{b}_a = G_0(s)$ and, on the other hand, that $G_0(s)$ can be always made SPR by a proper choice of $H(s)$, which means that (62) and (63) are satisfied.

The parameters of the neural controller (57) are adjusted using the same update laws (26)–(28), except that $\underline{x}$ and $\underline{\phi}_{if}(\underline{x})$ are replaced by $\widehat{\underline{x}}$ and $\underline{\phi}_{if}(\widehat{\underline{x}})$, respectively. It can be verified, using the same analysis, that the update laws (26)–(28) (with the indicated changes) guarantee that $\|K_1\| \le \kappa_1$, $|\underline{k}_2| \le \kappa_2$ and $|\underline{k}_3| \le \kappa_3\ \forall t \ge 0$.

Consider the Lyapunov function

$$\begin{aligned}V &= \frac{1}{2}\underline{\phi}_3(\underline{x})\,\underline{k}_3^*\underline{v}^T P_a\underline{v} + \frac{1}{2\gamma_1}\,\mathrm{tr}\left[\widetilde{K}_1^T\widetilde{K}_1\right] \\ &\quad + \frac{1}{2\gamma_2}\widetilde{\underline{k}}_2^T\widetilde{\underline{k}}_2 + \frac{1}{2\gamma_3}\widetilde{\underline{k}}_3^T\widetilde{\underline{k}}_3, \qquad (64)\end{aligned}$$

where P_a is the solution of (62) and (63) for a given positive definite matrix Q_a.

The differentiation of (64) along the trajectory of (60) yields

$$\begin{aligned}\dot{V} &= -\frac{1}{2}\underline{\phi}_3(\underline{x})\,\underline{k}_3^*\underline{v}^T Q_a\underline{v} + \frac{1}{2}\dot{\underline{\phi}}_3(\underline{x})\,\underline{k}_3^*\underline{v}^T P_a\underline{v} \\ &\quad + b_m\underline{v}^T P_a\underline{b}_a\Big[\underline{\phi}_{1f}(\widehat{\underline{x}})\,\widetilde{K}_1\widehat{\underline{x}} \\ &\qquad + \underline{\phi}_{2f}(\widehat{\underline{x}})\,\widetilde{\underline{k}}_2 + \underline{\phi}_{3f}(\widehat{\underline{x}})\,\widetilde{\underline{k}}_3 r\Big] \\ &\quad + \underline{v}^T P_a\underline{b}_a\left[\underline{\alpha}_f\underline{v} + \zeta_f\right] + \frac{1}{\gamma_1}\,\mathrm{tr}\left[\widetilde{K}_1^T\dot{\widetilde{K}}_1\right] \\ &\quad + \frac{1}{\gamma_2}\widetilde{\underline{k}}_2^T\dot{\widetilde{\underline{k}}}_2 + \frac{1}{\gamma_3}\widetilde{\underline{k}}_3^T\dot{\widetilde{\underline{k}}}_3. \qquad (65)\end{aligned}$$

Then, exploiting (61), (65) can be arranged as

$$\begin{aligned}\dot{V} &= -\frac{1}{2}\underline{\phi}_3(\underline{x})\,\underline{k}_3^*\underline{v}^T Q_a\underline{v} + \frac{1}{2}\dot{\underline{\phi}}_3(\underline{x})\,\underline{k}_3^*\underline{v}^T P_a\underline{v} \\ &\quad + \underline{v}^T P_a\underline{b}_a\left[\underline{\alpha}_f\underline{v} + \zeta_f\right] \\ &\quad + \frac{1}{\gamma_1}\,\mathrm{tr}\Big[\widetilde{K}_1^T\Big(\gamma_1 b_m e_y\underline{\phi}_{1f}^T(\widehat{\underline{x}})\,\widehat{\underline{x}}^T - \dot{K}_1\Big)\Big] \\ &\quad + \frac{1}{\gamma_2}\widetilde{\underline{k}}_2^T\Big(\gamma_2 b_m e_y\underline{\phi}_{2f}^T(\widehat{\underline{x}}) - \dot{\underline{k}}_2\Big) \\ &\quad + \frac{1}{\gamma_3}\widetilde{\underline{k}}_3^T\Big(\gamma_3 b_m e_y\underline{\phi}_{3f}^T(\widehat{\underline{x}})\,r - \dot{\underline{k}}_3\Big). \qquad (66)\end{aligned}$$

Further, using the update laws (26)–(28) and the same analysis as that performed in the previous section, it can be shown that the last three terms in (66) are always nonpositive. Hence, (66) becomes

$$\begin{aligned}\dot{V} &\le -\frac{1}{2}\underline{\phi}_3(\underline{x})\,\underline{k}_3^*\underline{v}^T Q_a\underline{v} + \frac{1}{2}\dot{\underline{\phi}}_3(\underline{x})\,\underline{k}_3^*\underline{v}^T P_a\underline{v} \\ &\quad + \underline{v}^T P_a\underline{b}_a\underline{\alpha}_f\underline{v} + \underline{v}^T P_a\underline{b}_a\zeta_f, \qquad (67)\end{aligned}$$

which is upper bounded by

$$\begin{aligned}\dot{V}_2 &\le -\frac{1}{2}\kappa_3\underline{v}^T Q_a\underline{v} + \frac{1}{2}\varphi_0\kappa_3\underline{v}^T P_a\underline{v} \\ &\quad + \rho_0\underline{v}^T P_a\underline{v} + \frac{1}{2}\underline{v}^T P_a\underline{b}_a\underline{b}_a^T P_a\underline{v} + \frac{1}{2}\zeta_f^2, \qquad (68)\end{aligned}$$

where the upper bounds $\|\underline{b}_a\underline{\alpha}_f\| \le \rho_0$ and $|\underline{\phi}_3(x)| \le 1$ are used.

Further, (68) can be arranged as

$$\begin{aligned}\dot{V} &\le -\frac{1}{2}\kappa_3\underline{v}^T\Big(Q_a - \Big(\varphi_0 + 2\frac{\rho_0}{\kappa_3}\Big)P_a - \frac{1}{\kappa_3}P_a\underline{b}_a\underline{b}_a^T P_a\Big)\underline{v} \\ &\quad + \frac{1}{2}\zeta_f^2. \qquad (69)\end{aligned}$$

Hence, if the matrices Q_a and P_a are chosen such that

$$Q_a - \left(\varphi_0 + 2\frac{\rho_0}{\kappa_3}\right)P_a - \frac{1}{\kappa_3}P_a\underline{b}_a\underline{b}_a^T P_a \ge Q_2 \qquad (70)$$

for some positive definite matrix Q_2, then

$$\dot{V} \le -\frac{1}{2}\kappa_3\underline{v}^T Q_2\underline{v} + \frac{1}{2}\zeta_f^2. \qquad (71)$$

Hence, (71) can be upper bounded by

$$\dot{V} \le -\frac{1}{2}\kappa_3\lambda_{Q_2}|\underline{v}|^2 + \frac{1}{2}\zeta_f^2, \qquad (72)$$

where λ_{Q_2} is the smallest eigenvalue of Q_2.

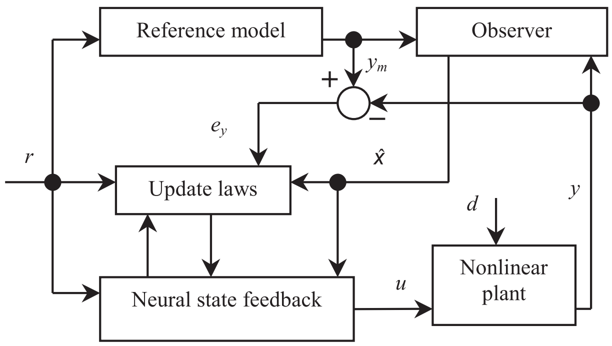


Fig. 3. Observer-based neural adaptive control structure.

Then, $\dot{V} \le 0$ outside the bounded region defined by

$$|\underline{v}| \le \frac{1}{\sqrt{\kappa_3 \lambda_{Q_2}}} |\zeta_f| . \tag{73}$$

The properties of the observer-based neural adaptive approach are summarized by the following result:

Theorem 2. *The control system of Fig. 3, composed of the nonlinear system (7)–(8) satisfying P1–P2, the reference model (9)–(10), the neural controller (57) satisfying C1–C3, the state observer (53)–(54), the matrices P_a and Q_a satisfying (62)–(63) and (70), and the update laws (26)–(28) (with the indicated changes) guarantee the following:*

(i)

$$|\underline{\widehat{x}}| \le |\underline{x}_m| + c_5 |\zeta_f| , \tag{74}$$

$$|\underline{x}| \le |\underline{x}_m| + c_6 |\zeta_f| , \tag{75}$$

(ii)

$$|u| \le \kappa_1 |\underline{x}_m| + \kappa_2 + \kappa_3 |r| + c_7 |\zeta_f| , \tag{76}$$

(iii)

$$\int_0^T |\underline{\widehat{e}}|^2 \, \mathrm{d}t \le c_8 + c_9 \int_0^T |\zeta_f|^2 \, \mathrm{d}t, \tag{77}$$

$$\int_0^T |\underline{e}|^2 \, \mathrm{d}t \le c_{10} + c_{11} \int_0^T |\zeta_f|^2 \, \mathrm{d}t. \tag{78}$$

Proof. (i) Define the following Lyapunov function:

$$V_o = \frac{1}{2} \underline{\widehat{e}}^T P_o \underline{\widehat{e}}, \tag{79}$$

where, for some positive definite matrix Q_o, P_o is a solution of the equation

$$\left[A - \underline{l}\underline{c}^T\right]^T P_o + P_o \left[A - \underline{l}\underline{c}^T\right] = -Q_o . \tag{80}$$

The differentiation of (79) along (55) yields

$$\dot{V}_o = -\frac{1}{2} \underline{\widehat{e}}^T Q_o \underline{\widehat{e}} + \underline{\widehat{e}}^T P_o \underline{l}\underline{c}^T \underline{e} \tag{81}$$

which is upper bounded by

$$\dot{V}_o \le -\frac{1}{2} \lambda_{Q_o} |\underline{\widehat{e}}|^2 + \lambda_{P_o} |\underline{l}| \, |\underline{e}| \, |\underline{\widehat{e}}| , \tag{82}$$

where λ_{Q_o}, λ_{P_o} are the smallest and the largest eigenvalues of Q_o and P_o, respectively. Then, $\dot{V}_o \le 0$ whenever $|\underline{\widehat{e}}|$ is outside the region defined by

$$|\underline{\widehat{e}}| \le 2 \frac{\lambda_{P_o}}{\lambda_{Q_o}} |\underline{l}| \, |\underline{e}| . \tag{83}$$

On the other hand, we have $|\underline{v}| = \sqrt{|\underline{e}|^2 + |\underline{\widehat{e}}|^2}$. If combined with (83), this yields

$$|\underline{e}| \le \frac{1}{\sqrt{1 + 4\left(\frac{\lambda_{P_o}}{\lambda_{Q_o}} |\underline{l}|\right)^2}} |\underline{v}| . \tag{84}$$

Further, (73) and (84) yield the following upper bound:

$$|\underline{e}| \le \frac{1}{\sqrt{\kappa_3 \lambda_{Q_2} \left(1 + 4\left(\frac{\lambda_{P_o}}{\lambda_{Q_o}} |\underline{l}|\right)^2\right)}} |\zeta_f| . \tag{85}$$

Then, introducing (85) in (83) gives

$$|\underline{\widehat{e}}| \le \frac{2\lambda_{P_o} |\underline{l}| / \lambda_{Q_o}}{\sqrt{\kappa_3 \lambda_{Q_2} \left(1 + 4\left(\frac{\lambda_{P_o}}{\lambda_{Q_o}} |\underline{l}|\right)^2\right)}} |\zeta_f| . \tag{86}$$

From (86) and the fact that $|\underline{\widehat{x}}| \le |\underline{x}_m| + |\underline{\widehat{e}}|$ we get

$$|\underline{\widehat{x}}| \le |\underline{x}_m| + \frac{2\lambda_{P_o} |\underline{l}| / \lambda_{Q_o}}{\sqrt{\kappa_3 \lambda_{Q_2} \left(1 + 4\left(\frac{\lambda_{P_o}}{\lambda_{Q_o}} |\underline{l}|\right)^2\right)}} |\zeta_f| , \tag{87}$$

which yields (74), with

$$c_5 = \frac{2\lambda_{P_o} |\underline{l}| / \lambda_{Q_o}}{\sqrt{\kappa_3 \lambda_{Q_2} \left(1 + 4\left(\frac{\lambda_{P_o}}{\lambda_{Q_o}} |\underline{l}|\right)^2\right)}} .$$

Now, from $|\underline{x}| \le |\underline{e}| + |\underline{x}_m|$ and (85), it follows that

$$|\underline{x}| \le |\underline{x}_m| + \frac{1}{\sqrt{\kappa_3 \lambda_{Q_2} \left(1 + 4\left(\frac{\lambda_{P_o}}{\lambda_{Q_o}} |\underline{l}|\right)^2\right)}} |\zeta_f| . \tag{88}$$

Then, (75) is satisfied for

$$c_6 = \frac{1}{\sqrt{\kappa_3 \lambda_{Q_2} \left(1 + 4\left(\frac{\lambda_{P_o}}{\lambda_{Q_o}} |\underline{l}|\right)^2\right)}} .$$

$$P_a > 0, \tag{94}$$

$$\begin{bmatrix} -A_a^T P_a - P_a A_a - \left(\varphi_0 + 2\frac{\rho_0}{\kappa_3}\right) P_a - \frac{1}{\kappa_3} P_a \underline{b}_a \underline{b}_a^T P_a & P_a \underline{b}_a - \underline{c}_a^T \\ \underline{b}_a^T P_a - \underline{c}_a & 0 \end{bmatrix} > 0. \tag{95}$$

(ii) Using (87) and $|u| \le \kappa_1 |\hat{\underline{x}}| + \kappa_2 + \kappa_3 |r|$, we get

$$|u| \le \kappa_1 |\underline{x}_m| + \kappa_2 + \kappa_3 |r| + \kappa_1 c_5 |\zeta_f| . \tag{89}$$

Hence (76) is proved with $c_7 = \kappa_1 c_5$.

(iii) Introducing (85) in (82) gives

$$\dot{V}_1 \le -\frac{1}{2}\lambda_{Q_o} |\hat{\underline{e}}|^2 + \frac{1}{2}\lambda_{Q_o} c_5^2 |\zeta_f|^2 . \tag{90}$$

Then integrating both sides of (90) yields

$$\int_0^T |\hat{\underline{e}}|^2 \, \mathrm{d}t \le \frac{2}{\lambda_{Q_o}} \left[|V_1(0)| + |V_1(T)| \right] + c_5^2 \int_0^T |\zeta_f|^2 \, \mathrm{d}t, \tag{91}$$

which gives (77) with $c_8 = \frac{2}{\lambda_{Q_o}} [|V_1(0)| + |V_1(T)|]$ and $c_9 = c_5^2$.

To prove (78), integrating both sides of (72) yields

$$\int_0^T |\underline{v}|^2 \, \mathrm{d}t \le \frac{2}{\kappa_3 \lambda_{Q_2}} \left[|V(0)| + |V(T)| \right] + \frac{1}{\kappa_3 \lambda_{Q_2}} \int_0^T |\zeta_f|^2 \, \mathrm{d}t. \tag{92}$$

Further, using (84), we get

$$\int_0^T |\underline{e}|^2 \, \mathrm{d}t \le \frac{1}{1 + 4\left(\frac{\lambda_{P_o}}{\lambda_{Q_o}} |\underline{l}|\right)^2} \int_0^T |\underline{v}|^2 \, \mathrm{d}t. \tag{93}$$

Then, combining (92) and (93) yields (78), with

$$c_{10} = \frac{2\left[|V(0)| + |V(T)|\right]}{\kappa_3 \lambda_{Q_2} \left(1 + 4\left(\frac{\lambda_{P_o}}{\lambda_{Q_o}} |\underline{l}|\right)^2\right)}$$

and

$$c_{11} = \frac{1}{\kappa_3 \lambda_{Q_2} \left(1 + 4\left(\frac{\lambda_{P_o}}{\lambda_{Q_o}} |\underline{l}|\right)^2\right)}.$$

■

Remarks:

1. From (77) and (78) we have that if $\zeta_f \in L_2$ then, by Barbalat's lemma, the tracking and estimated tracking errors converge to zero.
2. Other observers such as a high gain observer may be used, but in this case the control input should be saturated to avoid the effect of large transient values of the estimated states.
3. Parts (i)–(iii) of Theorem 1 and 2 point out the following facts: First, the tracking and estimation errors are bounded and their bounds depend on some design parameters that can be tuned by the designer. Second, the input control is also bounded and the amount of energy can be controlled by the design parameters. Third, if the uncertainty terms are square integrable, then the tracking and estimation errors converge to zero.
4. The satisfaction of the stability conditions (62), (63) and (70) can be expressed as an LMI problem of the form (94) and (95). The feasibility of (94) and (95) depends on the choice of the reference model (9) and (10), the filter $H(s)$ and the observer dynamics (53) and (54). The reference model and observer eigenvalues influence the magnitudes of the eigenvalues of matrix P, which affects the positivity of (95). On the other hand, an increase in the norm of $H(s)$ reduces the norm of the uncertain terms $\underline{\alpha}_{4f}$ and $\underline{\alpha}_{5f}$ which reduces the magnitude of ρ_0 but increases the filtering dynamics.

5. Simulation

The validity of the proposed neural adaptive control scheme is verified on an inverted pendulum (Fig. 4), described by

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= f(x_1, x_2) + g(x_1)u + d, \end{aligned} \tag{96}$$

with

$$f(x_1, x_2) = \frac{(m_c + m) g \sin x_1 - m l x_2^2 \cos x_1 \sin x_1}{l\left(\frac{4}{3}(m_c + m) - m \cos^2 x_1\right)},$$

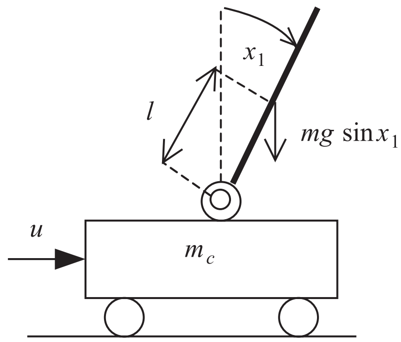


Fig. 4. Inverted pendulum.

$$g(x_1) = \frac{\cos x_1}{l\left(\frac{4}{3}(m_c+m) - m\cos^2 x_1\right)},$$

where x_1, x_2 and u are the pendulum angular position, velocity and input force, respectively. Moreover, d is an uncertainty term. Remark that $f(x)$ can be written as $f(\underline{x}) = a(\underline{x})\underline{x}$, with $\underline{x}^T = [\ x_1 \quad x_2\ ]$ and

$$a_1(\underline{x}) = \frac{(m_c+m)g}{l\left(\frac{4}{3}(m_c+m) - m\cos^2 x_1\right)}\frac{\sin x_1}{x_1},$$

$$a_2(\underline{x}) = -\frac{mlx_2\cos x_1 \sin x_1}{l\left(\frac{4}{3}(m_c+m) - m\cos^2 x_1\right)}.$$

For the simulation, the system parameters are chosen as follows: $m_c = 1$ Kg, $m = 0.1$ Kg, $l = 0.5$ m and $g = 9.8$ m/s^2. The reference model is given by $a_m = [\ 1 \quad 2\ ]$ and $b_m = 1$. The control system is discretized using the sampling step $T = 2$ ms. We note that since T is small, the stability conclusions drawn on the continuous-time basis remain valid. If this is not the case, the proposed approach should be redesigned in a discrete-time framework.

The steps of the observer-based adaptive neural control design are as follows:

1. For the observer design, select $\underline{l}^T = [\ 8 \quad 16\ ]$ and, to ensure the SPR condition, select $H(s) = (s+2)$.
2. Regarding the reference model dynamics, the reference variables are bounded within the interval $[-1,1]\times[-2,2]$ (along x_1 and x_2, respectively). As for the form of $f(\underline{x})$ and $g(x_1)$, the control input is given by

$$u = \underline{\phi}_1(x_1,\widehat{x}_2)K_1\widehat{\underline{x}} + \underline{\phi}_3(x_1), \underline{k}_3 r, \qquad (97)$$

where the first term in (97) is constructed by a two-output RBF network defined by nine RBF functions, i.e., $\underline{\phi}_1(x_1,\widehat{x}_2) \in \mathbb{R}^{1\times 9}$, having standard deviations of $1/\sqrt{2}$ and $\sqrt{2}$ and using mesh spacings of 1 and 2 with respect to x_1 and x_2, respectively. The second term in (97) is constructed by a single-output RBF network with three RBFs, i.e., $\underline{\phi}_3(x_1) \in \mathbb{R}^{1\times 3}$, having standard deviation of $1/\sqrt{2}$ and using mesh spacings of 1 with respect to x_1. $K_1 \in \mathbb{R}^{9\times 2}$ and $\underline{k}_3 \in \mathbb{R}^{3\times 1}$ are the two adjustable parameters of neural networks. Note that the pendulum velocity x_2 is replaced by the estimated one $\widehat{x}_2$.
3. The required bounds are fixed as $\kappa_1 = 60$ and $\kappa_3 = 30$, which implies $|a_1(\underline{x})| < 16$, $|a_2(\underline{x})| < 0.5$ and $0.98 < g(x_1) < 1.47$, respectively. The evaluation of $\dot{\phi}_3(x_1)$ over the interval defined for x_1 and x_2 yields $|\dot{\phi}_3(x_1)| \le \varphi_0 = 1.81$. The upper bounds on the approximation errors are fixed as $\epsilon_1 = 0.05$ and $\epsilon_3 = 0.01$.
4. Using the above bounds, the solution of the LMIs (94) and (95) yields

$$P_a = \begin{bmatrix} 305.1491 & -45.1804 & 153.2330 & -76.6165 \\ -45.1804 & 56.4105 & -13.0517 & 6.5258 \\ -153.2330 & -13.0517 & 222.3894 & -110.6947 \\ 76.6165 & 6.5258 & -110.6947 & 55.3473 \end{bmatrix} > 0,$$

which gives, with $\underline{b}_a^T = [\ 0 \quad 0 \quad 1 \quad 2\ ]$ and $\underline{c}_a = [\ 0 \quad 0 \quad 1 \quad 0\ ]$, the following result:

$$\underline{b}_a^T P_a - \underline{c}_a = 10^{-8}\left[\ 0.1340 \quad 0.0392 \quad -0.1298 \quad -0.0335\ \right],$$

which is practically zero.
5. The free control parameters are adjusted using (26) and (28), with $\gamma_1 = 500$ and $\gamma_3 = 100$.

The tracking performance for the nominal plant (i.e., no uncertainty is introduced) is shown in Fig. 5. It is clear that the tracking errors are rapidly reduced, and the control input is moderate and smooth. Further, the pendulum position and velocity are rapidly estimated by the observer used. Figure 6 shows the tracking performance under the disturbance term $d = 2\text{sign}(x_2) + 2x_2$, where, after a short transient period, the tracking errors converge to small bounded values. Figure 7 illustrates the effect of parametric uncertainties, where the masses and length are reduced by 50% (Case (a)) and increased by 50% (Case (b)). Finally, Fig. 8 shows the combined effect of the disturbance term with two parametric uncertainty cases (a) and (b). Those results illustrate the neural controller robustness against disturbances and uncertainties, and its effectiveness in both tracking the reference model and estimating the pendulum state vector. As predicted by theory, the evolution of the norms of the neural adaptive controller parameters converges to bounded values.

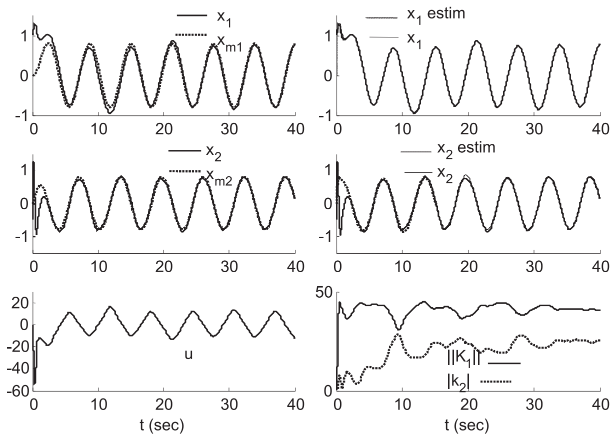


Fig. 5. Nominal plant tracking performance.

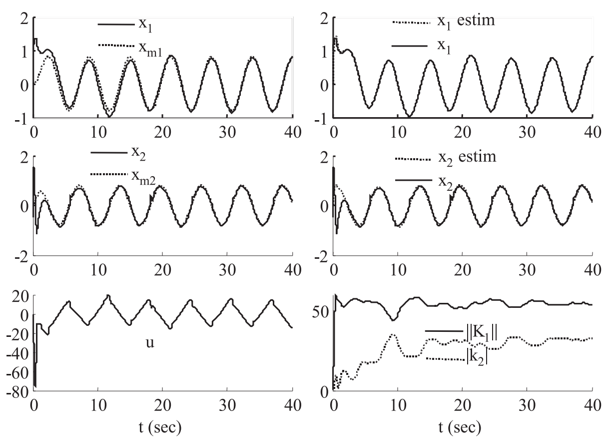


Fig. 6. Effect of disturbance.

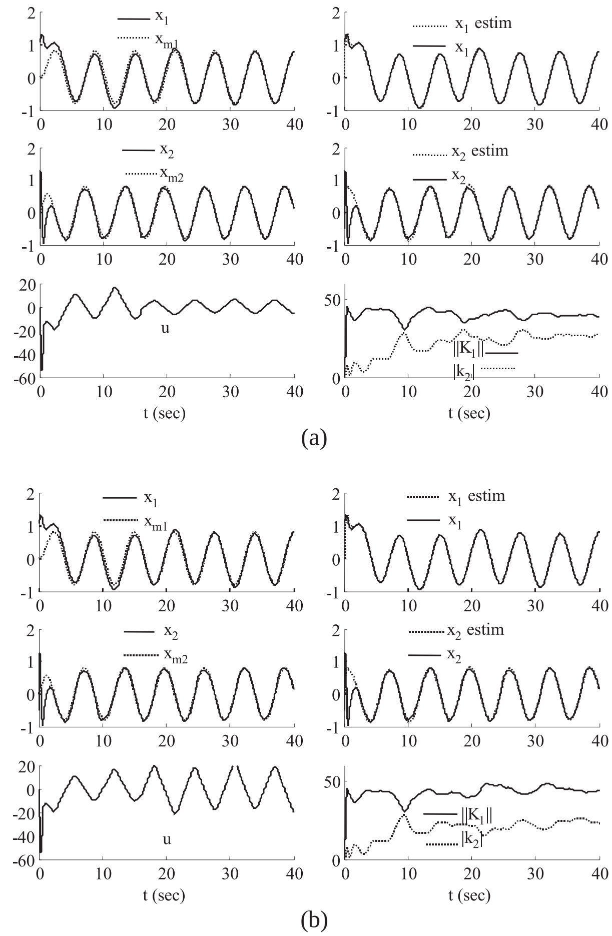


Fig. 7. Effect of parametric uncertainties.

6. Conclusion

This paper has presented an extension of the model reference control approach to nonlinear plants using adaptive neural networks. Both full state and observer-based approaches were analyzed, and their stability was proved. The advantages of this approach over the previously designed ones are as follows: no supervisory control is needed to ensure closed loop stability, and no knowledge of a high-frequency gain derivative bound is required. The simulation results conformed the effectiveness and simplicity of the proposed algorithm. Future work is directed to extend this control design to multivariable nonlinear systems.

References

Billings S.A., Jamaluddin H.B. and Chen S. (1992): *Properties of neural networks with applications to modeling nonlinear dynamical systems*. — Int. J. Contr., Vol. 55, No. 1, pp. 193–224.

Cybenko G. (1989): *Approximations by superpositions of a sigmoidal function*. — Math. Signals Syst., Vol. 2, pp. 303–314.

Cotter M.E. (1990): *The Stone-Weierstrass theorem and its applications to neural nets*. — IEEE Trans. Neural Netw., Vol. 1, No. 4, pp. 290–295.

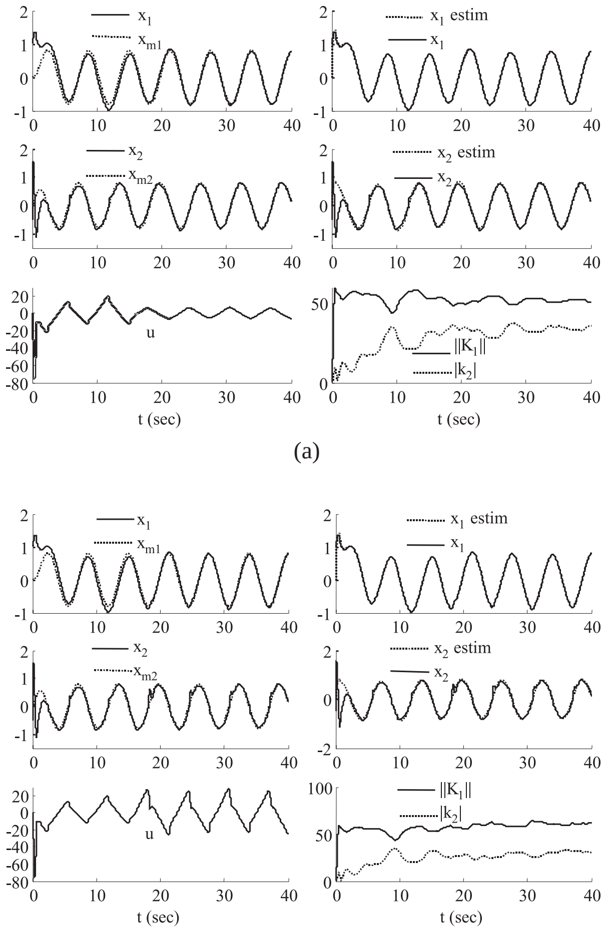
Chang Y-C. and Yen H-M. (2005): *Adaptive output feedback tracking control for a class of uncertain nonlinear systems using neural networks*. — IEEE Trans. Syst. Man Cybern., Part B, Vol. 35, No. 6, pp. 1311–1316.

Ferrari S. and Stengel R.F. (2005): *Smooth function approximation using neural networks*. — IEEE Trans. Neural Netw., Vol. 16, No. 1, pp. 24–38.

Ge S.S., Hang C.C. and Zhang T. (1999): *A direct method for robust adaptive nonlinear control with guaranteed transient performance*. — Syst. Contr. Lett., Vol. 37, pp. 275–284.

Huang S.N., Tan K.K. and Lee T.H. (2006): *Nonlinear adaptive control of interconnected systems using neural networks*. — IEEE Trans. Neural Netw., Vol. 17, No. 1, pp. 243–246.

Landau Y.D. (1979): *Adaptive Control: The Model Reference Approach*. — New York: Marcel Dekker.



(b)

Fig. 8. Effect of combined disturbance and uncertainties.

Narendra K.S. and Parthasarathy K. (1990): *Identification and control of dynamical systems using neural networks*. — IEEE Trans. Neural Netw., Vol. 1, No. 1, pp. 4–27.

Neidhoefer J.C., Cox C.J. and Saeks R.E. (2003): *Development and application of a Lyapunov synthesis based neural adaptive controller*. — IEEE Trans. Syst. Man Cybern., Vol. 33, No. 1, pp. 125–137.

Park J. and Sandberg I.W. (1991): *Universal approximation using radial basis function networks*. — Neural Comput., Vol. 3, No. 2, pp. 246–257.

Poznyak A.S., You W., Sanchez E.E. and Perez J.P. (1999): *Nonlinear adaptive trajectory tracking using dynamic neural networks*. — IEEE Trans. Neural Netw., Vol. 10, No. 6, pp. 1402–1411.

Patino H.D. and Liu D. (2000): *Neural network-based model reference adaptive control system*. — IEEE Trans. Syst. Man Cybern., Vol. 30, No. 1, pp. 198–204.

Plett G.L. (2003): *Adaptive inverse control of linear and nonlinear systems using dynamic neural networks*. — IEEE Trans. Neural Netw., Vol. 14, No. 2, pp. 360–376.

Rivals I. and Personnaz L. (2000): *Nonlinear internal model control using neural networks: Application to processes with delay and design issues*. — IEEE Trans. Neural Netw., Vol. 11, No. 1, pp. 80–90.

Sastry S. and Bodson M. (1989): *Adaptive Control: Stability, Convergence, and Robustness*. — Upper Saddle River, NJ: Prentice-Hall.

Sanner R.M. and Slotine J.E. (1992): *Gaussian networks for direct adaptive control*. — IEEE Trans. Neural Netw., Vol. 3, No. 6, pp. 837–863.

Spooner J.T. and Passino K.M. (1996): *Stable adaptive control using fuzzy systems and neural networks*. — IEEE Trans. Neural Netw., Vol. 4, No. 3, pp. 339–359.

Seshagiri S. and Khalil H. (2000): *Output feedback control of nonlinear systems using RBF neural networks*. — IEEE Trans. Neural Netw., Vol. 11, No. 1, pp. 69–79.

Yesildirek A. and Lewis F.L. (1995): *Feedback linearization using neural networks*. — Automatica, Vol. 31, No. 11, pp. 1659–1664.

Yu S.-H. and Annaswamy A.M. (1997): *Adaptive control of nonlinear dynamic systems using adaptive neural networks*. — Automatica, Vol. 33, No. 11, pp. 1975–1995.

Zhihong M., Wu H.R. and Palaniswami M. (1998): *An adaptive tracking controller using neural networks for a class of nonlinear systems*. — IEEE Trans. Neural Netw., Vol. 9, No. 5, pp. 947–955.

Zhang Y., Peng P.Y. and Jiang Z.P. (2000): *Stable neural controller design for unknown nonlinear systems using backstepping*. — IEEE Trans. Neural Netw., Vol. 11, No. 6, pp. 1347–1359.

Received: 1 September 2005
Revised: 16 April 2006