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ROUGH SETS BASED ON GALOIS CONNECTIONS

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Rough set theory is an important tool to extract knowledge from relational databases. The original definitions of approximation operators are based on an indiscernibility relation, which is an equivalence one. Lately, different papers have motivated the possibility of considering arbitrary relations. Nevertheless, when those are taken into account, the original definitions given by Pawlak may lose fundamental properties. This paper proposes a possible solution to the arising problems by presenting an alternative definition of approximation operators based on the closure and interior operators obtained from an isotone Galois connection. We prove that the proposed definition satisfies interesting properties and that it also improves object classification tasks.

Keywords: rough sets, Galois connections, approximation operators.

1. Introduction

Rough set theory was introduced by Pawlak (1982) as a formal tool to acquire and model knowledge from the information contained in databases. Due to its capability of dealing with uncertainty, it has been successfully applied to solve practical tasks related to medical diagnosis (Varma *et al*., 2015), pattern recognition (Hassanien *et al*., 2009) and image processing (Hassanien *et al*., 2010), among others. Moreover, it has been related to other areas of knowledge such as fuzzy sets (Yao, 1998a), mathematical morphology (Bloch, 2000), the theory of evidences (Tan *et al*., 2018), belief functions (Yao and Lingras, 1998) and formal concept analysis (Medina, 2012b). For a wider compendium of applications and related areas of rough set theory, we refer to the reader to Zhang *et al*. (2016).

The original goal of this theory was the definition of vague concepts by means of the construction of two approximation operators. The essence of the construction of those approximation operators lies of the idea of distinguishing objects according to a set of attributes. Keeping up with this idea, several extensions of the original rough set theory have already been published. Specifically, we can distinguish four main classes of generalized rough sets: those based on the combination of rough set theory with different areas such as fuzzy set theory (Cornelis *et al*., 2014) or probability theory (Ziarko, 2008); those based on granularity or coverings (Couso and Dubois, 2011; Han, 2019; Yao, 1998; 2018), those based on degrees of inclusion (Skowron *et al*., 2004), and those based on neighbourhood operators or arbitrary binary relations (Slowinski and Vanderpooten, 1997; Yao and Yao, 2012).

As for the latter class, we focus on the generalized rough sets introduced by Yao (1996). In this paper, the author presents an interesting generalization of approximation operators considering arbitrary relations, which has widely been referred to in the literature. However, the examination of the approximation operators of Yao (1996) may lead to the following drawback: the lower one might not be contained in the original set and

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the upper approximation might not contain the original set (see Yao and Yao, 2012, Theorem 1). Moreover, Zhu (2007) proves that some restrictions on the relations should be imposed, in order to obtain some desirable properties for the approximation operators given by Yao (1996). However, the consideration of restrictions on the relations might be a strong assumption in some cases, in view of the current trends in information processing, leading to the management of imperfect information (i.e., incomplete and/or inconsistent information) (Grant and Hunter, 2006; Madrid and Ojeda-Aciego, 2011a; Pawlak, 1991). For that reason, instead of imposing restrictions on the relations, we propose to overcome those drawbacks making use of the theory of Galois connections (Birkhoff, 1967; Davey and Priestley, 2002).

Two dual versions of the notion of the Galois connection exist: an antitone Galois connection (or simply Galois connection) and an isotone Galois connection (or adjunction); both are composed of a pair of mappings and used in multiple areas of computer science, such as fuzzy logic (Hajek, 1998; Novák *et al.*, 1999; Medina *et al.*, 2004; Cornejo *et al.*, 2018a; Madrid and Ojeda-Aciego, 2011b), fuzzy sets (Bustince *et al.*, 2015; Madrid and Ojeda-Aciego, 2017), formal concept analysis (Wille, 1982; 2005; Ganter and Wille, 1999; Medine *et al.*, 2009), fuzzy transforms (Perfilieva, 2006; Madrid, 2017), fuzzy relation equations (Sanchez, 1976; Di Nola *et al.*, 1989; Cornejo *et al.*, 2017a; Díaz-Moreno *et al.*, 2017; Medina, 2017), mathematical morphology (Ronse and Heijmans, 1991; Alcalde *et al.*, 2017; Madrid *et al.*, 2019) and the geometry of approximation (Pagliani and Chakraborty, 2008). In this paper, we will consider the so-called *isotone Galois connection*. The composition of both mappings appearing in the Galois connection gives rise to an interior operator and a closure operator, which are suitable to play the roles of the lower and upper approximation operators, respectively. In fact, the original approximation operators of Pawlak (1982) form an isotone Galois connection, and also are an interior operator and a closure operator, due to the consideration of equivalence relations. Moreover, rough set theory has been related to many of the above mentioned theories based on Galois connections (Bloch, 2000; Díaz-Moreno and Medina, 2013; Medina, 2012b; Perfilieva *et al.*, 2017; Yao, 1998a, 2004; Yao and Chen, 2006; Benítez-Caballero *et al.*, 2020)

In this work, we modify slightly the definition given by Yao (1996) to define two Galois connections which are used later to introduce two different lower and upper approximations motivated by the previously mentioned approaches. Specifically, the proposed approximation operators are the interior and closure operators obtained from the composition of operators in an isotone Galois connection. As a result, we solve the above-mentioned drawbacks, that is, the lower approximation is always contained in the original set and the upper approximation always contains the original set, independently of the properties satisfied by the relation considered. This is not the first time some authors consider this possibility to define approximation operators.

In particular, concept-forming operators, which form Galois connections, have been considered to define approximation operators within formal concept analysis (Shao *et al.*, 2007) or to study the lattice of rough sets (Järvinen *et al.*, 2009). In the work of Pagliani (2016) approximation operators based on the composition of operators in the Galois connection are dismissed in favor of the definition given by Yao (1996) because of topological properties of the latter. Nevertheless, Pagliani manifests the importance of the dismissed definition and focuses on determining cases where both definitions coincide. In contrast to Pagliani, we opt for the composition of operators because it increases the granularity, that is, the number of classes in which the set of objects is split into. Hence, we show that the obtained approximations are more accurate and, in classification tasks, the number of objects that can be classified increases. Providing a better classification for objects when the relation has no restrictions is fundamental for real applications, since general relations arise in different real problems when, for example, the notion of indiscernible objects needs to be relaxed. In this paper, we will also present and analyze a possible relaxed definition of the indiscernibility relation.

The structure of the paper is the following. Some preliminary notions are recalled in Section 2. The proposed definitions of approximation operators based on isotone Galois connections and different properties are presented in Section 3. In Section 4, we propose a tolerance relation that models conveniently the notion of indiscernibility, and then we apply the proposed approximation operators to decision systems, showing some examples and more properties. The work ends in Section 5 with some conclusions and prospect for future work.

2. Preliminaries

2.1. Rough set theory. First of all, we recall some notions related to rough set theory.

Definition 1. An *approximation space* is a pair (U, R), where U is a set (called universe) and R is a binary relation over U.

The previous definition was introduced by Yao (1996) as a generalization of the original definition given by Pawlak (1982). Specifically, when the given relation R is an equivalence one, the approximation space is called the *Pawlak approximation space*. Moreover, depending on the nature of the relation R, we obtain different kinds of approximation spaces. For example, when R is a tolerance relation (reflexive and symmetric), then we have the approximation space given by Zakowski (1983), or if R is reflexive and transitive the approximation space derives in a topological space (Kortelainen, 1994).

In the following, we recall the definition of the upper and lower approximations of a set. These definitions are given in a general environment when the relation R is an arbitrary one.

Definition 2. (*Yao, 1996*) Let (U, R) be an approximation space and $A \subseteq U$. The *lower approximation* and the *upper approximation of* A are defined respectively as

$$R{\downarrow}\, A = \{y \in U \mid Ry \subseteq A\} \tag{1}$$

and

$$R{\uparrow}\, A = \{y \in U \mid Ry \cap A \neq \varnothing\}, \tag{2}$$

where Ry is a set defined as $Ry = \{x \in U \mid (x, y) \in R\}$, which is called the R-foreset.

In the literature, we can find equivalent ways of defining $R \downarrow A$ and $R \uparrow A$ (Cornelis *et al.*, 2014; Stell, 2007; Yao, 2004). Once we have recalled the previous notions, we can introduce the definition of a rough set.

Definition 3. Let (U, R) be an approximation space and $A \subseteq U$. The *rough set associated with* A is defined as the pair $(R{\downarrow}\, A, R{\uparrow}\, A)$.

Let us recall some interesting properties:

1. $R{\downarrow}\, (A \cap B) = R{\downarrow}\, A \cap R{\downarrow}\, B$.
2. $R{\uparrow}\, (A \cup B) = R{\uparrow}\, A \cup R{\uparrow}\, B$.
3. $R{\downarrow}\, (U) = U$.
4. $R{\uparrow}\, (\varnothing) = \varnothing$.
5. If $A \subseteq B$, then $R{\downarrow}\, A \subseteq R{\downarrow}\, B$.
6. If $A \subseteq B$, then $R{\uparrow}\, A \subseteq R{\uparrow}\, B$.

When R is reflexive, the following properties are also satisfied:

7. $R{\downarrow}\, A \subseteq A \subseteq R{\uparrow}\, A$.
8. $R{\downarrow}\, A \subseteq R{\uparrow}\, (R{\downarrow}\, A)$.
9. $R{\downarrow}\, (R{\uparrow}\, A) \subseteq R{\uparrow}\, A$.
10. $R{\downarrow}\, (R{\downarrow}\, A) \subseteq R{\downarrow}\, A$.
11. $R{\uparrow}\, A \subseteq R{\uparrow}\, (R{\uparrow}\, A)$.

When R is symmetric, we have that

12. $R{\uparrow}\, (R{\downarrow}\, A) \subseteq A \subseteq R{\downarrow}\, (R{\uparrow}\, A)$.

Moreover, when R is an equivalence relation, the following equalities hold:

13. $R{\uparrow}\, (R{\downarrow}\, A) = R{\downarrow}\, A$.
14. $R{\downarrow}\, (R{\uparrow}\, A) = R{\uparrow}\, A$.

In addition, the accuracy of the approximations of a finite set A, in an approximation space, can be measured as follows.

Definition 4. An *accuracy measure of the finite set* $A \subseteq U$ in the approximation space (U, R) is defined as

$$\mu_R(A) = \frac{\mathrm{Card}(R{\downarrow}\, A)}{\mathrm{Card}(R{\uparrow}\, A)}, \tag{3}$$

where $\mathrm{Card}(A)$ represents the cardinality of the set A.

Data rarely appear as approximation spaces. Instead, they are usually structured from tables that relate objects to attributes; this kind of structure is called an information system.

Definition 5. An *information system* $(U, \mathcal{A})$ is a tuple such that $U = \{x_1, x_2, \ldots, x_n\}$ and $\mathcal{A} = \{a_1, a_2, \ldots, a_m\}$ are finite, non-empty sets of objects and attributes, respectively. Each $a \in \mathcal{A}$ is associated with a mapping $\bar{a}\colon U \to V_a$, where V_a is the value set of a over U.

If $V_a = \{0, 1\}$ for each $a \in \mathcal{A}$, we say that $(U, \mathcal{A})$ is a *Boolean information system*.

There are different ways of defining approximation spaces from a given information system. The standard procedure to obtain an approximation space from an information system is defining an indiscernibility relation over the set of objects. Given a subset B of $\mathcal{A}$, the *indiscernibility relation with respect to* B, Ind_B, is defined as

$$\mathrm{Ind}_B = \{(x, y) \in U \times U \mid \text{for all } a \in B, \bar{a}(x) = \bar{a}(y)\}. \tag{4}$$

Observe that Ind_B is an equivalence relation and, consequently, we can consider the equivalence classes which are written as $[x]_B = \{y \mid (x, y) \in \mathrm{Ind}_B\}$. In addition, Ind_B produces a partition on U denoted as $U/\mathrm{Ind}_B = \{[x]_B \mid x \in U\}$.

The obtained approximation space (U, Ind_B) will be called *the induced approximation space from the information system* $(U, \mathcal{A})$ and used to discern elements. In this way, when $(x, y) \in \mathrm{Ind}_B$, we say that x and y are *indiscernible* in B, otherwise we say that they are *discernible*.

In this paper, we also consider a special kind of information system called a decision system.

Definition 6. A *decision system* $(U, \mathcal{A} \cup \{d\})$ is a kind of information system in which $d \notin \mathcal{A}$ is called the decision attribute.

The equivalence classes obtained from the relation shown in Eqn. (4), with respect to $\{d\}$, are called decision classes and are denoted as $[x]_d$. The standard task in decision systems is to describe the decision classes from the attributes in $\mathcal{A}$ in order to classify objects abroad U. Definition 6 has been straightforwardly extended in different papers to consider a non-singleton set of decision attributes (Luo *et al.*, 2018; Yang *et al.*, 2017).

Given a subset of attributes $B \subseteq \mathcal{A}$, the B-positive region is formed by those equivalence classes, induced by Ind_B, with the same decision value. In addition, the dependency of B over the decision attribute d is defined in order to determine how well classification of objects can be performed according to the decision attribute d, taking into account the information of B.

Definition 7. Let $(U, \mathcal{A} \cup \{d\})$ be a decision system, and $B \subseteq \mathcal{A}$ and (U, Ind_B) its induced approximation space. The *B-positive region*, denoted as POS_B, is defined as

$$POS_B = \bigcup_{x \in U} \mathrm{Ind}_B \!\downarrow [x]_d,$$

and the degree of dependency of d over B, γ_B, as

$$\gamma_B = \frac{\mathrm{Card}(POS_B)}{\mathrm{Card}(U)},$$

where $[x]_d$ represents the equivalence class of the object $x \in U$ with respect to the indiscernibility relation Ind_d of Eqn. (4).

2.2. Galois connections and adjunctions.

There exist two dual versions of the notion of the Galois connection (Birkhoff, 1967; Davey and Priestley, 2002). In this work we consider only one of those, the so-called *isotone Galois connection*. For more detailed information about isotone and antitone Galois connections, we refer the reader to Denecke *et al.* (2004) or Davey and Priestley (2002).

Definition 8. Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be posets. A pair (φ, ψ) of mappings $\varphi\colon P \to Q$, $\psi\colon Q \to P$ is called an *isotone Galois connection* between P and Q if the following equivalence is satisfied, for all $p \in P$ and $q \in Q$:

$$\varphi(p) \leq_Q q \quad \text{if and only if} \quad p \leq_P \psi(q).$$

This notion is also called *adjunction*. The mapping φ is called a *lower* (or *left*) *adjoint* of ψ and the mapping ψ an *upper* (or *right*) *adjoint* of φ.

The definition of an isotone Galois connection has the following characterization, which plays a key role in our approach.

Proposition 1. *Let $\varphi\colon P \to Q$ and $\psi\colon Q \to P$ be two maps between the posets $(P, \leq_P)$ and $(Q, \leq_Q)$. The pair (φ, ψ) is an* isotone Galois connection *if and only if*

- *ψ and φ are order-preserving;*
- *$p \leq_P \psi(\varphi(p))$, for all $p \in P$;*
- *$\varphi(\psi(p)) \leq_Q q$, for all $q \in Q$.*

Galois connections are very much related to the interior and closure operators. In order to analyze such a relationship, let us introduce formally the proper definitions.

Definition 9. A map $\Gamma\colon P \to P$ defined on a poset $(P, \leq_P)$ is called a *closure operator* if the following properties are satisfied:

- $\Gamma(\Gamma(p)) = \Gamma(p)$ (idempotent);
- if $p_1 \leq_P p_2$ then, $\Gamma(p_1) \leq_P \Gamma(p_2)$ (order-preserving);
- $p \leq_P \Gamma(p)$ (extensive)

for all $p, p_1, p_2 \in P$. Similarly, a map $\Theta\colon Q \to Q$ on a poset $(Q, \leq_Q)$ is called an *interior operator* if the following properties are satisfied:

- $\Theta(\Theta(q)) = \Theta(q)$ (idempotent);
- if $q_1 \leq_Q q_2$ then, $\Theta(q_1) \leq_Q \Theta(q_2)$ (order-preserving);
- $\Theta(q) \leq_Q q$ (anti-extensive)

for all $q, q_1, q_2 \in Q$.

Note that Proposition 1 encourages us to link Galois connections with the interior and closure operators. Specifically, the following results shows that we can construct a closure and an interior operator by composition of mappings in every Galois connection.

Proposition 2. *Let (φ, ψ) be an isotone Galois connection between the posets $(P, \leq_P)$ and $(Q, \leq_Q)$. Then $\varphi \circ \psi$ and $\psi \circ \varphi$ form an interior operator and a closure operator in P and Q, respectively.*

The following result provides other useful properties of isotone Galois connections.

Theorem 1. *Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be ordered sets. Then the following statements hold:*

(i) If $\varphi\colon P \to Q$, $\psi\colon Q \to P$ form an isotone Galois connection (φ, ψ), then φ preserves suprema and ψ preserves infima.

(ii) If $(P, \leq_P)$ is a complete lattice and $\varphi\colon P \to Q$ preserves suprema, then the function $\psi\colon Q \to P$, defined as

$$\psi(q) = \bigvee \{p \in P \mid \varphi(p) \leq_Q q\}$$

for all $q \in Q$, is the unique right adjoint of φ.

(iii) If $(Q, \leq_Q)$ is a complete lattice and $\psi\colon Q \to P$ preserves infima, then the function $\varphi\colon P \to Q$, defined as

$$\varphi(p) = \bigwedge\{q \in Q \mid p \leq_P \psi(q)\}$$

for all $p \in P$, is the unique left adjoint of ψ.

Remark 1. In Properties 5, 6, 12, 13 and 14, Definition 3, and Proposition 1, the reader can observe a direct relationship between isotone Galois connections and approximation operators of rough sets, when R is an equivalence relation. Moreover, by these properties and Definition 9, the notions of the closure and interior operators are also related to approximation operators. Specifically, in the original approach of Pawlak, thanks to the consideration of equivalence relations, approximation operators, $R\uparrow$ and $R\downarrow$, are the closure and interior operators which form an isotone Galois connection. Hence, approximation operators in the original approach of Pawlak have both roles, although the idea underlying the lower and upper operators in rough set theory is closer to the interior and closure operators than to isotone Galois connections.

However, the properties mentioned above do not need to be satisfied for arbitrary relations. Specifically, only Properties 1–6 hold. Therefore, when U is finite and R is arbitrary, by Theorem 1, we only have that $R\uparrow$ and $R\downarrow$ determine a left and right adjunction, although they may not be adjunctions of the same Galois connection (for that the symmetry of R is needed). Moreover, $R\uparrow$ and $R\downarrow$ are neither closure nor interior operators, in general, which also weakens the justification of using $R\downarrow$ and $R\uparrow$ as the lower and upper operators. Thus, there exists a shortcoming of the definition given by Yao (1996) when arbitrary relations are considered. The following section proposes a solution to this problem.

3. Approximation operators based on the Galois connection

The original idea of Pawlak is to represent sets by pairs of a lower and an upper approximation. The consideration of equivalence relations in the original definition (Pawlak, 1982; 1991) was crucial for a coherent definition of such approximations. Subsequently, Yao (1996) defined a lower and an upper approximation for the case of arbitrary relations (recalled in Definition 2), which coincide with the usual ones when the relation is an equivalence one. However, such definitions have two noteworthy shortcomings, which are summarized below:

- When the relation R is not reflexive, the lower approximation of a set $A \subseteq U$ may not be contained in A. Similarly, the upper approximation may not contain A. In this case, the obtained approximations are senseless.
- If there exists an object $x \in U$ such that $(x, y) \notin R$ for all $y \in U$, i.e., x is not related to any object, then $x \in R\downarrow A$ and $x \notin R\uparrow A$ for all subset $A \subseteq U$. In this case, lower approximations are not contained in upper approximations.

Despite the previous items being certainly inconceivable when R is an indiscernibility relation, they may be feasible when R represents another relation between objects. For example, on a Boolean information system we define the relation R given as follows: an object x is related to an object y (xRy) if the number of attributes of x is strictly greater than that of attributes of y. Then, R is not reflexive and there is at least an object which is not related to any object (the ones with the least number of attributes).

The following example illustrates both of the previous questions.

Example 1. Consider $U = \{x_1, x_2, x_3\}$ and the binary relation R on U given by the table

R	x_1	x_2	x_3
x_1		$\times$	
x_2	$\times$		
x_3			

.

Setting $A = \{x_1\}$, we have that $R\downarrow A = \{x_2, x_3\}$ and $R\uparrow A = \{x_2\}$. Then

$$R\downarrow A \nsubseteq A \quad\text{and}\quad A \nsubseteq R\uparrow A \quad\text{and}\quad R\downarrow A \nsubseteq R\uparrow A.$$

Therefore, $R\downarrow$ and $R\uparrow$ cannot be considered a lower or an upper approximation, respectively. ♦

In order to overcome the previously mentioned issues and provide convenient definitions of the lower and upper approximations, the first step is to generalize the definition of the R-foreset, in order to distinguish such a set when the relation does not satisfy the symmetry.

Definition 10. Let (U, R) be an approximation space; the sets defined as

$$xR = \{y \in U \mid (x, y) \in R\}$$

and

$$Ry = \{x \in U \mid (x, y) \in R\}$$

are the *R-right-foreset of* $x \in U$ and the *R-left-foreset of* $y \in U$, respectively.

Taking into consideration the two generalizations of the R-foreset, we can define four different approximation operators.

Definition 11. Let (U, R) be an approximation space and $A \subseteq U$. We define the following operators:

- $R\downarrow^r A = \{x \in U \mid xR \subseteq A\}$,

- $R\uparrow_r A = \{x \in U \mid xR \cap A \neq \varnothing\}$,
- $R\downarrow^\ell A = \{y \in U \mid Ry \subseteq A\}$,
- $R\uparrow_\ell A = \{y \in U \mid Ry \cap A \neq \varnothing\}$.

It is convenient to note that $R\downarrow^\ell$ and $R\uparrow_\ell$ coincide with the approximation operators given by Yao (1996), i.e., with $R\downarrow$ and $R\uparrow$ given in Definition 2. Moreover, when the relation is symmetric, also the equalities $R\downarrow^r = R\downarrow^\ell = R\downarrow$ and $R\uparrow_r = R\uparrow_\ell = R\uparrow$ hold. From now on, for the sake of simplicity, in the case of considering a symmetric relation, we will write $R\downarrow$ and $R\uparrow$ instead of $R\downarrow^r$, $R\downarrow^\ell$ and $R\uparrow_r$, $R\uparrow_\ell$, respectively. Before introducing generalized rough sets, it is convenient to recall that the pairs $(R\uparrow_r, R\downarrow^\ell)$ and $(R\uparrow_\ell, R\downarrow^r)$ are isotone Galois connections (Yao, 1998a; Medina, 2012b). As a consequence, the operators $R\uparrow_r$ and $R\uparrow_\ell$ preserve the empty set, and $R\downarrow^r$ and $R\downarrow^\ell$ preserve the universe U.

The following example shows that, in general, the pairs of operators given by Yao (1996) (i.e., $(R\uparrow_\ell, R\downarrow^\ell)$ and $(R\downarrow^\ell, R\uparrow_\ell)$) are not isotone Galois connections.

Example 2. Consider the approximation space (U, R), where $U = \{x_1, x_2\}$ and the relation R is given in the following table:

R	x_1	x_2
x_1	$\times$	$\times$
x_2		$\times$

Let us show that there exist $A, B, A', B' \subseteq U$ such that none of the following equivalences hold:

$$R\uparrow_\ell (A) \subseteq B \quad \text{if and only if} \quad A \subseteq R\downarrow^\ell (B).$$
$$R\downarrow^\ell (A') \subseteq B' \quad \text{if and only if} \quad A' \subseteq R\uparrow_\ell (B').$$

Consequently, according to Definition 8, neither the pair $(R\uparrow_\ell, R\downarrow^\ell)$ nor $(R\downarrow^\ell, R\uparrow_\ell)$ is an isotone Galois connection.

On the one hand, we consider the subsets $A = B = \{x_2\}$. In this way we have that

$$R\downarrow^\ell \{x_2\} = \{y \in U \mid Ry \subseteq \{x_2\}\} = \varnothing$$

and

$$R\uparrow_\ell \{x_2\} = \{y \in U \mid Ry \cap \{x_2\} \neq \varnothing\} = \{x_2\}$$

since $Rx_1 = \{x_1\}$ and $Rx_2 = \{x_1, x_2\}$. In other words, $R\uparrow_\ell \{x_2\} \subseteq \{x_2\} = B$, but $A = \{x_2\} \nsubseteq R\downarrow^\ell \{x_2\}$. Therefore, we can assert that the pair $(R\downarrow^\ell, R\uparrow_\ell)$ is not an adjunction.

On the other hand, we consider the subsets $A' = \{x_1, x_2\}$ and $B' = \{x_1\}$. Then, we have that

$$R\uparrow_\ell \{x_1\} = \{y \in U \mid Ry \cap \{x_1\} \neq \varnothing\} = \{x_1, x_2\}$$

and

$$R\downarrow^\ell \{x_1, x_2\} = \{y \in U \mid Ry \subseteq \{x_1, x_2\}\} = \{x_1, x_2\}.$$

In this way, $A' = \{x_1, x_2\} \subseteq R\uparrow_\ell \{x_1\}$ but $R\downarrow^\ell \{x_1, x_2\} \nsubseteq \{x_1\} = B'$. Consequently, the pair $(R\uparrow_\ell, R\downarrow^\ell)$ is not an adjunction either. ♦

An advantage of considering isotone Galois connections is that we can define the interior and closure operators from them, which can be used to generalize the original definition of a rough set given by Pawlak. In the rest of this section, we prove that the approximations given by Yao (1996) can be improved considering these operators, namely, $R\uparrow_r (R\downarrow^\ell)$, $R\downarrow^\ell (R\uparrow_r)$, $R\uparrow_\ell (R\downarrow^r)$ and $R\downarrow^r (R\uparrow_\ell)$. Note that they always define a lower or an upper approximation for any set. Specifically, given a relation R and a set $A \subseteq U$, the following chains are satisfied because the two pairs $(R\uparrow_r, R\downarrow^\ell)$ and $(R\uparrow_\ell, R\downarrow^r)$ are isotone Galois connections:

$$R\uparrow_r (R\downarrow^\ell (A)) \subseteq A \subseteq R\downarrow^\ell (R\uparrow_r (A)), \tag{5}$$
$$R\uparrow_\ell (R\downarrow^r (A)) \subseteq A \subseteq R\downarrow^r (R\uparrow_\ell (A)). \tag{6}$$

In other words, the notion of a rough set is extended to arbitrary relations by using the following definition.

Definition 12. Let (U, R) be an approximation space and $A \subseteq U$. The *lower approximations of* A are defined as

$$R\uparrow_r (R\downarrow^\ell (A)) \quad \text{and} \quad R\uparrow_\ell (R\downarrow^r (A))$$

and the *upper approximations of* A are defined as:

$$R\downarrow^\ell (R\uparrow_r (A)) \quad \text{and} \quad R\downarrow^r (R\uparrow_\ell (A)).$$

A set $A \subseteq U$ is called *a generalized rough set* if is different from the two lower approximations and from the two upper approximations.

Recall that the operators in Definition 11 coincide with the approximation operators provided by Yao (1996) when R is symmetric. However, it is interesting to remark that our approach is different from that of Yao (1996) even when R is symmetric, because here we do not consider the operators given in Definition 11 that of approximation operators, but their composition.

The use of the interior and closure operators does not only guarantee the construction of lower and upper approximations for any set and any relation R. Moreover, the following properties are directly obtained by the theory of the Galois connection.

Theorem 2. *Let (U, R) be an approximation space and $A, B \subseteq U$, then*

- *If $A \subseteq B$ then $R\uparrow_r (R\downarrow^\ell (A)) \subseteq R\uparrow_r (R\downarrow^\ell (B))$;*
- *If $A \subseteq B$ then $R\uparrow_\ell (R\downarrow^r (A)) \subseteq R\uparrow_\ell (R\downarrow^r (B))$;*

- *If* $A \subseteq B$ *then* $R{\downarrow^\ell}(R{\uparrow_r}(A)) \subseteq R{\downarrow^\ell}(R{\uparrow_r}(B))$*;*
- *If* $A \subseteq B$ *then* $R{\downarrow^r}(R{\uparrow_\ell}(A)) \subseteq R{\downarrow^r}(R{\uparrow_\ell}(B))$*;*
- $R{\uparrow_r}(R{\downarrow^\ell}(A)) \subseteq A \subseteq R{\downarrow^r}(R{\uparrow_\ell}(A))$*;*
- $R{\uparrow_r}(R{\downarrow^\ell}(A)) \subseteq A \subseteq R{\downarrow^\ell}(R{\uparrow_r}(A))$*;*
- $R{\uparrow_\ell}(R{\downarrow^r}(A)) \subseteq A \subseteq R{\downarrow^\ell}(R{\uparrow_r}(A))$*;*
- $R{\uparrow_\ell}(R{\downarrow^r}(A)) \subseteq A \subseteq R{\downarrow^r}(R{\uparrow_\ell}(A))$*;*
- $R{\uparrow_r}(R{\downarrow^\ell}(R{\uparrow_r}(R{\downarrow^\ell}(A)))) = R{\uparrow_r}(R{\downarrow^\ell}(A))$*;*
- $R{\uparrow_\ell}(R{\downarrow^r}(R{\uparrow_\ell}(R{\downarrow^r}(A)))) = R{\uparrow_\ell}(R{\downarrow^r}(A))$*;*
- $R{\downarrow^\ell}(R{\uparrow_r}(R{\downarrow^\ell}(R{\uparrow_r}(A)))) = R{\downarrow^\ell}(R{\uparrow_r}(A))$*;*
- $R{\downarrow^r}(R{\uparrow_\ell}(R{\downarrow^r}(R{\uparrow_\ell}(A)))) = R{\downarrow^r}(R{\uparrow_\ell}(A))$.

Proof. The previous items are direct consequences of the fact that $R{\uparrow_r}(R{\downarrow^\ell}(A))$ and $R{\uparrow_\ell}(R{\downarrow^r}(A))$ are interior operators, and $R{\downarrow^r}(R{\uparrow_\ell}(A))$ and $R{\downarrow^\ell}(R{\uparrow_r}(A))$ are closure operators (Definition 9). ■

The following result shows that the lower and upper approximations are related also through the complement of sets. Specifically, the following proposition shows that they are dual with respect to the complement.

Theorem 3. *Let* (U, R) *be an approximation space and* $A \subseteq U$*. Then*

- $R{\uparrow_\ell}(R{\downarrow^r}(A)) = \left(R{\downarrow^\ell}(R{\uparrow_r}(A^c))\right)^c$,
- $R{\uparrow_r}(R{\downarrow^\ell}(A)) = (R{\downarrow^r}(R{\uparrow_\ell}(A^c)))^c$,

where X^c *is the complement of any subset* X *in* U*.*

Proof. Let us prove firstly that $R{\downarrow^r}(A^c) = (R{\uparrow_r} A)^c$ and $R{\downarrow^\ell}(A^c) = (R{\uparrow_\ell} A)^c$. Given $A \subseteq U$, we have that

$$\begin{aligned} R{\downarrow^r}(A^c) &= \{x \in U \mid xR \subseteq A^c\} \\ &= \{x \in U \mid xR \cap A = \varnothing\} \\ &= \{x \in U \mid xR \cap A \neq \varnothing\}^c = (R{\uparrow_r} A)^c, \end{aligned}$$

and similarly $R{\downarrow^\ell}(A^c) = (R{\uparrow_\ell} A)^c$. As a consequence, the following equalities are satisfied:

$$\begin{aligned} R{\uparrow_\ell}(R{\downarrow^r}(A)) &= \left(R{\downarrow^\ell}(R{\uparrow_r}(A^c))\right)^c \\ R{\uparrow_r}(R{\downarrow^\ell}(A)) &= (R{\downarrow^r}(R{\uparrow_\ell}(A^c)))^c. \end{aligned}$$

■

Note that Definition 12 generalizes the original definition given by Pawlak (1982) since, for any equivalence relation R, the following equalities hold:

$$\begin{aligned} R{\downarrow}(A) &= R{\uparrow_r}(R{\downarrow^\ell}(A)) = R{\uparrow_\ell}(R{\downarrow^r}(A)) \\ R{\uparrow}(A) &= R{\downarrow^\ell}(R{\uparrow_r}(A)) = R{\downarrow^r}(R{\uparrow_\ell}(A)). \end{aligned}$$

In addition, when R is reflexive, the following chains are satisfied for all $A \subseteq U$:

$$R{\downarrow^\ell}(A) \subseteq R{\uparrow_r}(R{\downarrow^\ell}(A)) \subseteq A \tag{7}$$
$$A \subseteq R{\downarrow^r}(R{\uparrow_\ell}(A)) \subseteq R{\uparrow_\ell}(A). \tag{8}$$

Consequently, the approximation operators in Definition 12 provide closer approximations to the original set than the approximation operators of Yao (1996), since the latter coincide with $R{\downarrow^\ell}(A)$ and $R{\uparrow^\ell}(A)$. Notice that, when the relation R is not reflexive, the inequality $R{\downarrow^\ell}(A) \subseteq A \subseteq R{\uparrow_\ell}(A)$ might not be satisfied, as we saw in Example 1. Therefore, the comparison of these approximation operators is senseless the case when the relation considered is not reflexive.

It is worth highlighting a result of Pagliani (2014) that characterizes when the approximation operators in Definition 12 coincide with those of Yao (1996).

Theorem 4. (Pagliani, 2014, Theorem 1) *Let* (U, R) *be an approximation space. The following assertions are equivalent:*

- $R{\uparrow_\ell}(R{\downarrow^r}(A)) = R{\downarrow^r}(A)$*, for all* $A \subseteq U$.
- $R{\uparrow_r}(R{\downarrow^\ell}(A)) = R{\downarrow^\ell}(A)$*, for all* $A \subseteq U$.
- R *is a preorder (i.e.,* R *is reflexive and transitive).*

As a consequence of the previous result and Eqns. (7) and (8), when the relation in an approximation space (U, R) is not transitive, the lower approximations given by the operators in Definition 12 are strictly more accurate than the approximations given by the operators in the work of Yao (1996). The following example illustrates this fact by considering a reflexive and symmetric relation, that is, a *tolerance relation*. Note that the use of tolerance relations as indiscernibility ones is very interesting when reducing some possible noise in the data, since it is a weaker definition than the equivalence relation.

Example 3. In this example we consider an information system $(U, \mathcal{A})$, where $U = \{x_1, x_2, x_3, x_4\}$, $\mathcal{A} = \{a_1, a_2, a_3, a_4\}$, and the relationship among them is given by means of the following table:

	a_1	a_2	a_3	a_4
x_1	×		×	
x_2		×	×	
x_3		×		×
x_4				×

.

In this example, the indiscernibility relation R assumes that two objects are indiscernible when they differ by two or fewer attributes, that is, xRy if there are at most two attributes $a_i, a_j \in \{a_1, a_2, a_3, a_4\}$ such that $a_i(x) \neq a_i(y)$ and $a_j(x) \neq a_j(y)$. In this way, the values of this indiscernibility relation over the set of objects are displayed in the following table:

R	x_1	x_2	x_3	x_4
x_1	×	×		
x_2	×	×	×	
x_3		×	×	×
x_4			×	×

It is easy to check that the indiscernibility relation R is reflexive and symmetric. Hence, as stated above, we use the notation $R\downarrow$ and $R\uparrow$ instead of $R\downarrow^r$, $R\downarrow^\ell$ and $R\uparrow_r$, $R\uparrow_\ell$, respectively. In addition, note that R is not transitive since, for example, we have that the pairs $(x_1, x_2), (x_2, x_3) \in R$, but $(x_1, x_3) \notin R$.

Let us consider the set $A = \{x_1, x_2\}$; then we have

$$R\!\downarrow A = \{x_1\} \quad \text{and} \quad R\!\uparrow A = \{x_1, x_2, x_3\},$$

which are the lower and upper approximations obtained by the approach of Yao (1996). Considering the approximations operators of Definition 12, we obtain the following sets:

$$R\!\uparrow (R\!\downarrow A) = \{x_1, x_2\} \quad \text{and} \quad R\!\downarrow (R\!\uparrow A) = \{x_1, x_2\}$$

as the lower and upper approximation of A. Hence, we have

$$R\!\downarrow A = \{x_1\} \subset \{x_1, x_2\} = R\!\uparrow (R\!\downarrow A)$$
$$R\!\downarrow (R\!\uparrow A) = \{x_1, x_2\} \subset \{x_1, x_2, x_3\} = R\!\uparrow A.$$

Thus, the subset A is not a generalized rough set, according to Definition 12. ♦

The previous example highlights the fact that the approximations given by Definition 12 are more precise than those given by Definition 2. The following definition shows how to measure the degree of accuracy of the approximations of a subset A.

Definition 13. The *accuracy measure of a finite set* $A \subseteq U$ in the approximation space (U, R), denoted as $\mu^*_R(A)$, is defined as

$$\frac{\max\{\mathrm{Card}(R\!\uparrow_r (R\!\downarrow^\ell (A))), \mathrm{Card}(R\!\uparrow_\ell (R\!\downarrow^r (A)))\}}{\min\{\mathrm{Card}(R\!\downarrow^\ell (R\!\uparrow_r (A))), \mathrm{Card}(R\!\downarrow^r (R\!\uparrow_\ell (A)))\}}.$$

Note that, since for any pair of approximations of a set A, the lower approximation is less than or equal to the upper (Eqns. (5) and (6)), we have that $0 \le \mu^*_R(A) \le 1$.

From the previous comments, when the relation R is reflexive, the accuracy measures μ_R and μ^*_R, given in Definitions 4 and 13 are related.

Proposition 3. *Let (U, R) be an approximation space, where the relation R is reflexive*[1] *and $A \subseteq U$; then*

$$\mu_R(A) \le \mu^*_R(A).$$

Proof. The result is obtained from the fact that, when the relation R is reflexive, the chains shown in Eqns. (7) and (8) are satisfied. ■

4. Applications to decision systems

In this section we consider decision systems, recalled in Definition 6, in order to continue illustrating the convenience of our approach. The standard task in a decision system, $(U, \mathcal{A} \cup \{d\})$, is to define a relation R in U (i.e., an approximation space), by using a subset of attributes $B \subseteq \mathcal{A}$, capable of classifying objects in U with respect to the decision attribute d.

In our approach, we intend to use more general relations than those defined by means of Eqn. (4). The following one weakens the original definition of an indiscenibility relation.

Definition 14. Given an information system $(U, \mathcal{A})$, $s \in \mathbb{N}$ and $B \subseteq \mathcal{A}$, the *s-indiscernibility relation with respect to B, R^s_B*, is defined as follows:

Two objects $x, y \in U$ belong to R^s_B if and only if there are at most s attributes $\{a_1, \dots, a_s\} \subseteq B$ such that $\overline{a}_k(x) \neq \overline{a}_k(y)$ for all $k \in \{1, \dots, s\}$.

If $(x, y) \in R^s_B$, we say that x and y are s-indiscernible in B. When $B = \mathcal{A}$, we simply say that x and y are s-indiscernible and the relation is denoted as R^s.

Note that the indiscernibility relation given in Eqn. (4) coincides with the 0-indiscernibility relation. Another example of an *s-indiscernibility relation* was already considered in Example 3, where the 2-indiscernibility relation of the corresponding decision system was discussed. Below, we introduce another example and compare the use of different values of s in the s-indiscernibility relation.

Example 4. Let $(U, \mathcal{A})$ be the information system given by the set of objects $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and attributes $\mathcal{A} = \{$Forecast, Temperature, Humidity, Wind, Sport$\}$, related according to the following table:

	Forecast	Temperature	Humidity	Wind	Sport
x_1	sunny	hot	high	weak	no
x_2	sunny	hot	high	strong	no
x_3	cloudy	hot	high	weak	yes
x_4	rainy	warm	high	weak	yes
x_5	rainy	cold	normal	weak	yes
x_6	rainy	cold	normal	strong	no

It is easy to check that every pair of objects are discernible (from each other) under the standard

[1]Note that, as we commented at the beginning of Section 3, Definition 2 may have no sense when the relation considered is not reflexive, and $\mu_R(A)$ could be greater than 1.

0-indiscernibility relation. If we consider the 1-indiscernibility relation, we obtain that x_1 and x_2 are 1-indiscernible, since they differ in only one attribute (Wind). Moreover, since the rest of objects differ in more than one attribute, the pair x_1 and x_2 is the only one of 1-indiscernible objects.

If we take into account the 2-indiscernibility relation, we obtain that x_1 and x_2 are 2-indiscernible objects again. In addition, in this case, the objects x_1 and x_3 become 2-indiscernible as well. Note that the 2-indiscernibility relation is not transitive, since $(x_2, x_1), (x_1, x_3) \in R^2$, but $(x_2, x_3) \notin R^2$. ♦

From the previous example it is obvious that there is a kind of monotonicity with respect to the value s. In other words, if two objects are s-indiscernible for $s \in \mathbb{N}$, then they are t-indiscernible for all $t \geq s$ as well.

Proposition 4. *Let $(U, \mathcal{A})$ be an information system and $s, t \in \mathbb{N}$ such that $s \leq t$. If $(x, y) \in R^s$, then $(x, y) \in R^t$ for all $x, y \in U$.*

Proof. This property results directly from Definition 14. ■

From Example 4 is also evident that the s-indiscernibility relation might not be transitive. Nevertheless, the s-indiscernibility is always a tolerance relation (reflexive and symmetric).

Proposition 5. *Let $(U, \mathcal{A})$ be an information system; the s-indiscernibility relation R^s is a tolerance relation for all $s \in \mathbb{N}$.*

Proof. This is a direct consequence of Definition 14. ■

Traditionally, the *positive region* is defined for a subset $B \subseteq \mathcal{A}$ since the indiscernibility relation Ind_B given in Eqn. (4) is determined once the subset $B \subseteq \mathcal{A}$ is fixed. However, in our approach, given a subset $B \subseteq \mathcal{A}$, we may be interested in considering a more general relation R_B different from the equivalence relation Ind_B. For that reason, it is more convenient to define the positive regions associated with the relation R_B instead of the subset B; the use of subscripts is used to emphasize that the relation R_B is defined by using only the information concerning the set of attributes B.

The following definition aims at measuring how well a relation R_B defined from the information of a subset of attributes B can be used to discern between the different classes of Ind_d (recall that Ind_d is an equivalence relation). Specifically, the positive region of the relation R_B contains the set of objects for which the relation R_B is able to foretell their decision classes unequivocally. Notice that, since R_B may be not symmetric, we can define two positive regions related to the capability of predicting the decision classes. Moreover, we can also measure the predictive ability of the relation R_B with respect to d, by means of the value $\gamma^*_{R_B}$.

Definition 15. Let $(U, \mathcal{A} \cup \{d\})$ be a decision system, $B \subseteq \mathcal{A}$, and (U, R_B) be a derived approximation space. The *R_B-left positive and R_B-right positive regions* with respect to R_B, denoted as $POS^{\ell}_{R_B}$ and $POS^{r}_{R_B}$ respectively, are defined as

$$POS^{\ell}_{R_B} = \bigcup_{x \in U} R_B {\uparrow_r} \left(R_B {\downarrow^{\ell}} [x]_d\right)$$

$$POS^{r}_{R_B} = \bigcup_{x \in U} R_B {\uparrow_{\ell}} \left(R_B {\downarrow^{r}} [x]_d\right),$$

and the degree of dependency of d over R_B, $\gamma^*_{R_B}$, as

$$\gamma^*_{R_B} = \frac{\max\left\{\mathrm{Card}(POS^{\ell}_{R_B}), \mathrm{Card}(POS^{r}_{R_B})\right\}}{\mathrm{Card}(U)},$$

where $[x]_d$ represents the equivalence class of the object $x \in U$ with respect to the indiscernibility relation Ind_d given by

$$\mathrm{Ind}_d = \{(x, y) \in U \times U \mid \bar{d}(x) = \bar{d}(y)\}.$$

Note that in the previous definition we used a standard indiscernibility relation Ind_d to define equivalent classes according to the decision attribute d. The sets $POS^{\ell}_{R_B}$ and $POS^{r}_{R_B}$ represent the set of objects in U that can be properly classified by using the relation R_B, defined from the attributes in B. Hence, in the case of $\gamma^*_{R_B} = 1$, the decision classes can be fully determined by the relation R_B. Moreover, if $A, B \subseteq \mathcal{A}$ and $\gamma^*_{R_A} = \gamma^*_{R_B}$, the use of one or another set of attributes (A or B) behaves similarly for the classification provided by the decision attribute d. However, in such a case, we cannot say that the information provided by A and B is the same, but the percent of correctly classified objects is so.

Example 5. Consider the decision system given by the following relation:

	a_1	a_2	a_3	a_4	a_5	a_6	d
x_1	×		×			×	×
x_2			×	×		×	×
x_3		×	×	×	×		×
x_4	×	×			×		
x_5	×	×	×	×			

,

and the 1-indiscernibility relation with respect to the subset of attributes $B = \{a_2, a_4, a_5, a_6\}$, which is displayed in the table below:

R^1_B	x_1	x_2	x_3	x_4	x_5
x_1	×	×			
x_2	×	×			
x_3			×	×	×
x_4			×	×	
x_5			×		×

.

Note that the relation Ind_d provides only two decision classes, $[x_1]_d = \{x_1, x_2, x_3\}$ and $[x_4]_d = \{x_4, x_5\}$. Then, in order to determine the positive regions, we only have to apply the lower approximations to both sets. In addition, since R_B^1 is symmetric, both lower approximations coincide. Hence, we have

$$R_B^1 {\uparrow_r} \left(R_B^1 {\downarrow^\ell} [x_1]_d\right) = R_B^1 {\uparrow_\ell} \left(R_B^1 {\downarrow^r} [x_1]_d\right) = \{x_1, x_2\}$$

$$R_B^1 {\uparrow_r} \left(R_B^1 {\downarrow^\ell} [x_4]_d\right) = R_B^1 {\uparrow_\ell} \left(R_B^1 {\downarrow^r} [x_4]_d\right) = \varnothing$$

and, as a consequence, $\gamma^*_{R_B^1} = 2/5$. ♦

The following result relates the measure of the degree of dependency of d over R_B to the existence of related objects in different classes.

Theorem 5. *Let $(U, \mathcal{A} \cup \{d\})$ be a decision system and a relation R_B with $B \subseteq \mathcal{A}$. If $\gamma^*_{R_B} = 1$; then, for all $x, y \in U$ such that $(x, y) \in R_B$ and $[x]_d \neq [y]_d$, there exists*

- *$z \in R_B {\downarrow^r} ([x]_d)$ with $(z, x) \in R_B$ and $(z, y) \notin R_B$*

or/and

- *$z \in R_B {\downarrow^\ell} ([y]_d)$ with $(y, z) \in R_B$ and $(x, z) \notin R_B$.*

Proof. Since $\gamma^*_{R_B} = 1$, we have $\mathrm{Card}(POS^r_{R_B}) = \mathrm{Card}(U)$ or/and $\mathrm{Card}(POS^\ell_{R_B}) = \mathrm{Card}(U)$. Consider the case $\mathrm{Card}(POS^r_{R_B}) = \mathrm{Card}(U)$. From this last equality and using the property shown in Eqn. (6) we can infer, for all $x \in U$, that

$$R_B {\uparrow_\ell} \left(R_B {\downarrow^r} ([x]_d)\right) = [x]_d.$$

Given $x, y \in U$ such that $(x, y) \in R_B$ with $[x]_d \neq [y]_d$, by the definition of the approximation operators we have

$$R_B {\uparrow_\ell} \left(R_B {\downarrow^r} ([x]_d)\right) = \{t \in U \mid Rt \cap R_B {\downarrow^r} ([x]_d) \neq \varnothing\}.$$

Since $R_B {\uparrow_\ell} (R_B {\downarrow^r} ([x]_d)) = [x]_d$, we get $x \in R_B {\uparrow_\ell} (R_B {\downarrow^r} ([x]_d))$ and, consequently, there exists $z \in U$ such that $z \in R_B x$ (i.e., $(z, x) \in R_B$) and $z \in R_B {\downarrow^r} ([x]_d)$.

It remains to prove that $(z, y) \notin R_B$. Let us assume by *reductio ad absurdum* that $(z, y) \in R_B$, which is equivalent to $y \in z R_B$. By definition, $R_B {\downarrow^r} ([x]_d) = \{t \in U \mid t R_B \subseteq [x]_d\}$ and, since $y \in z R_B$ and $y \notin [x]_d$, we have that $z R_B \nsubseteq [x]_d$. As a consequence, we obtain the contradiction $z \notin R_B {\downarrow^r} ([x]_d)$.

If we consider the case $\mathrm{Card}(POS^\ell_{R_B}) = \mathrm{Card}(U)$ we obtain, following a similar reasoning, the existence of $z \in R_B {\downarrow^\ell} ([y]_d)$ such that $(y, z) \in R_B$ and $(x, z) \notin R_B$. ■

As a consequence of the previous result, the relation R_B cannot be transitive when there exists a pair of related objects which belong to different decision classes and $\gamma^*_{R_B} = 1$.

The following results analyze the effects of adding new properties to the discussed relation in the previous theorem.

Corollary 1. *Let $(U, \mathcal{A} \cup \{d\})$ be a decision system, $B \subseteq \mathcal{A}$ and (U, R_B) be an approximation space such that R_B is reflexive and $\gamma^*_{R_B} = 1$. Then, for all $x, y \in U$ such that $(x, y) \in R_B$ and $[x]_d \neq [y]_d$, there exists*

- *$z \in [x]_d$ such that $(z, x) \in R_B$ and $(z, y) \notin R_B$*

or/and

- *$z \in [y]_d$ such that $(y, z) \in R_B$ and $(x, z) \notin R_B$.*

Proof. It is a direct consequence of the previous theorem by noting that, by reflexivity, $z \in R_B {\downarrow^r} ([x]_d)$ implies $z \in [x]_d$. ■

Proposition 6. *Let $(U, \mathcal{A} \cup \{d\})$ be a decision system, $B \subseteq \mathcal{A}$ and (U, R_B) be an approximation space such that R_B is symmetric and $\gamma^*_{R_B} = 1$. Then, for all $x, y \in U$ such that $(x, y) \in R_B$ and $[x]_d \neq [y]_d$, there exist two different elements:*

- *$z_x \in R_B {\downarrow^r} ([x]_d)$ such that $(z_x, x) \in R_B$ and $(z_x, y) \notin R_B$,*

and

- *$z_y \in R_B {\downarrow^\ell} ([y]_d)$ such that $(y, z_y) \in R_B$ and $(x, z_y) \notin R_B$.*

Proof. The proof follows a similar reasoning to the one given to Theorem 5 noting that, when the relation is symmetric, the two lower approximation operators coincide. Then, $POS^r_{R_B} = POS^\ell_{R_B}$ holds and, consequently,

$$\mathrm{Card}(POS^r_{R_B}) = \mathrm{Card}(POS^\ell_{R_B}) = \mathrm{Card}(U).$$

Therefore, both the cases in the proof of Theorem 5 are satisfied, and z_x is different from z_y because they belong to different classes, that this, we have in particular that $z_x \in [x]_d$, $z_y \in [y]_d$, and $[x]_d \neq [y]_d$. ■

Corollary 2. *Let $(U, \mathcal{A} \cup \{d\})$ be a decision system, $B \subseteq \mathcal{A}$ and (U, R_B) be an approximation space such that R_B is a tolerance relation and $\gamma^*_{R_B} = 1$. Then, for all $x, y \in U$ such that $(x, y) \in R_B$ and $[x]_d \neq [y]_d$, there exist two different elements:*

- *$z_x \in [x]_d$ such that $(z_x, x) \in R_B$ and $(z_x, y) \notin R_B$*

and

- *$z_y \in [y]_d$ such that $(y, z_y) \in R_B$ and $(x, z_y) \notin R_B$.*

Proof. This result is a direct consequence of Corollary 1 and Proposition 6. ■

Finally, another consequence of Theorem 5 is given when the indiscernibility relation R_B is an equivalence one.

Corollary 3. *Let $(U, \mathcal{A} \cup \{d\})$ be a decision system, $B \subseteq \mathcal{A}$, and (U, R_B) be an approximation space such that R_B is an equivalence relation. If $\gamma^*_{R_B} = 1$ and $(x, y) \in R_B$, then $[x]_d = [y]_d$.*

The following result justifies the existence of representative objects within each decision class when $\gamma^*_{R_B} = 1$. Specifically, such representative objects are only related to objects in its decision class when R_B is a reflexive relation.

Theorem 6. *Let $(U, \mathcal{A} \cup \{d\})$ be a decision system, $B \subseteq \mathcal{A}$, and R_B be a reflexive relation such that $\gamma^*_{R_B} = 1$. Then, for each class $[x]_d$,*

- *there exists $z \in [x]_d$ such that $(z, y) \notin R_B$, for all $y \in U \setminus [x]_d$*

or/and

- *there exists $z \in [x]_d$ such that $(y', z) \notin R_B$ for all $y' \in U \setminus [x]_d$.*

Proof. Let us assume by *reductio ad absurdum* that there exists an object $x \in U$ such that, for every $z \in [x]_d$, there exists $y \in U \setminus [x]_d$ with $(z, y) \in R_B$, and there exists $y' \in U \setminus [x]_d$ with $(y', z) \in R_B$.

Note that, since $R_B{\uparrow_\ell}$ and $R_B{\uparrow_r}$ preserve the empty set, if we prove that $R_B{\downarrow^r}\,([x]_d)$ and $R_B{\downarrow^\ell}\,([x]_d)$ are the empty set, then

$$R_B{\uparrow_\ell}\,(R_B{\downarrow^r}\,[x]_d) = R_B{\uparrow_r}\,\big(R_B{\downarrow^\ell}\,[x]_d\big) = \emptyset \neq [x]_d,$$

which contradicts the fact that $\gamma^*_B = 1$ (by the definition of the positive region) and we would finish the proof.

Let us prove that $R_B{\downarrow^r}\,([x]_d)$ is the empty set. Since R_B is reflexive, we have that $R_B{\downarrow^r}\,([x]_d) \subseteq [x]_d$ and hence, in order to prove that $R_B{\downarrow^r}\,([x]_d)$ is the empty set, we only have to demonstrate that if $z \in [x]_d$ then $z \notin R_B{\downarrow^r}\,([x]_d)$. Given $z \in [x]_d$, by assumption we have that there exists $y \in U \setminus [x]_d$ with $(z, y) \in R_B$ and, as a consequence, $zR_B \nsubseteq [x]_d$. Therefore $z \notin R_B{\downarrow^r}\,([x]_d)$.

Assuming that there exists $y' \in U \setminus [x]_d$ with $(y', z) \in R_B$, a similar procedure can be given in order to demonstrate that $R_B{\downarrow^\ell}\,([x]_d)$ is the empty set as well. ■

The following result is a direct consequence of the previous theorem.

Corollary 4. *Let $(U, \mathcal{A} \cup \{d\})$ be a decision system, $B \subseteq \mathcal{A}$, and R_B be a tolerance relation such that $\gamma^*_{R_B} = 1$. Then, for each class $[x]_d$ there exists at least one $z \in [x]_d$ such that $(z, y) \notin R_B$ for all $y \in U \setminus [x]_d$.*

The previous results show the existence of a kind of representative objects in each class. In this respect, each object related by R_B to one of those representative objects belongs to the same class and, as a result, those elements can be used as the center of the clusters in a classification procedure. Note that the representative objects may not be unique, that is, there may exist more than one representative object in each class. Moreover, in such a case, it could be necessary to consider all of the representative objects of one class to define correctly the respective class. Note also that in the case of the s-indiscernibility relation, such representative objects are those with at least $s + 1$ attributes different from all the objects that do not belong to their classes. We illustrate these ideas by means of the following example.

Example 6. Consider again the decision system given in Example 5 and the 1-indiscernibility relation with respect to the subset of attributes $C = \{a_1, a_2, a_3\}$, that is, the relation R^1_C given by

R^1_C	x_1	x_2	x_3	x_4	x_5
x_1	×	×			×
x_2	×	×	×		
x_3		×	×		×
x_4				×	×
x_5	×		×	×	×

.

Note that R^1_C is symmetric. Therefore both lower approximations coincide. Hence, for the decision class $[x_1]_d = \{x_1, x_2, x_3\}$ we have that

$$R^1_C{\downarrow^\ell}\,[x_1]_d = R^1_C{\downarrow^r}\,[x_1]_d = \{x_2\},$$

and then

$$R^1_C{\uparrow_r}\,\big(R^1_C{\downarrow^\ell}\,[x_1]_d\big) = R^1_C{\uparrow_\ell}\,\big(R^1_C{\downarrow^r}\,[x_1]_d\big) = \{x_1, x_2, x_3\}.$$

Similarly, for the decision class $[x_4]_d = \{x_4, x_5\}$ we have that

$$R^1_C{\downarrow^\ell}\,[x_4]_d = R^1_C{\downarrow^r}\,[x_4]_d = \{x_4\},$$

and then

$$R^1_C{\uparrow_r}\,\big(R^1_C{\downarrow^\ell}\,[x_4]_d\big) = R^1_C{\uparrow_\ell}\,\big(R^1_C{\downarrow^r}\,[x_4]_d\big) = \{x_4, x_5\}.$$

As a result, we have $\gamma^*_{R^1_C} = 1$.

By Theorem 6, we know that at least two representative elements $x \in [x_1]_d$ and $x' \in [x_4]_d$ exist which are related only to objects of their own decision classes. In this example, those elements are x_2 and x_4, respectively. Moreover, since

$$R^1_C{\uparrow_r}\,(\{x_2\}) = \{x_1, x_2, x_3\} = [x_1]_d$$

and

$$R^1_C \uparrow_r (\{x_4\}) = \{x_4, x_5\} = [x_4]_d,$$

we can classify correctly objects in U according to the decision attribute $\{d\}$, only verifying whether the object is related either to x_2 or x_4.

In addition, in the case of including new objects in our decision system with a missing decision attribute $\{d\}$, we only need to check to which object, x_2 or x_4, it is related by means of R^1_C, in order to classify those. For example, consider two new objects; x_6 with the attributes $\{a_2, a_3, a_4\}$ and x_7 with the attributes $\{a_1, a_4, a_5, a_6\}$, for which the values for the decision attribute $\{d\}$ are missing. Our goal is to label these objects according to the attribute d. In this case, the 1-indiscernibility relation with respect to the subset of attributes C is depicted in the following table:

R^1_C	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_1	×	×			×	×	
x_2	×	×	×			×	
x_3		×	×		×	×	
x_4				×	×		×
x_5	×		×	×	×		
x_6	×	×	×			×	
x_7				×			×

.

From the previous table it is clear that x_6 is related to the representative object x_2 and x_7 is related to x_4. Therefore, we can classify x_6 as an object satisfying the decision attribute d and x_7 as an object that does not satisfy the decision attribute d.

It is important to note that the object x_7 is not related to any object with respect to C when we consider the 0-indiscernibility relation. Therefore, from the examination of the usual indiscernibility relation, this object could not be labeled within any decision class.

This example shows that the examination of general relations increases the number of objects that our system is able to classify. ♦

The following result shows that the accuracy obtained by means of our approach is always greater than or equal to the accuracy obtained by the standard rough sets classification by Yao (1996) (Definition 2). Note that Definition 7 was originally introduced for the relation Ind_B. However, it can be straightforwardly extended to an arbitrary relation R_B. Under such an assumption, we compare the degrees of dependency given by Definitions 7 and 15.

Proposition 7. *Let $(U, \mathcal{A} \cup \{d\})$ be a decision system, $B \subseteq \mathcal{A}$, and R_B be a tolerance relation. Then $\gamma_{R_B} \leq \gamma^*_{R_B}$.*

Proof. By Eqns. (7) and (8), we have that $R\downarrow^\ell ([x]_d) \subseteq R \uparrow_r (R \downarrow^\ell ([x]_d))$ for all $x \in U$. Consequently, $\mathrm{Card}(POS_{R_B}) \leq \mathrm{Card}(POS^\ell_{R_B})$, and therefore $\gamma_{R_B} \leq \gamma^*_{R_B}$. ■

Corollary 5. *Let $(U, \mathcal{A} \cup \{d\})$ be a decision system, $B \subseteq \mathcal{A}$, and R_B be a tolerance relation such that $\gamma_{R_B} = 1$. Then, $\gamma^*_{R_B} = 1$ as well.*

The following example shows that the converse of the previous result does not hold.

Example 7. Example 6 reveals that $\gamma^*_{R^1_B} = 1$ for $B = \{a_1, a_2, a_3\}$. However, since

$$R^1_B \downarrow^\ell [x_1]_d = R^1_B \downarrow^r [x_1]_d = \{x_2\}$$

and

$$R^1_B \downarrow^\ell [x_4]_d = R^1_B \downarrow^r [x_4]_d = \{x_4\},$$

we have that $\gamma_{R_B} = 2/5 \neq 1$. ♦

From the previous proposition and example, we conclude that our approach needs to consider fewer attributes for classification than the traditional one based on standard rough sets (Pawlak, 1982; Yao, 1996).

5. Conclusions and future work

In this work we have motivated the use of an alternative definition of approximation operators in rough set theory. The main difference with respect to other definitions in the literature is that it uses properties of an isotone Galois connection to define approximation operators as the interior and closure operators, from arbitrary (indiscernibility) relations. We have shown that our definition satisfies suitable properties in order to be considered for the needs of lower and upper approximations. Among those properties we point out that the lower approximation is always contained in the original set and the upper one always contains the original set, independently of the properties of the indiscernibility relation considered; in other words, reflexivity, symmetry or transitivity are not required. Other remarkable properties of approximation operators are idempotence, monotonicity and duality (in terms of the set complement).

Moreover, we have applied approximation operators to a classification task. We have proven that the proposed approximation operators perform better classification than standard approximation operators when the indiscernibility one is not an equivalence relation and, in addition, it is capable of classifying new objects according to the similarity with respect to some representative objects in each class.

For future work, we have different goals. For instance, the labeled procedure described in Example 6 should be further studied. For this goal, firstly we have to analyze two natural cases: the one where all new objects can be classified and the one where some new objects cannot be classified. Secondly, we should also analyze the role of representative objects and how reliable they are. Last but not least, we will apply this procedure to practical examples.

Other future goals address the analysis of more arbitrary relations than those devoted to indiscernibility (as order relations), the study of the relationship of our approach to the generalization of rough sets based on granularity or covering (Zakowski, 1983; Yao, 1998b) and the extension of our approach to fuzzy settings (Cornelis *et al*., 2014; Medina, 2012a).

In addition, due to the close relationship between rough sets theory and formal concept analysis, it would be interesting to analyze the possible influence of the results introduced in this paper, on the classification of objects and attributes of formal contexts and on (fuzzy) attribute reduction (Cornejo *et al.*, 2017b; 2018b).

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