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PARAMETER IDENTIFIABILITY FOR NONLINEAR LPV MODELS

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Linear parameter varying (LPV) models are being increasingly used as a bridge between linear and nonlinear models. From a mathematical point of view, a large class of nonlinear models can be rewritten in LPV or quasi-LPV forms easing their analysis. From a practical point of view, that kind of model can be used for introducing varying model parameters representing, for example, nonconstant characteristics of a component or an equipment degradation. This approach is frequently employed in several model-based system maintenance methods. The identifiability of these parameters is then a key issue for estimating their values based on which a decision can be made. However, the problem of identifiability of these models is still at a nascent stage. In this paper, we propose an approach to verify the identifiability of unknown parameters for LPV or quasi-LPV state-space models. It makes use of a parity-space like formulation to eliminate the states of the model. The resulting input-output-parameter equation is analyzed to verify the identifiability of the original model or a subset of unknown parameters. This approach provides a framework for both continuous-time and discrete-time models and is illustrated through various examples.

Keywords: identifiability, parameter estimation, linear parameter varying models, parity space approach, null space.

1. Introduction

Parameter identifiability studies are motivated by the need for well-posed problems in several applications (Verdière *et al.*, 2005; Coll and Sánchez, 2019). For instance, estimating a parameter through optimization (Beelen and Donkers, 2017) needs a unique set of parameters to satisfy the inputs and outputs of the experiment (at least locally). It is in this respect that the distinguishability property of parameters is defined and forms the basis of the parameter identifiability analysis.

Consider the general nonlinear system model of the form

$$\Sigma_\theta : \begin{cases} \dot{x}(t) = f(x(t), u(t), \theta), & \text{(1a)} \\ y(t) = h(x(t), u(t), \theta) & \text{(1b)} \end{cases}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ and constant parameters $\theta \in \mathbb{R}^q$. The distinguishability property of a model structure refers to (Ljung and Glad, 1994)

$$\tilde{y}(t|\theta') \equiv \tilde{y}(t|\theta'') \Rightarrow \theta' = \theta'', \tag{2}$$

where $\tilde{y}$ stands for the output (1b) computed as the solution of the system (1) for an input $\tilde{u}$ and θ' (or θ'') as the parameter. The essence of distinguishability is captured by the property of parameter identifiability, which refers to whether the model parameter(s) can be uniquely identified by a set of input-output data. It is to be noted that the parameter identifiability property assumes an error-free model and noise-free data and hence is not a sufficient condition for the existence of a solution.

To understand the relevance of this property for applications such as fault diagnosis or prognosis, consider the problem of equipment degradation estimation in large-scale systems. Maintenance activities are rare and typically involve gathering data by deploying temporary sensors (e.g., hand-held monitors). Consequently, only a finite amount of data is available to estimate relevant parameters that are indicative of the underlying degradation phenomena. If a model-based estimation is used, the procedure requires that the available input-output data admit a unique solution for the set of unknown parameters, at least within a known range of

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parameter values. This constraint can be reformulated as the local parameter identifiability of the model.

To start with, in Section 2, we review the literature on the available methods to verify identifiability, focusing on continuous nonlinear state-space models. This review includes the limited, but relevant works for the identifiability of discrete-time models as well. This is followed, in Section 3, by a reminder of the important definitions on which the proposed work will be based. Indeed, the literature contains several definitions for identifiability and this section outlines those definitions of interest and their relationships. This section ends with the statement of the hypotheses relating in particular to the models studied (LPV and quasi-LPV models). Section 4 presents the proposed algorithm in a step-by-step manner for continuous-time models with examples illustrating the procedure. The discrete-time models are treated similarly, with a focus on the steps different from those of continuous-time models, in Section 5. A discussion on a systematic implementation of the proposed approach is given in Section 6 followed by some concluding remarks in Section 7.

2. Relevant literature

To characterize identifiability of a nonlinear model encompassed by (1), there are several definitions in the literature. These definitions and the corresponding characterizations vary due to several factors, including the following:

- the characteristics of the functions f and h (e.g., analytic, homogeneous, meromorphic),
- the characteristics of the inputs (sufficiently continuous/differentiable, piecewise continuous, etc.),
- the assumptions on the initial (state) conditions,
- the neighborhood of the identifiability characterization (local around a particular θ or global).

There are also nuances associated with strong and weak notions of identifiability. For example, Němcová (2010) notes that her identifiability definitions are weaker compared with those of Xia and Moog (2003) because a system is considered to be structurally identifiable if its outputs corresponding to two different parameter values differ for all inputs of an open dense subset of the set of all admissible inputs whereas, in their own work, they just require the existence of at least one such input [1]

[1]However, at the same time, Němcová (2010) considers piece-wise continuous inputs, which are more representative in biological system identification.

2.1. Identifiability of continuous-time models.

Starting with the study of structural identifiability of linear models by Bellman and Åström (1970), several methods have explored the problem in the following decades. In a broad sense, the methods to verify identifiability could be classified as those that perform:

(i) analysis of observables,

(ii) analysis of the system map.

The term *observables* has been borrowed from Chis *et al.* (2011) and roughly refers to the outputs and the parameter information embedded in them. That is, the first classification refers to verifying directly, whether the outputs (and inputs) provide a way to validate the distinguishability property in (2). Methods such as the Taylor series approach and the generating series approach fall into this category. The second class of methods look at some specific properties of the system model to check for identifiability. Isomorphism-based approaches or approaches that consider identifiability as an extended observability property belong to the second class. This classification is not strict as several methods cross over. For example, the implicit function theorem based approach (Xia and Moog, 2003) and the differential algebraic tools based approach (Bellu *et al.*, 2007) exploit system model properties to eliminate the latent (state) variables and then analyze the observables.

2.1.1. Taylor series and generating series approaches.

The Taylor series approach is one of the first proposed for identifiability analysis by Pohjanpalo (1978). By considering the system output as an analytic function of time, it exploits the fact that their derivatives should hold all possible information about the unknown parameters θ. The uniqueness of the Taylor series expansion of this function is an indication of the system identifiability. If the test fails, more coefficients are to be computed and verified again.

The generating series approach by Walter and Lecourtier (1982) is conceptually similar to the Taylor series approach and is applicable to control affine models of the form

$$\begin{aligned}\dot{x}(t) &= f(x(t),\theta) + g(x(t),\theta)u(t),\\ x(t_0) &= x_0(\theta),\\ y(t) &= h(x(t),\theta).\end{aligned} \tag{3}$$

Instead of using derivatives, the Lie derivative expansions of the output functions along the vector fields f and g are computed. The coefficients of the output functions and their Lie derivatives are termed the *exhaustive summary*. By verifying the uniqueness of the exhaustive summary, the structural global identifiability of the model is validated. A drawback of both these approaches lies in the need to know the number of output derivatives over which the identifiability can be verified. To mitigate some of these issues, iterative approaches have been suggested. The *identifiability tableau* proposed by Balsa-Canto *et al.* (2010) was applied for the Taylor series approach while Chis *et al.* (2011) used it for the generating series technique to develop the genSSI MATLAB toolbox.

2.1.2. Isomorphism based approach. The isomorphism based approach answers the distinguishability question by analyzing the relationship between state-space realizations. This method exploits the fact that, under certain conditions, indistinguishable state space models have locally isomorphic state spaces. Thus the identifiability analysis works to show that state isomorphisms must have certain properties within the class of state space systems considered. This helps to parameterize indistinguishable state space models and if the isomorphism can be shown to be the identity, then global identifiability is also verified.

One of the earliest works to analyze the identifiability property through the state-space realization theory is the paper by Glover and Willems (1974). For nonlinear systems, the local state isomorphism was investigated by Vajda and Rabitz (1989), which was followed up by the works of Joly-Blanchard and Denis-Vidal (1998) for uncontrolled systems, Peeters and Hanzon (2005) for homogeneous systems, and Němcová (2010) for polynomial and rational models provided that piece-wise continuous inputs are applied. These approaches assume that the system model is minimal. However, minimality is not a necessary condition for identifiability. One critique of these approaches is the lack of a systematic method to verify identifiability. While this was mitigated to some extent in the systematic solution proposed by Denis-Vidal and Joly-Blanchard (1996), this continues to remain an issue when using this method for complex nonlinear models (Chis *et al.*, 2011).

2.1.3. Differential algebraic approach. The potential of the differential algebraic tools for identifiability problems was discussed in the seminal work by Ljung and Glad (1994). The authors deploy Ritt's algorithm to find the characteristic set of the polynomial ideal generated by the system model (assuming that it can be written in a polynomial form). Given the system model Σ_θ in (1), the idea is to rewrite the state-space model as a set of polynomials,

$$g_i\left(\frac{\mathrm{d}}{\mathrm{d}t}, u(t), y(t), \theta\right) = 0, \quad i = 1, 2, \dots \tag{4}$$

where $\frac{\mathrm{d}}{\mathrm{d}t}$ stands for all the higher order derivatives of the inputs and outputs and along with $\dot{\theta} = 0$. Then, after a careful choice of ranking of the variables, the characteristic set of the ideal generated by the set of polynomials (4) is obtained through Ritt's algorithm. The structure of the characteristic set provides an inference of the identifiability characteristic of the original model (local, global, non-identifiable). The authors also show that the identifiability and the estimation of the parameters are guaranteed to succeed when, for each parameter θ_j, a linear regression form,

$$P_j\left(\frac{\mathrm{d}}{\mathrm{d}t}, u(t), y(t)\right) + \theta_j Q_j\left(\frac{\mathrm{d}}{\mathrm{d}t}, u(t), y(t)\right) = 0 \tag{5}$$

is obtained.

The implementation of the differential algebra approach has different flavors. One approach, proposed by Saccomani *et al.* (1997), uses a differential ring that does not consider θ, a strategy framed by Ollivier (1990). This proves useful for biological systems. Since these models have a large number of parameters, including them in the differential ring incurs significant computational efforts. This is elaborated by Audoly *et al.* (2001) who develop identifiability tools for biological systems.

2.1.4. Differential geometric approach. Tunali and Tarn (1987) employ the authors characterize identifiability as an extended observability problem, where the parameters are added to the state vector and the observability of the new model is evaluated. These results are local in nature, but have an intuitive appeal to it that it lead to the development of the toolbox STRIKE-GOLDD (Villaverde *et al.*, 2016).

Xia and Moog (2003) employ the implicit function theorem as a means to derive local identifiability results. In particular, local structural identifiability is formulated as algebraic identifiability and illustrated. While relationships between the various local identifiability characterizations are clearly given, the actual computation steps to validate identifiability is slightly ambiguous and the example provided seems not well-handled as was also noted by Saccomani (2011).

2.2. Discrete-time identifiability. For the discrete-time case, the identifiability results are limited. Anstett *et al.* (2006) formulated the cryptographic key's ability to be cracked as an identifiability problem and then reused the continuous-time results in the discrete-time context. Anstett *et al.* (2008) developed a discrete-time version of the local state isomorphism theorem and used it to establish identifiability results for discrete-time systems with polynomial nonlinearities.

Nõmm and Moog (2004) develop discrete-time local identifiability results using the implicit function theorem similar to that employed for continuous time by Xia and Moog (2003). These results, however, do not provide any specific systematic procedure, neither do the examples provide any insights into the procedure.

2.3. Identifiability of LPV models. For identifiability of LPV models, all the works consider models with static dependences on the scheduling variables. Lee and Poolla (1997) derive some perspectives on those models that could be represented using the linear fractional transformation (LFT) approach. They provide an identifiability characterization of such models using the existence of a similarity transformation between two realizations. Dankers *et al.* (2011) deal with the dual problems of identifiability and informativity that concerns the parameter estimation (informativity being defined as a weaker version of the condition of the persistence of excitation for a given signal). The models considered are the input-output models with an LPV-ARX structure in contrast to the state-space models of interest in this paper. Identifiability of a discrete-time affine LPV model is discussed by Alkhoury *et al.* (2017), who use the realization theory developed by Petreczky and Mercère (2012). The authors provide a systematic procedure that culminates in a rank condition that would verify the presence of an isomorphism between the realizations. The results are necessary and sufficient for local structural identifiability and sufficient for global structural identifiability.

2.4. Software packages for identifiability evaluation. While there are several systematic approaches to validate identifiability for different systems, implementation of each of those methods for comparison purposes is difficult. In this respect, the continuous-time results obtained in this paper are compared with those produced by DAISY (Bellu *et al.*, 2007).

DAISY is a package developed on the REDUCE platform and implements the ideas that originated by Saccomani *et al.* (1997) and were elaborated by Audoly *et al.* (2001). The package uses Ritt's algorithm to eliminate the system states and compute the characteristic set associated with the differential ideal generated by the system differential equations. The differential ring used is $\mathbb{R}[x, y, u]$ (instead of $\mathbb{R}[x, y, u, \theta]$ as in the work of Ljung and Glad (1994)) and hence a normalized input–output relation is obtained from the characteristic set. The exhaustive summary (Walter and Lecourtier, 1982) is extracted from the normalized input–output relations by gathering the functions of parameters that appear as coefficients. Further, the authors assign random numerical values to the parameters and employ the Buchberger algorithm to compute their Gröbner basis (Buchberger, 2006). Depending upon the number of solutions it admits, the original system is globally, locally or non-identifiable.

There are also other packages such as genSSI (Chis *et al.*, 2011) and STRIKE-GOLDD (Villaverde *et al.*, 2016) developed in MATLAB. The package genSSI evaluates the identifiability of models in the control affine form using the generating series approach from Walter and Lecourtier (1982). STRIKE-GOLDD is based on the extended observability approach of Tunali and Tarn (1987). It computes $n + q$ derivatives of the output and then evaluates the Jacobian of the resulting equation with respect to the extended state vector that includes the unknown parameters.

2.5. Motivation for the present work. The above summary of the literature illustrates a wide range of works that have been carried out for identifiability of nonlinear models. However, the literature is limited when it comes to LPV models. The key objectives addressed in this paper are the following:

- to develop a procedure to verify identifiability of LPV (and quasi-LPV) models,
- to explore a unified procedure for a class of both continuous-time and discrete-time LPV models,
- to utilize and exploit the theoretical and applied results already in the literature.

With this in mind and the wide-ranging models that can be represented through LPV/quasi-LPV models, the elimination strategy seems appropriate in this context as it can be applied, to some extent, irrespective of the underlying model. One of the underlying themes in the literature of elimination techniques is to arrive at the exhaustive summary of a model. In the work of Walter and Lecourtier (1982), it is through a generating series, whereas in that of Audoly *et al.* (2001), it is through differential algebra (Ritt's algorithm). In this work, this is achieved using a parity-space based approach.

Before the approach is discussed, some important definitions related to the parameter identifiability of nonlinear models must be recalled and the assumptions about the structure and the properties of the models studied must be made; this is the subject of the following section.

3. Definitions and assumptions

In this section, the definitions of identifiability of interest and the assumptions underlying the proposed procedure are given.

The above definition formalizes the distinguishability property through structural identifiability. To *a priori* verify identifiability using standard mathematical tools, more tangible definitions are required. In this respect two approaches of interest are discussed below, namely, structural identifiability by Audoly *et al.* (2001) and algebraic identifiability by Xia and Moog (2003). This characterization requires the following notations and terminologies:

An **exhaustive summary** (Walter and Lecourtier, 1982) of an experiment is a set of functions, $\Pi(\theta)$, if it contains only, but all, the information about θ that can be extracted from the knowledge of u and y. That is, they embody the parameter dependence of the input-output model completely. These are also referred to as the observational parameter vector by Jacquez and Greif (1985). Some authors use a slightly different terminology, for example, Audoly *et al.* (2001) refer to the set of equations

$$\Pi(\theta) = \Pi(\tilde{\theta}) \tag{6}$$

as exhaustive summary, where $\tilde{\theta}$ refers to the specific instance of θ used to verify if $\Pi(\theta))$ admits only one solution, that is, θ. In this work, we use *exhaustive summary* to refer to the generic set of equations denoted by $\Pi(\theta))$ whereas (6) would be referred to as *exhaustive summary evaluation*.

Identifiability equations (Xia and Moog, 2003) are q equations which are functions of the known and measured variables along with their derivatives, and the unknown parameters. They are of the form:

$$\Phi(\theta, y, \dot{y}, \ddot{y}, \dots, u, \dot{u}, \dots) = 0.$$

Given a system model (1), it is possible to obtain its exhaustive summary through various methods and then validate the number of solutions in θ admitted by it. This characterization is formalized as follows. Note that the use of $y(\Pi(\theta), t)$ is in reference to the experiment which provides a set of measurement (outputs) that depend on the exhaustive summary.

Definition 1. (*Structural identifiability* (*Audoly* et al.*, 2001*)) A parameter θ_i is

- globally (or uniquely) identifiable if and only if, for almost any $\tilde{\theta}$, the following system has only one solution, $\theta_i = \tilde{\theta}_i$, $i \in \{1, \dots, q\}$:

$$y(\Pi(\theta), t) = y(\Pi(\tilde{\theta}), t); \tag{7}$$

- locally (nonuniquely) identifiable if and only if, for almost any $\tilde{\theta}$, the system (7) has (for θ_i) more than one, but a finite number of solutions;

- non-identifiable if and only if, for almost any $\tilde{\theta}$, the system (7) has (for θ_i) an infinite number of solutions.

The second definition of interest is the algebraic identifiability by Xia and Moog (2003).

Definition 2. (*Algebraic identifiability* (*Xia and Moog, 2003*)) . The system model Σ_θ is said to be algebraically identifiable if there exist a $T > 0$, a positive integer k and a meromorphic function $\Phi : \mathbb{R}^q \times \mathbb{R}^{(k+1)m} \times \mathbb{R}^{(k+1)p} \to \mathbb{R}^q$ such that

$$\det \frac{\partial \Phi}{\partial \theta} \neq 0 \tag{8}$$

and

$$\Phi(\theta, y, \dot{y}, \dots, y^{(k)}, u, \dot{u}, \dots, u^{(k)}) = 0 \tag{9}$$

on $[0, T]$, for all $(\theta, u, \dots, u^{(k)}, y, \dot{y}, \dots, y^{(k)})$, where (θ, x_0, u) belong to an open and dense subset.

The relationship between the algebraic identifiability and the local structural identifiability is clarified in the following proposition. This is done by reiterating the characterization of the two definitions to illustrate the equivalence.

Proposition 1. *For a system model of type Σ_θ, the definitions of algebraically identifiable and locally structurally identifiable are equivalent.*[2]

To verify this equivalence, the first step would be to consider how one can obtain a set of q equations in the form of Φ. In the procedure described in this paper as well as those by Audoly *et al.* (2001), the elimination of states would yield p equations (denote them by Ψ). If $p < q$, then one could obtain more equations by differentiating Ψ to obtain further equations until $p = q$. With this set, one can readily verify Proposition 1 by checking that the exhaustive summary satisfies the two definitions (locally structurally or algebraic identifiable) if only if and the set of identifiability equations also satisfy them. The objective of this proposition would be put to use when we consider equivalent approaches to verify identifiability locally.

Assumption 1. (*Model structure*) The models of interest are those nonlinear models that could be written in an LPV or a quasi-LPV form with affine parametrization. That is,

$$\begin{aligned} \dot{x}(t) &= A(\rho(t), \theta)x(t) + B(\rho(t), \theta)u(t), \\ \dot{y}(t) &= C(\rho(t), \theta)x(t) + D(\rho(t), \theta)u(t) \end{aligned} \tag{10}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, $\rho \in \mathbb{R}^\xi$, $\theta \in \mathbb{R}^q$ with the appropriate dimensions for the system matrices $X \in \{A, B, C, D\}$ which are of the form,

$$X(\rho(t), \theta) = X_0(\rho(t)) + \sum_{j=1}^{q} \theta_j \bar{X}_j(\rho(t)). \tag{11}$$

The scheduling or premise variable, $\rho(t)$, is either composed of external variables with static dependences (in this case the model is LPV) or that of system variables such as inputs, states and outputs (in this case the model is quasi-LPV).

[2]Generically, for almost all cases.

Assumption 2. (*Premise variables*) The premise variables of the quasi-LPV model are known or measured.

Remark 1. The nonlinear models of the form (1) can be rewritten in quasi-LPV forms using several of the existing embedding techniques (see, e.g., Ohtake *et al.*, 2003; Kwiatkowski *et al.*, 2006; Abbas *et al.*, 2014). The quasi-LPV representation is not unique, and one might obtain models with different types of premise variables. In this work, only those models with known or measured premise variables are considered.

Assumption 3. (*Characteristics of f and h in* (1)) The state and the output functions, f and h, respectively, are assumed to be meromorphic. Further,

$$\text{rank}\left(\frac{\partial h(x,\theta,u)}{\partial x}\right) = p. \tag{12}$$

The assumptions on the model functions are a superset to the assumptions given by Audoly *et al.* (2001) as well as Xia and Moog (2003) to complete Definitions 1 as well as 2. In terms of the quasi-LPV model, this condition requires the following:

- the nonlinearities that appear in the matrices $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, and $D(\cdot)$ are meromorphic
- the rows of the matrix $C(\cdot)x + D(\cdot)u$ are locally independent, that is,

$$\text{rank}\left(\frac{\partial}{\partial x}\left[C(\cdot)x + D(\cdot)u\right]\right) = p.$$

Assumption 4. (*Initial conditions*) The initial state conditions are arbitrary.

Assumption 5. (*Inputs*) Higher-order derivatives of the inputs are defined and are known.

This assumption is required at least up to the order required for identifiability analysis so that it is possible to formulate $\Phi(\cdot)$ in (9).

Remark 2. (*Discrete-time case*) The discussion in this section has focused on continuous-time models, though it holds for the discrete-time case with the exchange of shift in discrete-time for derivatives in continuous time as commented by Anstett *et al.* (2006).

4. Parameter identifiability for continuous-time models

In this section, an overview of the proposed parity-space based identifiability analysis method is given. The method is illustrated with a set of examples and the results obtained are compared with that from DAISY.

4.1. Step-by-step description.

The procedure for the identifiability analysis proposed is inspired by the parity-space approach of Chow and Willsky (1984) as a means to eliminate the states of the system. The procedure could be summarized as follows.

Step 1: Formulation of algebraic equations. The LPV/quasi-LPV model in (10) is rewritten as

$$\begin{aligned} x^{(1)} &= A^{(0)}x^{(0)} + B^{(0)}u^{(0)},\\ y^{(0)} &= C^{(0)}x^{(0)} + D^{(0)}u^{(0)}. \end{aligned} \tag{13}$$

The superscript refers to the order of derivatives, i.e.,

$$A^{(j)} = \frac{\mathrm{d}^j\left(A(\rho(t),\theta)\right)}{\mathrm{d}t^j},$$

where the dependences on time, premise variables and parameters are omitted for the sake of simplicity. If the model has to be considered up to the second-order derivatives of the output, it is possible to rewrite the above as (with known and measured parts on the left-hand side),

$$\begin{bmatrix} y^{(0)}\\ y^{(1)}\\ y^{(2)}\\ \mathbf{0}\\ \mathbf{0}\end{bmatrix} + \begin{bmatrix} -D^{(0)} & \mathbf{0} & \mathbf{0}\\ -D^{(1)} & -D^{(0)} & \mathbf{0}\\ -D^{(2)} & -2D^{(1)} & -D^{(0)}\\ B^{(0)} & \mathbf{0} & \mathbf{0}\\ B^{(1)} & B^{(0)} & \mathbf{0}\end{bmatrix}\begin{bmatrix} u^{(0)}\\ u^{(1)}\\ u^{(2)}\end{bmatrix}$$

$$= \begin{bmatrix} C^{(0)} & \mathbf{0} & \mathbf{0}\\ C^{(1)} & C^{(0)} & \mathbf{0}\\ C^{(2)} & 2C^{(1)} & C^{(0)}\\ -A^{(0)} & I_n & \mathbf{0}\\ -A^{(1)} & -A^{(0)} & I_n\end{bmatrix}\begin{bmatrix} x^{(0)}\\ x^{(1)}\\ x^{(2)}\end{bmatrix}. \tag{14}$$

More generally, for up to an order w of the output derivative,

$$\begin{bmatrix}\mathbb{Y}\\ \mathbf{0}_{w\times n}\end{bmatrix} + \begin{bmatrix}-\mathbb{D}(\theta)\\ \mathbb{B}(\theta)\end{bmatrix}\mathbb{U} = \begin{bmatrix}\mathbb{C}(\theta)\\ \mathbb{A}(\theta)\end{bmatrix}\mathbb{X}, \tag{15}$$

with the left hand side containing known and measured terms. The presence of θ indicates the explicit appearance of the parameter in the matrices. Notice, however, that all the elements except $\mathbb{U}$ are indirectly dependent on the parameter θ. Here

$$\begin{aligned} \mathbb{Y} &= \begin{bmatrix}(y^{(0)})^T & (y^{(1)})^T & \cdots & (y^{(w)})^T\end{bmatrix}^T,\\ \mathbb{U} &= \begin{bmatrix}(u^{(0)})^T & (u^{(1)})^T & \cdots & (u^{(w)})^T\end{bmatrix}^T,\\ \mathbb{X} &= \begin{bmatrix}(x^{(0)})^T & (x^{(1)})^T & \cdots & (x^{(w)})^T\end{bmatrix}^T, \end{aligned} \tag{16}$$

and matrices $\mathbb{B}$, $\mathbb{D}$, $\mathbb{C}$ and $\mathbb{A}$ are given by Eqns. (17)–(20), respectively.

As is apparent, each of these matrices forms a Pascal's triangle with the increasing order of derivatives.

$$\mathbb{B}(\theta)=\begin{bmatrix} B^{(0)} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0}\\ B^{(1)} & B^{(0)} & \mathbf{0} & \cdots & \mathbf{0}\\ B^{(2)} & 2B^{(1)} & B^{(0)} & \cdots & \mathbf{0}\\ B^{(3)} & 3B^{(2)} & 3B^{(1)} & \cdots & \mathbf{0}\\ \vdots & \vdots & \vdots & & \vdots\\ B^{(w-1)} & \binom{w}{2}B^{(w-2)} & \binom{w}{3}B^{(w-3)} & \cdots & \mathbf{0}\end{bmatrix}, \tag{17}$$

$$\mathbb{D}(\theta)=\begin{bmatrix} D^{(0)} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0}\\ D^{(1)} & D^{(0)} & \mathbf{0} & \cdots & \mathbf{0}\\ D^{(2)} & 2D^{(1)} & D^{(0)} & \cdots & \mathbf{0}\\ D^{(3)} & 3D^{(2)} & 3D^{(1)} & \cdots & \mathbf{0}\\ \vdots & \vdots & \vdots & & \vdots\\ D^{(w)} & \binom{w+1}{2}D^{(w-1)} & \binom{w+1}{3}D^{(w-2)} & \cdots & D^{(0)}\end{bmatrix}, \tag{18}$$

$$\mathbb{C}(\theta)=\begin{bmatrix} C^{(0)} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0}\\ C^{(1)} & C^{(0)} & \mathbf{0} & \cdots & \mathbf{0}\\ C^{(2)} & 2C^{(1)} & C^{(0)} & \cdots & \mathbf{0}\\ C^{(3)} & 3C^{(2)} & 3C^{(1)} & \cdots & \mathbf{0}\\ \vdots & \vdots & \vdots & & \vdots\\ C^{(w)} & \binom{w+1}{2}C^{(w-1)} & \binom{w+1}{3}C^{(w-2)} & \cdots & C^{(0)}\end{bmatrix}, \tag{19}$$

$$\mathbb{A}(\theta)=\begin{bmatrix} -A^{(0)} & I_n & \mathbf{0} & \cdots & \mathbf{0}\\ -A^{(1)} & -A^{(0)} & I_n & \cdots & \mathbf{0}\\ -A^{(2)} & -2A^{(1)} & -A^{(0)} & \cdots & \mathbf{0}\\ -A^{(4)} & -3A^{(2)} & -3A^{(1)} & \cdots & \mathbf{0}\\ \vdots & \vdots & \vdots & & \vdots\\ -A^{(w-1)} & -\binom{w}{2}A^{(w-2)} & -\binom{w}{3}A^{(w-3)} & \cdots & I_n\end{bmatrix}. \tag{20}$$

This is useful when the algorithm is implemented. The representation in (15) is further simplified to indicate the dependence on the unknown parameter

$$\mathbb{Y}_0+\mathbb{G}(\theta)\mathbb{U}=\mathbb{O}(\theta)\mathbb{X}. \tag{21}$$

Notice that the matrix $\mathbb{O}(\theta)$ has a dimension of $(wp+(w-1)n)\times((w-1)n)$.

Step 2: Computation of the null space. Once a set of algebraic equations are formulated, the next step is to eliminate the state variables and their derivatives. This is achieved by computing the left null space of $\mathbb{O}(\theta)$, that is, to find a matrix $\Omega(\theta)$, such that,

$$\Omega^T(\theta)\mathbb{O}(\theta)=\mathbf{0}.$$

For a given $\mathbb{O}(\theta)$, the null-space $\Omega(\theta)$, if it exists, can be computed using symbolic computations (e.g., using the Symbolic Math Toolbox of MATLAB). The existence of the null-space is directly related to the output and the state matrices which populate $\mathbb{O}(\theta)$.

Step 3: Formulation of the input-output-parameter (I-O-P) equations. Once the null-space has been obtained, one can compute from (21),

$$\Omega^T(\theta)\left(\mathbb{Y}_0+\mathbb{G}(\theta)\mathbb{U}\right)=\mathbf{0}, \tag{22}$$

which can alternatively be represented as

$$\Psi(\theta,y,\cdots,y^{(w)},u,\cdots,u^{(w)})=\mathbf{0}, \tag{23}$$

where $\Psi(\cdot)$ is termed the input-output-parameter (I-O-P) equations to signify its dependence on inputs, outputs and parameters, and their derivatives.

Step 4: Identifiability evaluation. Once the I-O-P equations are obtained, identifiability is verified through one of the following approaches:

- Following the final step in the DAISY package of Bellu *et al.* (2007):
  - Extract the coefficients of $\Psi(\cdot)$ considering as polynomials in inputs, outputs, and their derivatives. Those coefficients that depend on the parameters θ form the exhaustive summary $\Pi(\theta)$.
  - Assign symbolic values to each of the parameters $\{\theta_1,\ldots,\theta_q\}$ and evaluate the

exhaustive summary to obtain $\Pi(\tilde{\theta})$. For large-scale problems, symbolic values can be replaced with numerical values.

  - Apply the Buchberger algorithm on $\Pi(\tilde{\theta})$ to obtain all the solutions. Depending on the number of solutions $\Pi(\tilde{\theta})$ admits, identifiability can be evaluated using Definition 1. In the case of the numerical approach, the last two steps are repeated several times. Since the results are *generic*, that is, valid for almost all numerical values except for a set of measure zero, this repetition would help to avoid reaching conclusions based on possibly choosing a numerical value of this set of measure zero.

- A local identifiability verification using Jacobians (inspired by Proposition 1):

  - Obtain the set of q identifiability equations $\Phi(\cdot)$. Note that the number of I-O-P equations is equal to the number of outputs p; thus, if $p < q$, one has to differentiate the I-O-P equations and set $\dot{\theta} = 0$ to obtain q equations (i.e., Φ).
  - Compute the Jacobian of $\Psi(.)$ with respect to the parameters θ, that is,

$$\operatorname{rank}\left(\frac{\partial \Psi}{\partial \theta}\right) = q.$$

  - If the rank is q, then local identifiability is verified.

4.2. Illustrative examples. In this section, several examples are given to show the steps involved in the parity-space approach. The results obtained from the parity-space approach are validated by comparing them with those produced by DAISY, STRIKE-GOLDD and genSSI software packages. Further, the intermediate step of the exhaustive summary is compared with that obtained using DAISY.

Example 1. This example is used to show all the steps of the proposed approach. Further, it also concerns a model which is not identifiable. Consider the second-order nonlinear model

$$\begin{aligned}\dot{x}_1 &= \theta_1 x_1 + \theta_2 x_2 u,\\ \dot{x}_2 &= \theta_3 x_1 - x_2,\\ y &= x_1 u,\end{aligned}$$

A quasi-LPV equivalent form with $x = [x_1 \quad x_2]^T$ and $\rho = u$ is

$$\dot{x} = \begin{bmatrix}\theta_1 & \theta_2 u\\ \theta_3 & -1\end{bmatrix} x, \qquad y = \begin{bmatrix}u & 0\end{bmatrix} x$$

corresponding to (10) with $B = 0$ and $D = 0$. Based on the specifications of the model structure required, the genSSI software cannot handle this example. Using the parity-space approach with output up to $\ddot{y}$, we obtain the following representation corresponding to that in (21),

$$\begin{bmatrix}y\\ \dot{y}\\ \ddot{y}\\ 0\\ 0\\ 0\\ 0\end{bmatrix} = \begin{bmatrix}u & 0 & 0 & 0 & 0 & 0\\ \dot{u} & 0 & u & 0 & 0 & 0\\ \ddot{u} & 0 & 2\dot{u} & 0 & u & 0\\ -\theta_1 & -\theta_2 u & 1 & 0 & 0 & 0\\ -\theta_2 & -\theta_3 & 0 & 1 & 0 & 0\\ 0 & -\theta_2\dot{u} & -\theta_1 & -\theta_2 u & 1 & 0\\ 0 & 0 & -\theta_2 & \theta_3 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \dot{x}_1\\ \dot{x}_2\\ \ddot{x}_1\\ \ddot{x}_2\end{bmatrix},$$

where the matrix $\mathbb{G}(\theta)$ is equal to 0. The left null space of the matrix $\mathbb{O}(\theta)$ is given by

$$\Omega^T(\theta) = \begin{bmatrix}\dfrac{\theta_2^2 u^3 + u\ddot{u} - 3\dot{u}^2 + \theta_1\theta_3 u^2 - 2\theta_1 u\dot{u} + \theta_3 u\dot{u}}{u^3}\\ \dfrac{3\dot{u} + \theta_1 u - \theta_3 u)}{u^2}\\ -\dfrac{1}{u}\\ \dfrac{-(\dot{u} - \theta_3 u)}{u}\\ \theta_2 u\\ 1\\ 0\end{bmatrix}^T$$

which leads to the I-O-P equation

$$\begin{aligned}\Psi(\cdot) = {}& \theta_1 u^2 y - 3\dot{u}^2 y - u^2\ddot{y} - u^2\dot{y} + \theta_1 u^2\dot{y}\\ &+ u\dot{u}y + 3u\dot{u}\dot{y} + u\ddot{u}y - 2\theta_1 u\dot{u}y + \theta_2\theta_3 u^3 y.\end{aligned}$$

The exhaustive summary is obtained by extracting the coefficients considering $\Psi(\cdot)$ as a polynomial in inputs, outputs and their derivatives. Considering only those coefficients that depend on various inputs, outputs, their derivatives and their product combinations, the following is obtained:

$$\Pi(\theta) = \{1 - 2\theta_1,\ \theta_1 - 1,\ \theta_1,\ \theta_2\theta_3\}. \tag{24}$$

To verify the number of solutions admitted by this exhaustive summary, a Gröbner basis analysis is performed. One strategy is to assign a symbolic value to for each of the parameters ($\tilde{\theta}_1 = a$, $\tilde{\theta}_2 = b$, $\tilde{\theta}_3 = c$) and evaluate the exhaustive summary to obtain the specific exhaustive summary,

$$\{2\theta_1 - 2a,\ \theta_1 - a,\ \theta_1 - a,\ \theta_2\theta_3 - bc\}.$$

For this simple example, it is easy to see that only θ_1 is identifiable as it admits a unique solution and the other two parameters θ_2 and θ_3 may have several solutions. Hence the model is not identifiable. To formally verify this, these polynomial equations were given as input to the Buchberger algorithm implemented in MuPAD CAS under MATLAB. The Gröbner basis for this set is

$$\{\theta_1 - a,\ \theta_2\theta_3 - bc\}$$

which, if the equations admit a unique solution should have returned $\theta_i = \tilde{\theta}_i$ for $i = 1, 2, 3$. However, it is not the case here and so the model is not identifiable (or only θ_1 is identifiable).

Comparison with DAISY. The normalized input-output equation obtained through the DAISY package is given by

$$\Gamma = \ddot{u}u^4y - 3\dot{u}^2u^3y + 3\dot{u}\dot{y}u^4 + \dot{u}u^4y(-2\theta_1 + 1) - \ddot{y}u^5 + \dot{y}u^5(\theta_1 - 1) + u^6y\theta_2\theta_3 + u^5y\theta_1,$$

which has the same set of exhaustive summary as given in (24). The DAISY package results also verify those inferred above. ♦

Example 2. In this example, the case of local identifiability is illustrated.

$$\dot{x} = \begin{bmatrix}\theta_1 & \theta_2 u\\ \theta_2 & -\theta_3\end{bmatrix} x,$$
$$y = \begin{bmatrix}u & 0\end{bmatrix} x.$$

Using the parity-space approach, the I-O-P equation obtained is

$$\Psi(\cdot) = \theta_1u^2\dot{y} - u^2\ddot{y} - 3\dot{u}^2y - \theta_3u^2\dot{y} + \theta_2^2u^3y + 3u\dot{u}\dot{y} + u\ddot{u}y - 2\theta_1u\dot{u}y + \theta_3u\dot{u}y + \theta_1\theta_3u^2y$$

which has the exhaustive summary of

$$\Pi(\theta) = \{\theta_3 - 2\theta_1,\ \theta_1 - \theta_3,\ \theta_1\theta_3,\ \theta_2^2\}.$$

The Gröbner basis for this summary with the symbolic assignment of $\tilde{\theta}_1 = a$, $\tilde{\theta}_2 = b$ and $\tilde{\theta}_3 = c$ was obtained as

$$\{\theta_1 - a,\ \theta_3 - c,\ \theta_2^2 - b^2\},$$

which indicates that while θ_1 and θ_3 are identifiable, θ_2 is only locally identifiable. The results and the exhaustive summary compare with those obtained from DAISY. ♦

Example 3. (*Air handling unit*) Consider a simple model of a heat exchanger used by Srinivasarengan *et al.* (2016). The model has been simplified by assuming that the inlet air temperature and the water temperature are known and constant. A quasi-LPV representation of that model is

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-\theta_1u_1 - \theta_2 & \theta_2\\ \theta_4 & -\theta_3u_2 - \theta_4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\theta_1 & 0\\ 0 & 5\theta_3\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix},$$
$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}.$$

The exhaustive summary obtained from the parity space approach for this model is[3]

$$\Pi(\theta) = \{3 - \theta_4 - \theta_2,\ \theta_2 + \theta_4 - 2,\ \theta_1,\ -\theta_1,\ 5\theta_2\theta_3,\ -\theta_3,\ \theta_3 - \theta_2\theta_3,\ \theta_1\theta_4,\ -\theta_1,\ 2\theta_1 - \theta_1\theta_4,\ \theta_1\theta_3,\ -\theta_1\theta_3\}.$$

For large-scale models, performing Gröbner analysis with symbolic values for the parameters could become intractable. In such cases, especially when practical applications are involved (where the range over which the parameters can take values is predictable), one can reliably use numerical values. By choosing arbitrary numerical values, $\tilde{\theta}_1 = 1$, $\tilde{\theta}_2 = 2$, $\tilde{\theta}_3 = 3$, $\tilde{\theta}_4 = 5$, the specific instance of the exhaustive summary was obtained and the Gröbner basis obtained is

$$\{\theta_2 - 2,\ \theta_4 - 5,\ \theta_3 - 3,\ \theta_1 - 1\}$$

indicating that the model is globally identifiable. These results comply with those obtained by DAISY both for the exhaustive summary and the eventual identifiability interpretation. Because a numerical value was used, the results are local in nature. Further, as suggested by Bellu *et al.* (2007), to obtain confidence on the obtained results, this analysis should be repeated for several sets of arbitrary numerical values. ♦

5. Parameter identifiability for discrete-time models

In practical scenarios, parameter estimation involves discrete time models. Hence it is vital to consider the identifiability of system models in discrete-time. In this section, a brief outline to extend the parity-space method to discrete time quasi-LPV models is given. It is to be noted that the effect of discretization on the identifiability is not treated here. Consider a discrete-time quasi-LPV model of the form

$$x_{k+1} = A(\rho_k, \theta)x_k + B(\rho_k, \theta),$$
$$y_k = C(\rho_k, \theta)u_k + D(\rho_k, \theta). \tag{25}$$

For this type of model, the procedure for parameter identifiability follows a similar path. The key difference is in the first step where the set of algebraic equations is obtained in a different way. For simplicity, in the following, A_k would be used in place of $A(\rho_k, \theta)$ and similarly for other matrices.

For the discrete-time case, the algebraic equations take a far simpler structure compared with that in the continuous-time case. The continuous-time algebraic

[3]The reader will notice that duplicate terms have not been removed here, but their presence does not change the obtained result.

equations in (15) are rewritten for the discrete-time case as

$$\begin{bmatrix}\mathbb{Y}_k\\ \mathbf{0}_{w\times n}\end{bmatrix}+\begin{bmatrix}-\mathbb{D}_k\\ \mathbb{B}_k\end{bmatrix}\mathbb{U}_k=\begin{bmatrix}\mathbb{C}_k\\ \mathbb{A}_k\end{bmatrix}\mathbb{X}_k, \tag{26}$$

where

$$\begin{aligned}\mathbb{Y}_k &= \begin{bmatrix}y_k^T & y_{k+1}^T & \cdots & y_{k+w}^T\end{bmatrix}^T,\\ \mathbb{U}_k &= \begin{bmatrix}u_k^T & u_{k+1}^T & \cdots & u_{k+w}^T\end{bmatrix}^T,\\ \mathbb{X}_k &= \begin{bmatrix}x_k^T & x_{k+1}^T & \cdots & x_{k+w}^T\end{bmatrix}^T,\end{aligned} \tag{27}$$

and

$$\mathbb{B}_k=\begin{bmatrix}B_k & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & B_{k+1} & \cdots & \mathbf{0} & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \mathbf{0} & \mathbf{0} & \cdots & B_{k+w-1} & \mathbf{0}\end{bmatrix}, \tag{28}$$

$$\mathbb{D}_k=\begin{bmatrix}D_k & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{0} & D_{k+1} & \cdots & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{0} & \mathbf{0} & \cdots & D_{k+w}\end{bmatrix}, \tag{29}$$

$$\mathbb{A}_k=\begin{bmatrix}-A_k & I_n & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & -A_{k+1} & I_n & \cdots & \mathbf{0} & \mathbf{0}\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & A_{k+w-1} & I_n\end{bmatrix}, \tag{30}$$

$$\mathbb{C}_k=\begin{bmatrix}C_k & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & C_{k+1} & \cdots & \mathbf{0} & \mathbf{0}\\ \vdots & \vdots & \ddots & & \vdots\\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & C_{k+w}\end{bmatrix}. \tag{31}$$

The other three steps in this case follow those of the continuous the time approach with the derivatives replaced with time shifts (Steps 2 to 4 in Section 4).

All the software packages in the literature are available only for continuous-time models. Hence, comparison of results with any the existing packages is not feasible. Hence, the examples are picked from the existing literature on discrete-time identifiability and the results compared with those obtained from methods proposed in this work.

Example 4. Consider the following example of the Henon map adapted from Anstett *et al.* (2006):

$$\begin{aligned}x_{1,k+1} &= \theta_1 x_{1,k}^2+\theta_2 x_{2,k}+u_k,\\ x_{2,k+1} &= \theta_3 x_{1,k}+\theta_4 u_k,\\ y_k &= x_{1,k}.\end{aligned}$$

The I-O-P equation is

$$\begin{aligned}\Psi(\cdot) = &-\theta_2\theta_3 y_k^2 - u_{k+1} + y_{k+2} - u_k(\theta_2\theta_4+\theta_1 y_{k+1})\\ &+\theta_1 y_{k+1}(u_k - y_{k+1}).\end{aligned}$$

The exhaustive summary obtained using the parity-space approach is

$$\Pi(\theta)=\{\theta_1,\ \theta_2\theta_4,\ \theta_2\theta_3\}.$$

The identifiability results comply with those from Anstett *et al.* (2006) in that only the parameter θ_1 is identifiable. ♦

Example 5. This is also an example of the Burgers map from Anstett *et al.* (2006)

$$\begin{aligned}x_{1,k+1} &= (1+\theta_1)x_{1,k}+x_{1,k}x_{2,k}+u_k,\\ x_{2,k+1} &= (1-\theta_2)x_{2,k}-x_{1,k}^2 u_k,\\ y_k &= x_{1,k}.\end{aligned}$$

The I-O-P equation obtained for this case is

$$\begin{aligned}\Psi(\cdot) = y_{k+2} &- u_{k+1} + (y_{k+1}-\theta_2 y_{k+1})(\theta_1+1)\\ &- u_k(1-y_{k+1}y_k^2+\theta_1)\\ &+\frac{(u_k-y_{k+1})(y_k+y_{k+1}+\theta_1 y_k-\theta_2 y_{k+1})}{y_k}.\end{aligned}$$

The exhaustive summary obtained for this example is

$$\Pi(\theta)=\{\theta_1,\theta_2,\theta_1-\theta_2-\theta_1\theta_2\}.$$

It is easy to see that the model is identifiable and agrees with the results of Anstett *et al.* (2006). ♦

6. Towards a systematic formulation

In this section, we discuss how to realize a systematic implementation of the proposed algorithm. This includes some algorithmic steps for sample scenarios. The implementation and realization as a toolbox is planned as a topic for future work. These formulations are for both continuous-time and discrete-time models with appropriate modifications, though the discussion focuses on continuous-time models.

6.1. Choice of the number of derivatives. The discussion in the preceding sections did not explicitly consider w, the number of derivatives (or shifts in discrete time) for which the null-space $\Omega^T(\theta)$ exists and hence the I-O-P equations and the exhaustive summary that follow. This corresponds to the observability index of the system model. A detailed discussion on the observability index of a nonlinear system was lead by Nijmeijer and Van der Schaft (1990) though a brief idea is given below. Consider a SISO system of the form (1); the observability index of this model is defined as $w > 0$, and in the neighborhood of x_0 we have

$$\text{rank}\left[L_f^{w-1}h, L_f^{w-2}h, \cdots, L_f^0 h\right] = w$$
$$\text{and rank}\left[L_f^{w}h, L_f^{w-1}h, \cdots, L_f^0 h\right] = w$$

where, $L_f h$ corresponds to the Lie derivative of h over f, that is,

$$L_f h \triangleq \frac{\partial h(x,u,\theta)}{\partial x} f(x,u,\theta)$$

and $L_f^i h$ refers to the i-th successive application of the Lie derivative. Essentially this means that, locally, the dimension of the space spanned by the model does not grow after $w - 1$ derivatives. For MIMO systems, the observability index is defined for each output. A discrete-time version of this issue is briefly discussed by Anstett (2006, Chapter 5).

This means that for a SISO system, using w derivatives would guarantee that $\Omega^T(\theta)$ exists and hence would provide the I-O-P equation corresponding to the output. The next question is to know whether the observability index has been connected to the system dimensions theoretically. That is, is it possible to obtain the index without verifying the rank condition. For linear systems, this is less than or equal to the number of states (easy to verify using the Cayley–Hamilton theorem). For nonlinear systems, for the following two classes:

- models of the form (1) where the functions are rational,
- models in a control-affine form (3) with analytical functions,

it has been shown that, locally, the observability index has an upper bound, equal to the number n of states of the system. This means that n derivatives of outputs are sufficient to guarantee that the null-space $\Omega^T(\theta)$ exists. For a more detailed discussion, see (Anguelova, 2007). This is an upper bound because: (i) the model may not be minimal and has unobservable spaces and hence the observability index is less than n; (ii) for MIMO systems, each output would have different observability indices and hence the total number of derivatives required to span the entire observability space can be less than n.

Consequently, for single-output systems, n derivatives of the output would guarantee that the null-space $\Omega^T(\theta)$ exists. Hence $w = n$ for SISO systems. For MIMO systems, this is further complicated. Each output's observability index has an upper bound of n, but is more likely to be lower than n. A systematic approach to handle this scenario is discussed later in this section.

6.2. Algorithm for parameter identifiability. The algorithm used for the analysis of the identifiability of the illustrative examples discussed in the previous section is summarized as Algorithm 1. The implementation was done in the MATLAB computing environment with the use of the Symbolic Math Toolbox and the MuPAD computer algebra system (CAS). Once the set of I-O-P equations is obtained and the exhaustive summary extracted, the Gröbner basis evaluation is performed through MuPAD CAS scripts. Hence, at this moment, there are components of the algorithms that require manual intervention. The first step in the algorithm chooses the upper bound on the number of derivatives to be n. This is feasible for the simple models chosen for illustrative examples, but not necessarily for more complicated situations. Further, for MIMO systems, since the observability index depends on individual outputs, a step-by-step analysis starting from 0 derivatives is considered. Further optimization is envisaged in this respect.

Algorithm 1. Analysis of parameter identifiability.

1: Choose an upper bound on the number of derivatives $w = n$, and matrices $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}$.
2: Evaluate the matrices (17)–(20) and their higher order (element-wise) derivatives.
3: **for** $w = 0$ to n **do**
4: Formulate $\mathbb{Y}_0 + \mathbb{G}(\theta)\mathbb{U} = \mathbb{O}(\theta)\mathbb{X}$ as in (21).
5: Compute $\Omega^T(\theta)$, the left null-space of $\mathbb{O}(\theta)$ using symbolic computation.
6: Obtain the I-O-P equations $\Psi(\cdot)$ and extract the coefficients to obtain the exhaustive summary $\Pi(\theta)$.
7: **for** $j = 1$ to number of iterations **do**
8: Choose random values for the parameters $\theta_1, \ldots, \theta_q$
9: Evaluate the Gröbner basis and verify the number of solutions admitted by the exhaustive summary
10: **end for**
11: **if** global or local identifiability is satisfied **then**
12: END
13: **end if**
14: **end for**

As mentioned in the end of Step 4 of the method presented in Section 4.1, the assignment of numerical values (rather than symbolic ones) to the parameters needs to be repeated for guaranteeing the obtained result. This iteration loop begins from Line 7 and ends in Line 10.

Analyzing outputs independently. One of the assumptions that is part of the problem specification (and adopted from Xia and Moog (2003)) is

$$\text{rank}\left(\frac{\partial h(x,u,\theta)}{\partial x}\right) = p.$$

Algorithm 2. Checking local structural identifiability.

```
Choose the maximum value for observability index for each output (w_1, ..., w_p) is n.
Evaluate the matrices (28)–(31) and their higher-order (element-wise) derivatives.
for i = 1 to p (for each output) do
   for w = 0 to n do
      Formulate 𝕐_0 + 𝔾(θ)𝕌 = 𝕆(θ)𝕏 as in (21).
      Compute Ω^T(θ), the left null space of 𝕆(θ) using symbolic computation.
      Obtain the I-O-P equation ψ_i(·).
      if one I-O-P equation is obtained then
         End of search for output i
      end if
   end for
   Add the I-O-P for output to the overall I-O-P, Ψ(·) = {Ψ(·), ψ_i(·)}
end for
Evaluate ∂Ψ/∂θ and compute the rank.
if rank ∂Ψ/∂θ = q then
   Model is locally structurally identifiable
else
   Model is not identifiable
end if
```

That is, the outputs are at least locally independent. This provides an opening to develop local structural identifiability analysis methods that can provide the following advantages:

- Obtain the local identifiability results through Jacobian analysis instead of the Buchberger algorithm to obtain the Gröbner basis.

- As suggested by Bellu *et al.* (2007), there are p normalized input-output equations. By considering one input at a time, the stopping criterion for the algorithm could be set as one I-O-P equation per output by considering the system with one output at a time.

Note that this approach makes sense only if $p \geq q$, as discussed in the paragraph following Proposition 1. Otherwise, one has to differentiate the I-O-P equations to obtain an appropriate set of identifiability equations $\Phi(\cdot)$, which involves combinatorial possibilities and is beyond the scope of the current attempt. This limited case is realized as Algorithm 2.

7. Concluding remarks and perspectives

In this paper, we proposed a procedure for verifying the identifiability of LPV and quasi-LPV models in continuous-time and discrete-time cases. The procedure exploits the parity-space approach to eliminate the states and uses the residual set of input-output-parameter equations to verify the model identifiability. Through several examples the procedures were illustrated and compared with the results obtained from the existing literature. With these preliminaries, the algorithm looks to be a useful candidate in the domain of LPV model analysis. Given the nature of this paper, there are several paths for improvements to realize a robust identifiability procedure.

Extending results to a newer class of system models. It was noted that the parity-space approach would work well for polynomial parametrization as well. This should be formally extended. While the initial conditions were considered arbitrary in this paper, there are cases where such an assumption can be detrimental (see Example 4 of Villaverde and Banga (2017) and Example 2 of Denis-Vidal *et al.* (1999)). These cases need to be carefully handled in the implementation to cover a larger spectrum of models and initial conditions. Further, the case of known initial conditions and partially known initial conditions will also be handled in this extension. The results were restricted to measured or known premise variables. This could be extended to LPV models which have unmeasured or estimated premise variables.

Systematic implementation. A systematic implementation of the procedure is an immediate topic for future work. The realization could be in the form of a toolbox in MATLAB similar to those reported by Chis *et al.* (2011) or Villaverde *et al.* (2016). This toolbox would provide a detailed strategy for the model input, options for evaluating local/global identifiability results, the final display of the results and other relevant information.

Numerical approach. The STRIKE-GOLDD toolbox (Villaverde *et al.*, 2016) evaluates identifiability either numerically or symbolically. In the numerical approach, a random set of initial conditions are chosen for the states and random values are associated with inputs and their derivatives (these random values are chosen as prime numbers to avoid undesirable cancellations). This significantly reduces the computational effort required to compute the Jacobian. It is to be noted that the numerical approach here is not completely numerical. It still requires computing symbolically the Lie derivatives to set up the so-called "observability-identifiability matrix" in their paper. A similar inclusion of numerical methods can reduce the symbolic computations. In Algorithm. 2, this would change the initial assignment step and the Jacobian computation step. The null-space computation steps could also be replaced by exploiting the works on polynomial null-space computation (e.g., Anaya and Henrion, 2009; Khare *et al.*, 2010).

Computational complexity and efficiency. The computational complexity analysis of the proposed algorithm would be a topic of future interest once the implementation is realized completely in a computational environment like MATLAB. This would include analysing the relative efficiencies of deploying the Buchberger algorithm for the Gröbner basis computations versus the Jacobian evaluation for local structural identifiability.

Another related interest is in the computational comparison with other methods. The DAISY software package envisioned to reduce the computational complexity of the approach by Ljung and Glad (1994) by the choice of a differential ring that does not contain θ. This works well for biological systems with a large number of parameters and a small number of states. However, in engineering systems one often encounters a model with a large number of states and relatively few parameters. An application such as DAISY suffers from the same type of computational overhead as Ljung and Glad (1994) had for biological systems. It is to be analyzed whether the parity-space approach can bring in any specific advantages. Similarly, to analyze the same type of models for same characteristics, the parity-space approach should also be compared with genSSI and STRIKE-GOLDD toolboxes. Further, as in the recent work by Joubert (2020), a numerical sensitivity analysis of the work would be imperative to better understand the numerical properties of the algorithm.

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References

Abbas, H.S., Tóth, R., Petreczky, M., Meskin, N. and Mohammadpour, J. (2014). Embedding of nonlinear systems in a linear parameter-varying representation, *IFAC Proceedings Volumes* **47**(3): 6907–6913.

Alkhoury, Z., Petreczky, M. and Mercère, G. (2017). Identifiability of affine linear parameter-varying models, *Automatica* **80**: 62–74.

Anaya, J.Z. and Henrion, D. (2009). An improved Toeplitz algorithm for polynomial matrix null-space computation, *Applied Mathematics and Computation* **207**(1): 256–272.

Anguelova, M. (2007). *Observability and Identifiability of Nonlinear Systems with Applications in Biology*, PhD thesis, Chalmers University of Technology, Gothenburg.

Anstett, F. (2006). *Les systèmes dynamiques chaotiques pour le chiffrement: Synthèse et cryptanalyse*, PhD thesis, Université Henri Poincaré-Nancy I, Nancy.

Anstett, F., Bloch, G., Millérioux, G. and Denis-Vidal, L. (2008). Identifiability of discrete-time nonlinear systems: The local state isomorphism approach, *Automatica* **44**(11): 2884–2889.

Anstett, F., Millérioux, G. and Bloch, G. (2006). Chaotic cryptosystems: Cryptanalysis and identifiability, *IEEE Transactions on Circuits and Systems I: Regular Papers* **53**(12): 2673–2680.

Audoly, S., Bellu, G., D'Angiò, L., Saccomani, M.P. and Cobelli, C. (2001). Global identifiability of nonlinear models of biological systems, *IEEE Transactions on Biomedical Engineering* **48**(1): 55–65.

Balsa-Canto, E., Alonso, A.A. and Banga, J.R. (2010). An iterative identification procedure for dynamic modeling of biochemical networks, *BMC Systems Biology* **4**: 11.

Beelen, H. and Donkers, T. (2017). Joint state and parameter estimation for discrete-time polytopic linear parameter-varying systems, *IFAC-PapersOnLine* **50**(1): 9778–9783.

Bellman, R. and Aström, K. J. (1970). On structural identifiability, *Mathematical Biosciences* **7**(3): 329–339.

Bellu, G., Saccomani, M.P., Audoly, S. and D'Angiò, L. (2007). Daisy: A new software tool to test global identifiability of biological and physiological systems, *Computer Methods and Programs in Biomedicine* **88**(1): 52–61.

Buchberger, B. (2006). Bruno Buchberger's PhD thesis 1965: An algorithm for finding the basis elements of the residue class ring of a zero dimensional polynomial ideal, *Journal of Symbolic Computation* **41**(3–4): 475–511.

Chis, O.-T., Banga, J.R. and Balsa-Canto, E. (2011). Structural identifiability of systems biology models: A critical comparison of methods, *PLOS ONE* **6**(11): e27755.

Chow, E. and Willsky, A. (1984). Analytical redundancy and the design of robust failure detection systems, *IEEE Transactions on Automatic Control* **29**(7): 603–614.

Coll, C. and Sánchez, E. (2019). Parameter identification and estimation for stage-structured population models, *International Journal of Applied Mathematics and Computer Science* **29**(2): 327–336, DOI:10.2478/amcs-2019-0024.

Dankers, A., Tóth, R., Heuberger, P. S., Bombois, X. and Van den Hof, P.M. (2011). Informative data and identifiability in LPV-ARX prediction-error identification, *50th IEEE Conference on Decision and Control and European Control Conference, CDC-ECC 2011, Orlando, USA*, pp. 799–804.

Denis-Vidal, L. and Joly-Blanchard, G. (1996). Identifiability of some nonlinear kinetics, *3rd Workshop on Modelling of Chemical Reaction Systems*, Heidelberg, Germany, pp. 1–8.

Denis-Vidal, L., Joly-Blanchard, G. and Noiret, C. (1999). Some results and applications about identifiability of non-linear systems, *European Control Conference, ECC 1999, Karlsruhe, Germany*, pp. 1232–1237.

Glover, K. and Willems, J. (1974). Parametrizations of linear dynamical systems: Canonical forms and identifiability, *IEEE Transactions on Automatic Control* **19**(6): 640–646.

Joly-Blanchard, G. and Denis-Vidal, L. (1998). Some remarks about an identifiability result of nonlinear systems, *Automatica* **34**(9): 1151–1152.

Joubert, D. (2020). *Structural Identifiability of Large Systems Biology Models*, PhD thesis, Wageningen University, Wageningen.

Khare, S.R., Pillai, H.K. and Belur, M.N. (2010). Algorithm to compute minimal nullspace basis of a polynomial matrix, *19th International Symposium on Mathematical Theory of Networks and Systems, MTNS 2010, Budapest, Hungary*, pp. 219–224.

Kwiatkowski, A., Boll, M.-T. and Werner, H. (2006). Automated generation and assessment of affine LPV models, *45th IEEE Conference on Decision and Control, CDC 2006, San Diego, USA,* pp. 6690–6695.

Lee, L.H. and Poolla, K. (1997). Identifiability issues for parameter-varying and multidimensional linear systems, *ASME 1997 Design Engineering Technical Conferences, Sacramento, USA*.

Ljung, L. and Glad, T. (1994). On global identifiability for arbitrary model parametrizations, *Automatica* **30**(2): 265–276.

Nõmm, S. and Moog, C. (2004). Identifiability of discrete-time nonlinear systems, *IFAC Proceedings Volumes* **37**(13): 333–338.

Němcová, J. (2010). Structural identifiability of polynomial and rational systems, *Mathematical Biosciences* **223**(2): 83–96.

Nijmeijer, H. and Van der Schaft, A. (1990). *Nonlinear Dynamical Control Systems*, Springer, New York.

Ohtake, H., Tanaka, K. and Wang, H.O. (2003). Fuzzy modeling via sector nonlinearity concept, *Integrated Computer-Aided Engineering* **10**(4): 333–341.

Ollivier, F. (1990). *Le problème de l'identifiabilité structurelle globale: Approche théorique, méthodes effectives et bornes de complexité*, PhD thesis, Ecole Polytechnique, Palaiseau.

Peeters, R.L. and Hanzon, B. (2005). Identifiability of homogeneous systems using the state isomorphism approach, *Automatica* **41**(3): 513–529.

Petreczky, M. and Mercère, G. (2012). Affine LPV systems: Realization theory, input-output equations and relationship with linear switched systems, *51st Annual Conference on Decision and Control, CDC 2012, Maui, USA,* pp. 4511–4516.

Pohjanpalo, H. (1978). System identifiability based on the power series expansion of the solution, *Mathematical Biosciences* **41**(1–2): 21–33.

Saccomani, M.P. (2011). An effective automatic procedure for testing parameter identifiability of HIV/AIDS models, *Bulletin of Mathematical Biology* **73**(8): 1734–1753.

Saccomani, M.P., Audoly, S., Bellu, G., Cobelli, C. (1997). Global identifiability of nonlinear model parameters, *IFAC Proceedings Volumes* **30**(11): 233–238.

Srinivasarengan, K., Ragot, J., Maquin, D. and Aubrun, C. (2016). Takagi–Sugeno model based nonlinear parameter estimation in air handling units, *4th IFAC International Conference on Intelligent Control and Automation Sciences, ICONS 2016, Reims, France*, pp. 188–193.

Tunali, E. T. and Tarn, T.-J. (1987). New results for identifiability of nonlinear systems, *IEEE Transactions on Automatic Control* **32**(2): 146–154.

Vajda, S. and Rabitz, H. (1989). State isomorphism approach to global identifiability of nonlinear systems, *IEEE Transactions on Automatic Control* **34**(2): 220–223.

Verdière, N., Denis-Vidal, L., Joly-Blanchard, G. and Domurado, D. (2005). Identifiability and estimation of pharmacokinetic parameters for the ligands of the macrophage mannose receptor, *International Journal of Applied Mathematics and Computer Science* **15**(4): 517–526.

Villaverde, A.F. and Banga, J.R. (2017). Structural properties of dynamic systems biology models: Identifiability, reachability, and initial conditions, *Processes* **5**(2): 29.

Villaverde, A.F., Barreiro, A. and Papachristodoulou, A. (2016). Structural identifiability of dynamic systems biology models, *PLOS Computational Biology* **12**(10): e1005153.

Walter, E. and Lecourtier, Y. (1982). Global approaches to identifiability testing for linear and nonlinear state space models, *Mathematics and Computers in Simulation* **24**(6): 472–482.

Xia, X. and Moog, C.H. (2003). Identifiability of nonlinear systems with application to HIV/AIDS models, *IEEE Transactions on Automatic Control* **48**(2): 330–336.

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