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OPTIMAL CONTROL PROBLEMS WITHOUT TERMINAL CONSTRAINTS: THE TURNPIKE PROPERTY WITH INTERIOR DECAY

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We show a turnpike result for problems of optimal control with possibly nonlinear systems as well as pointwise-in-time state and control constraints. The objective functional is of integral type and contains a tracking term which penalizes the distance to a desired steady state. In the optimal control problem, only the initial state is prescribed. We assume that a cheap control condition holds that yields a bound for the optimal value of our optimal control problem in terms of the initial data. We show that the solutions to the optimal control problems on the time intervals $[0, T]$ have a turnpike structure in the following sense: For large T the contribution to the objective functional that comes from the subinterval $[T/2, T]$, i.e., from the second half of the time interval $[0, T]$, is at most of the order $1/T$. More generally, the result holds for subintervals of the form $[r\,T, T]$, where $r \in (0, 1/2)$ is a real number. Using this result inductively implies that the decay of the integral on such a subinterval in the objective function is faster than the reciprocal value of a power series in T with positive coefficients. Accordingly, the contribution to the objective value from the final part of the time interval decays rapidly with a growing time horizon. At the end of the paper we present examples for optimal control problems where our results are applicable.

Keywords: optimal control, turnpike property, system with hyperbolic PDEs, interior decay.

1. Introduction

The turnpike property was initially investigated by P.A. Samuelson in mathematical economics in 1949 (Dorfman *et al*., 1958). Ever since the turnpike phenomenon for optimization problems has been studied in a variety of frameworks (see, e.g., Zaslavski, 2006; 2014). For systems with ordinary differential equations, the exponential turnpike property was examined by Trélat and Zuazua (2015). For optimal control problems with partial differential equations, see the works of Grüne *et al*. (2020), Porretta and Zuazua (2013), Trélat and Zhang (2018) or Gugat and Lazar (2023). Optimal boundary control problems are investigated, for example, by Gugat and Hante (2019). Manifold turnpikes are studied by Faulwasser *et al*. (2022). The connection of the turnpike property to greedy optimal control for elliptic problem is pointed out by Hernández-Santamaría *et al*. (2019). The turnpike phenomenon for optimal control problems with time-discrete systems is studied by Damm *et al*. (2014) or Grüne and Guglielmi (2018). In the work of Gugat (2021) the turnpike property with interior decay is introduced. It describes the situation where for sufficiently large time horizons the distance between the dynamic optimal control/state pair and the corresponding static solution in the interior of the time interval is very small with respect to an appropriate integral norm. In that same work problems with both given initial states and prescribed terminal states were studied under an exact controllability assumption. In contrast to this, in the present paper we study problems without terminal conditions. Hence we also do not need an exact controllability assumption.

We study systems that are governed by an abstract

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nonlinear semigroup. The optimal control problem concerns a process on a finite time interval $[0, T]$. In the optimal control problems, the initial state is given but no terminal condition is prescribed. The objective function is of integral type and penalizes the distance to the desired steady state on the time interval $[0,\, T]$. The solution of the corresponding static problem is the desired steady state.

In addition to the strict dissipativity, our main assumption is a cheap control condition, which requires that the optimal control cost can be bounded above in terms of the distance between the given initial state and the desired steady state.

We show that the optimal controls have a turnpike structure with interior decay in the following sense: The contribution to the objective functional that is generated from the subinterval $[T/2,\, T]$ of $[0,\, T]$ decays with the order $1/T$. Our result allows for pointwise-in-time control constraints and pointwise-in-time state constraints. The control constraints in our setting include switching constraints that only admit a finite set of control values. The result can be generalized to subintervals of the form $[r\,T, T]$, where $r \in (0, 1/2)$ is a real number.

Our turnpike results are helpful for computations, since they imply that for large time horizons T, approximations of the optimal control–state pairs should be close to the optimal steady state after the time $T/2$.

This paper has the following structure. In Section 1.1 the dynamic optimal control problem is defined. Section 2 contains the definition of the turnpike property with interior decay. In Section 3 the turnpike results are stated. The first theorem contains a sufficient condition for the turnpike property with interior decay. The second theorem gives a decay estimate that under additional assumptions implies that the interior decay is faster than any polynomial rate, see Corollary 1. For the proofs, several auxiliary results are used. At the end of the paper, our results are illustrated by some examples.

1.1. Definition of the optimal control problem. Let $\mathcal{U}$ and $\mathcal{Y}$ denote Banach spaces with the norms $\|\cdot\|_{\mathcal{U}}$ and $\|\cdot\|_{\mathcal{Y}}$, respectively. Let an initial state $y_0 \in \mathcal{Y}$ be given. The state y generated by the control u is denoted as

$$y = \Phi(a,\, y_0, u), \tag{1}$$

where for $a \in [0, \infty)$ and $t > a$ the mapping Φ is continuous from $\{a\} \times \mathcal{Y} \times L^2((a,\, t), \mathcal{U})$ to $C([a,\, t],\, \mathcal{Y})$ with $\Phi(a,\, y_0, u)(a) = y_0$. We assume that for any subinterval $(a_1,\, t_1)$ of $(a,\, t)$ we have

$$\Phi(a,\, y_0, u)|_{(a_1,\, t_1)} = \Phi(a_1,\, \Phi(a,\, y_0, u)(a_1), u|_{(a_1,\, t_1)}). \tag{2}$$

These properties are satisfied, for example, by strongly continuous semigroups (see Tucsnak and Weiss, 2009).

Let $u^{(\sigma)} \in \mathcal{U}$ denote a static control and let $y^{(\sigma)} \in \mathcal{Y}$ denote the corresponding static state, which is considered as a constant function in $C([a,\, t],\, \mathcal{Y})$. The static control $u^{(\sigma)}$ is considered as a constant control in $L^2((a,\, t), \mathcal{U})$. It satisfies

$$y^{(\sigma)}(s) = \Phi(a,\, y^{(\sigma)}, u^{(\sigma)})(s), \quad s \in [a,\, t].$$

Let $a,\, b \in \mathbb{R}$ with $a < b$ be given. Let $f_0 : \mathbb{R} \times \mathcal{Y} \times \mathcal{U} \to [0,\, \infty)$ be continuous. By the nonnegativity of f_0, we have for states $y \in L^2((a, b),\, \mathcal{Y})$ and controls $u \in L^2((a, b), \mathcal{U})$ and any subinterval $(a_1,\, b_1)$ of $(a,\, b)$ the inequality

$$\begin{aligned} 0 \le \int_{a_1}^{b_1} & f_0(t,\, y|_{(a_1,\, b_1)}(t),\, u|_{(a_1,\, b_1)}(t))\,\mathrm{d}t \\ & \le \int_a^b f_0(t,\, y(t),\, u(t))\,\mathrm{d}t. \end{aligned} \tag{3}$$

For states $y \in L^2((a, b),\, \mathcal{Y})$ and controls in $u \in L^2((a, b), \mathcal{U})$ define

$$J_{(a,\, b)}(u,\, y) = \int_a^b f_0(t,\, y(t),\, u(t))\,\mathrm{d}t \tag{4}$$

where y is the system state that is generated by the control function u. This type of objective functions has also been considered by Trélat *et al.* (2018).

For $y_0 \in \mathcal{Y}$, we consider a dynamic optimal control problem with the initial condition

$$y(a) = y_0. \tag{5}$$

Let F denote a closed subset of $\mathcal{Y}$ such that $y^{(\sigma)} \in F$. We define the state constraint

$$y(t) \in F, \quad t \in [a,\, b] \tag{6}$$

and the control constraint

$$u(t) \in U \quad \text{for } t \in (a,\, b) \text{ almost everywhere}, \tag{7}$$

where U is a nonempty closed subset of $\mathcal{U}$ such that $u^{(\sigma)} \in U$.

For $a,\, b \in \mathbb{R}$ with $b > a$ and an initial state $y_0 \in F$ define the optimization problem

$$\begin{aligned} P(a,\, b,\, y_0) : \min_u J_{(a,\, b)}(u,\, y) \\ \text{subject to (1), (5), (6) and (7).} \end{aligned} \tag{8}$$

Note that the existence of solutions of $P(a,\, b,\, y_0)$ can be shown under suitable assumptions using the direct method of the calculus of variations (see, e.g., Dacorogna, 2008); for example, if $\mathcal{U}$ and $\mathcal{Y}$ are reflexive Banach spaces, f_0 is time-independent, f_0, F and U are convex and f_0 satisfies a growth condition, e.g., $f_0(y,\, u) \ge \|u\|_{\mathcal{U}}^2$.

Let $\hat{y}(a,b,y_0)$ denote an optimal state and $\hat{u}(a,b,y_0)$ denote an optimal control for $P(a,b,y_0)$ in the sense that the constraints (1), (5), (6) and (7) are satisfied and

$$J_{(a,\,b)}(\hat{u}(a,b,y_0),\, \hat{y}(a,b,y_0)) = \hat{v}(a,\,b,\,y_0) \qquad (9)$$

where $\hat{v}(a,\,b,\,y_0)$ denotes the optimal value for $P(a,b,y_0)$.

For any subinterval $(a_1,\,t_1)$ of $(a,\,b)$, the assumption (2) and relation (3) imply

$$\hat{v}(a_1,\,t_1,\,\hat{y}(a,b,y_0)(a_1)) \le \hat{v}(a,b,\,y_0). \qquad (10)$$

In the subsequent analysis we need the following lemma. It is an adaptation of Lemma 1.1 of Gugat (2021) to the case without the terminal constraint.

Lemma 1. *For any subinterval* $(a_1,\,b)$ *of* $(a,\,b)$*, we have*

$$\begin{aligned} J_{(a_1,\,b)}(\hat{u}(a,\,b,y_0)|_{(a_1,\,b)},\, \hat{y}(a,\,b,y_0)|_{(a_1,\,b)}) \\ = \hat{v}(a_1,\,b,\,\hat{y}(a,\,b,y_0)(a_1)) \end{aligned} \qquad (11)$$

that is $(\hat{u}(a,\,b,y_0),\ \hat{y}(a,\,b,y_0))|_{(a_1,\,b)}$ *is an optimal control–state pair for the optimal control problem*

$$P(a_1,\,b,\,\hat{y}(a,\,b,y_0)(a_1)).$$

Proof. Since the proof is analogous to the proof of Lemma 1.1 in the work of Gugat (2021), we omit it here. ■

2. Measure turnpike property and the turnpike property with interior decay

Measure turnpike properties and an integral turnpike property are considered, e.g., by Trélat *et al.* (2018). An integral turnpike property for a boundary control problem with a hyperbolic system is shown by Gugat and Hante (2019). The measure turnpike property is defined as follows.

Definition 1. Problem $P(a,\,b,\,y_0)$ has the *measure turnpike property* at $(y^{(\sigma)},\,u^{(\sigma)}) \in F \times U$ if for all $\varepsilon > 0$ there is a $\Lambda(\varepsilon) > 0$ such that for all $b > a$ we have

$$\begin{aligned} \mu\,\{t \in [a,b] : \|\hat{y}_0(a,\,b,y_0,)(t) - y^{(\sigma)}\|_{\mathcal{Y}} \\ + \|\hat{u}_0(a,\,b,y_0)(t) - u^{(\sigma)}\|_{\mathcal{U}} > \varepsilon\,\} \le \Lambda(\varepsilon), \end{aligned}$$

where μ denotes the Lebesgue measure.

In the work of Trélat *et al.* (2018) the measure turnpike property is shown under the following strict dissipativity assumption.

Definition 2. Problem $P(a,\,b,\,y_0)$ is *strictly dissipative* at $(y^{(\sigma)},\,u^{(\sigma)}) \in F \times U$ if f_0 is time-independent and for all $(y,u) \in F \times U$ for the *supply rate function*

$$\omega(y,u) = f_0(y,\,u) - f_0(y^{(\sigma)},\,u^{(\sigma)}), \qquad (12)$$

there exists a bounded (both from below and above) *storage function* $S : F \to \mathbb{R}$ and a continuous and strictly increasing function $\alpha : [0,\infty) \to [0,\,\infty)$ with $\alpha(0) = 0$ such that for all $b > a$ the *dissipation inequality* holds, that is, for any admissible pair $(y(\cdot),u(\cdot))$ and for all $\tau \in [a,\,b]$ we have

$$\begin{aligned} S(y(a)) + \int_a^\tau \omega(y(t),\,u(t))\,\mathrm{d}t \ge S(y(\tau)) \\ + \int_a^\tau \alpha\left(\|y(t) - y^{(\sigma)}\|_{\mathcal{Y}} + \|u(t) - u^{(\sigma)}\|_{\mathcal{U}}\right)\,\mathrm{d}t. \end{aligned} \qquad (13)$$

Let M_S denote an upper bound for $|S(y)|$ for $y \in F$.

Remark 1. The relation between strict dissipativity and the turnpike property is discussed by Faulwasser *et al.* (2017) as well as Grüne and Müller (2016).

Note that, if (13) holds, the same inequality holds if ω is replaced by a function $\tilde{\omega}(y,\,u) \ge \omega(y,u)$, for example, by $\tilde{\omega}(y,\,u) := \omega(y,u) + f_0(y^{(\sigma)},\,u^{(\sigma)}) = f_0(y,\,u) \ge 0$. This implies that we can assume without restriction that $\omega \ge 0$.

Example 1. (*Strict dissipativity*) Let $\gamma \in (0,1]$ be given. For $f_0(y,\,u) = \|y - y^{(\sigma)}\|_{\mathcal{Y}}^2 + \gamma\,\|u - u^{(\sigma)}\|_{\mathcal{U}}^2$, problem $P(a,\,b,\,y_0)$ is strictly dissipative at $(y^{(\sigma)},\,u^{(\sigma)}) \in F \times U$ with $S = 0$ and $\alpha(z) = \frac{\gamma}{2}|z|^2$.

In the work of Trélat *et al.* (2018) an example for a problem of optimal distributed control of the heat equation is given that is strictly dissipative. ♦

The measure turnpike property can hold even if there exist real numbers $M > 0$ and $\Upsilon_0 > 0$ such that for all b sufficiently large (in the sense that $M < \frac{b-a}{2}$) and for all $t \in [\frac{a+b}{2},\,\frac{a+b}{2} + M]$ the inequality

$$\begin{aligned} \|\hat{y}(a,\,b,y_0)(t) - y^{(\sigma)}\|_{\mathcal{Y}} \\ + \|\hat{u}(a,\,b,y_0)(t) - u^{(\sigma)}\|_{\mathcal{U}} > \Upsilon_0 \end{aligned} \qquad (14)$$

holds. Yet such a situation contradicts the typical situation that close to the point $\frac{a+b}{2}$ in the middle of the time interval the distance between the dynamic optimum and the static optimum becomes very small, which holds, e.g., in the case of an exponential turnpike property, see also the the discussion after Definition 3.

To exclude such a situation, the *turnpike property with interior decay* is defined. Here the definition is stated in a different way than in the work of Gugat (2021) since we adapt it to the case where in the problem no terminal conditions (e.g., a terminal state) are prescribed. Our setting allows us to consider subintervals that have the terminal time as an upper limit.

Definition 3. Problem $P(a,\,b,\,y_0)$ has the *turnpike property with interior decay* at $(y^{(\sigma)},\,u^{(\sigma)}) \in F \times U$ if there exist $C_1 > 0$ and $\lambda_1 \in (0, 1)$ such that for all $b > a$ we have the inequality

$$\int_{\frac{a+b}{2}-\lambda_1\frac{b-a}{2}}^{b} \upsilon(t)\,\mathrm{d}t \le \frac{C_1}{b-a}, \tag{15}$$

where

$$\upsilon(t) = \alpha\Big(\|\hat{y}(a, b, y_0)(t) - y^{(\sigma)}\|_{\mathcal{Y}} + \|\hat{u}(a, b, y_0)(t) - u^{(\sigma)}\|_{\mathcal{U}}\Big).$$

If (14) is valid, the *turnpike property with interior decay* cannot hold. This can be seen as follows. If (15) holds, we have

$$\lim_{b\to\infty} \int_{\frac{a+b}{2}-\lambda_1\frac{b-a}{2}}^{b} \upsilon(t)\,\mathrm{d}t = 0. \tag{16}$$

If the inequality (14) holds, we have for b sufficiently large

$$\int_{\frac{a+b}{2}}^{\frac{a+b}{2}+M} \upsilon(t)\,\mathrm{d}t \ge M\,\Upsilon_0. \tag{17}$$

For b sufficiently large, we have

$$\left[\tfrac{a+b}{2},\, \tfrac{a+b}{2} + M\right] \subset \left[\tfrac{a+b}{2} - \lambda_1\tfrac{b-a}{2},\, b\right].$$

Hence (16) contradicts (17).

Thus the *turnpike property with interior decay* provides a more detailed picture of the behavior of the the optimal state and the optimal control in the interior of the time interval than the *measure turnpike*.

Remark 2. Also an *exponential turnpike property*, where the optimal state and the optimal control decay exponentially fast in the sense that there exist $C_1 > 0$ and $\mu > 0$ such that for all $T > 0$ for all $t \in [0, T]$ we have

$$\alpha\big(\|\hat{y}(a, b, y_0)(t) - y^{(\sigma)}\|_{\mathcal{Y}} + \|\hat{u}(a, b, y_0)(t) - u^{(\sigma)}\|_{\mathcal{U}}\big) \le C_2 \exp(-\mu\, t)$$

implies the *turnpike property with interior decay*.

3. Turnpike result

Throughout this section we assume that for all $(a, b) \subset (0, \infty)$ and all $y_0 \in F$ the problems $P(a, b, y_0)$ are *strictly dissipative* at $(y^{(\sigma)}, u^{(\sigma)}) \in F \times U$.

For the subsequent analysis, we replace the objective function in (4) by

$$J_{(a,b)}(u, y) = \int_a^b \omega(y(t), u(t))\,\mathrm{d}t, \tag{18}$$

with ω as in (13).

We show that under the strict dissipativity and a cheap control assumption that we define in (19) below, the solution of problem $P(a, b, y_0)$ has the turnpike property with interior decay. The cheap control assumption requires that the optimal values of the control problem are uniformly bounded with a bound that depends only on the distance between the initial state and $y^{(\sigma)}$.

In the subsequent analysis, we need the following lemma that adapts Lemma 3.1 of Gugat (2021) to the case without terminal constraint.

Lemma 2. *Let an interval $[a_1, b_1]$ and a nonempty subinterval $[a_2, b_2] \subset [a_1, b_1]$ be given.*

Then for any optimal state $\hat{y}(a_1, b_1, y_0)$ of $P(a_1, b_1, y_0)$ there exists a number $t_1 \in (a_2, b_2)$ such that

$$\begin{aligned}\alpha\big(&\|\hat{y}(a_1, b_1, y_0)(t_1) - y^{(\sigma)}\|_{\mathcal{Y}} + \|\hat{u}(a_1, b_1, y_0)(t_1) - u^{(\sigma)}\|_{\mathcal{U}}\big)\\ &\le \frac{\hat{v}(a_1, b_1, y_0) - \big[S(\hat{y}(a_1, b_1, y_0)(b_1)) - S(y_0)\big]}{b_2 - a_2}.\end{aligned}$$

Proof. Since the proof is similar to Lemma 3.1 in the work of Gugat (2021), we do not state it here. ■

Now we define the abstract *cheap control assumption* as follows.

Definition 4. We say that the *cheap control assumption* holds for $P(a, b, y_0)$, if there exist constants $\mu_0 > 0$ and $\varepsilon_0 > 0$ such that for all initial times a, all initial states $y_0 \in F$ and for all terminal times $b > a$ the inequality

$$\begin{aligned}\hat{v}(a, b, y_0) \le \mu_0\, \alpha\Big(\|y_0 - y^{(\sigma)}\|_{\mathcal{Y}}\Big) + \varepsilon_0\\ + S(\hat{y}(a, b, y_0)(b)) - S(y_0)\end{aligned} \tag{19}$$

holds.

Remark 3. A sufficient condition for (19) with $\varepsilon_0 = 0$ is that there exists a control function $u_1 \in L^2_{\mathrm{loc}}(a, \infty)$ such that $u_1(t) \in U$ for almost every $t > a$, $y_1 = \Phi(a, y_0, u_1)$ satisfies $y_1(t) \in F$ for $t > a$, and for all $t > a$ we have

$$\int_a^t f_0(y_1(s), u_1(s)) - f_0(y^{(\sigma)}, u^{(\sigma)})\,\mathrm{d}s \le \mu_0\|y_0 - y^{(\sigma)}\|^2_{\mathcal{Y}},$$

where the constant μ_0 is independent of t. This is the case if there exists a number $t_{\min} > a$ such that for $t > t_{\min}$, we have $y_1(t) = y^{(\sigma)}$ and $u_1(t) = u^{(\sigma)}$.

An example is a system that can be controlled exactly to $y^{(\sigma)}$ at the time $t_{\min}$ with the control $u_1|_{(a,t_{\min})}$ where the state remains equal to $y^{(\sigma)}$ with a constant control $u^{(\sigma)}$ for $t \ge t_{\min}$ (see also Section 4). Thus, the exact controllability implies that the cheap control assumption holds.

Now we analyze the optimal control problem $P(0, T, y_0)$ as defined in (8). Let a real number $\lambda \in (0, 1)$ be given.

Consider the interval

$$I_0(\lambda, T) = \big(0, (1-\lambda)\tfrac{T}{2}\big].$$

Then we have $I_0(\lambda, T) \subset (0, T/2)$.

By obtaining an upper bound for the part of the objective function (18) that comes from the subinterval $((1-\lambda)\frac{T}{2}, T) = (0, T)\backslash I_0(\lambda, T)$, we are able to obtain the following result.

Theorem 1. *Assume that for all $(a, b) \subset (0, \infty)$ and all $y_0 \in F$ the problems $P(a, b, y_0)$ are* strictly dissipative *at $(y^{(\sigma)}, u^{(\sigma)}) \in F \times U$. Assume that the cheap control assumption (19) holds with $\varepsilon_0 = 0$. Then for every $T > 0$ the problem $P(0, T, y_0)$ has the turnpike property with interior decay and the the measure turnpike property at $(y^{(\sigma)}, u^{(\sigma)})$.*

Proof. The cheap control assumption yields upper bounds for the optimal values of the problems $P(0, T, y_0)$. The combination of these bounds and the strict dissipation inequality (13) implies the measure turnpike property.

To show the turnpike property with interior decay, note that Lemma 2 implies that there exists a point $t_1^+ \in I_0(\lambda, T)$ such that

$$\begin{aligned}&\alpha\big(\|\hat y(0,T, y_0)(t_1^+) - y^{(\sigma)}\|_{\mathcal Y} \\ &\quad + \|\hat u(0,T, y_0)(t_1^+) - u^{(\sigma)}\|_{\mathcal U}\big) \\ &\le \frac{\hat v(0,T,y_0) - [S(\hat y(0,T, y_0)(T)) - S(y_0)]}{(1-\lambda)\,T/2}.\end{aligned} \tag{20}$$

Define

$$\begin{aligned}\Xi(s) = \alpha(&\|\hat y(0,T\, y_0)(s) - y^{(\sigma)}\|_{\mathcal Y} \\ &+ \|\hat u(0,T, y_0)(s) - u^{(\sigma)}\|_{\mathcal U}).\end{aligned}$$

The strict dissipation inequality (13) implies

$$\begin{aligned}A_0 := &\int_{(1-\lambda)\frac{T}{2}}^{T} \Xi(s)\,\mathrm{d}s \\ &\le \int_{t_1^+}^{T} \Xi(s)\,\mathrm{d}s \\ &\le S(\hat y(0,T, y_0)(t_1^+)) - S(\hat y(0,T, y_0)(T)) \\ &\quad + \int_{t_1^+}^{T} \omega(\hat y(0,T, y_0)(t), \hat u(0,T, y_0)(t))\,\mathrm{d}t \\ &\le \mu_0\,\alpha\left(\|\hat y(0,T, y_0)(t_1^+) - y^{(\sigma)}\|_{\mathcal Y}\right) + \varepsilon_0,\end{aligned}$$

where the last inequality is a consequence of the cheap control assumption (19). Combining the obtained bound with (20) we finally get

$$\begin{aligned}A_0 &\le \mu_0 \frac{\hat v(0,T, y_0) - [S(\hat y(0,T, y_0)(T)) - S(y_0)]}{(1-\lambda)\,T/2} + \varepsilon_0 \\ &\le \frac{\mu_0^2\alpha\|y_0 - y^{(\sigma)}\|_{\mathcal Y} + \mu_0\varepsilon_0}{(1-\lambda)\,T/2} + \varepsilon_0,\end{aligned} \tag{21}$$

where in the last step we used the cheap control assumption again.

Finally, taking into account the assumption $\varepsilon_0 = 0$, the definition of A_0 implies the assertion. ■

Now we consider the subintervals

$$\begin{aligned}I_1(\lambda, T) &= \big((1-\lambda)\tfrac{T}{2}, (1-\lambda^2)\tfrac{T}{2}\big], \\ I_2(\lambda, T) &= \big((1-\lambda^2)\tfrac{T}{2}, (1-\lambda^3)\tfrac{T}{2}\big]\end{aligned}$$

and, more generally, for $n \in \{0, 1, 2, \dots\}$

$$I_n(\lambda, T) = \big((1-\lambda^n)\tfrac{T}{2}, (1-\lambda^{n+1})\tfrac{T}{2}\big] \tag{22}$$

of $[0, T]$. The interval $I_n(\lambda, T)$ has length $(1-\lambda)\,\lambda^n\,T/2$. Note that the intervals $I_n(\lambda, T)$ are disjoint and $\bigcup_{n=0}^{\infty} I_n(\lambda, T) = (0, T/2)$.

We obtain upper bounds for the part of the objective function that comes from the integral over the subintervals

$$(0, T)\backslash \bigcup_{k=0}^{n} I_k(\lambda, T) = \big((1-\lambda^{n+1})\tfrac{T}{2}, T\big).$$

Our main turnpike result for the optimal control problem defined in (8) is stated in the following theorem.

Theorem 2. *Assume that the strict dissipativity and the cheap control assumption (19) hold for all initial states $y_0 \in Y$. Let $\lambda \in (0, 1)$ and a natural number $n \ge 0$ be given.*

For $k \in \{0, 1, 2, \dots\}$ let $\mu(I_k(\lambda, T)) = \lambda^k(1-\lambda)\,T/2$ denote the length of the interval $I_k(\lambda, T)$. Define the real number $g_0 = \mu_0/\mu(I_0(\lambda, T))$ and for $k \in \{0, 1, 2, \dots\}$ let

$$g_{k+1} = \frac{\mu_0}{\mu(I_{k+1}(\lambda, T))}\,(g_k + 1) = \frac{g_0}{\lambda^{k+1}}\,(g_k + 1).$$

Let

$$\begin{aligned}v(s) = \alpha(&\|\hat y(0,T, y_0)(s) - y^{(\sigma)}\|_{\mathcal Y} \\ &+ \|\hat u(0,T, y_0)(s) - u^{(\sigma)}\|_{\mathcal U}).\end{aligned}$$

Then for all real numbers $T > 0$ we have

$$\begin{aligned}A_n := &\int_{(1-\lambda^{n+1})\frac{T}{2}}^{T} v(s)\,\mathrm{d}s \\ &\le \frac{\mu_0^{n+2}}{\prod_{k=0}^{n}\mu(I_k(\lambda, T))}\alpha\left(\|y_0 - y^{(\sigma)}\|_{\mathcal Y}\right) \\ &\quad + \varepsilon_0\,(g_n + 1).\end{aligned} \tag{23}$$

The decay estimate (23) allows us to prove the following corollary, which states that the integral on the time interval $(T/2, T)$ corresponding to A_n decays faster than the reciprocal value of a power series with positive coefficients only.

Corollary 1. *Under the assumptions of Theorem 2 and the additional assumption $\varepsilon_0 = 0$, for $T > 0$ define*

$$A_*(T) := \int_{\frac{T}{2}}^{T} \alpha \Big(\|\hat{y}(0, T\, y_0)(s) - y^{(\sigma)}\|_{\mathcal{Y}} + \|\hat{u}(0, T,\, y_0)(s) - u^{(\sigma)}\|_{\mathcal{U}} \Big) \,\mathrm{d}s.$$

For all $n \in \{1, 2, \dots\}$ define

$$C_n = \frac{2^n\, \mu_0^{n+1}}{\lambda^{\frac{n(n-1)}{2}} (1-\lambda)^n}.$$

Then (23) *implies that*

$$A_*(T) \le \frac{C_n}{T^n}\, \alpha \Big(\|y_0 - y^{(\sigma)}\|_{\mathcal{Y}} \Big). \tag{24}$$

For $T > 0$ define the increasing function

$$F(T) = \sum_{n=1}^{\infty} \frac{1}{2^n\, C_n} T^n.$$

Then for all $T > 0$ we have

$$A_*(T) \le \frac{\alpha \left(\|y_0 - y^{(\sigma)}\|_{\mathcal{Y}} \right)}{F(T)}.$$

Proof. We have

$$\prod_{k=0}^{n} \mu(I_k(\lambda,\, T)) = \frac{T^{n+1}}{2^{n+1}} (1-\lambda)^{n+1} \lambda^{\frac{n\,(n+1)}{2}} = \frac{T^{n+1}}{C_{n+1}} \mu_0^{n+2}.$$

Hence (23) and the definition of C_n imply (24).

For all $n \in \{1, 2, \dots\}$ we have

$$\frac{1}{2^n} \frac{1}{A_*(T)} \ge \frac{T^n}{2^n\, C_n} \frac{1}{\alpha \left(\|y_0 - y^{(\sigma)}\|_{\mathcal{Y}} \right)}.$$

Since $1 = \sum_{n=1}^{\infty} 2^{-n}$, this implies that we also have

$$\frac{1}{A_*(T)} = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{A_*(T)} \ge \sum_{n=1}^{\infty} \frac{T^n}{2^n\, C_n} \frac{1}{\alpha \left(\|y_0 - y^{(\sigma)}\|_{\mathcal{Y}} \right)} = \frac{1}{\alpha \left(\|y_0 - y^{(\sigma)}\|_{\mathcal{Y}} \right)} F(T).$$

■

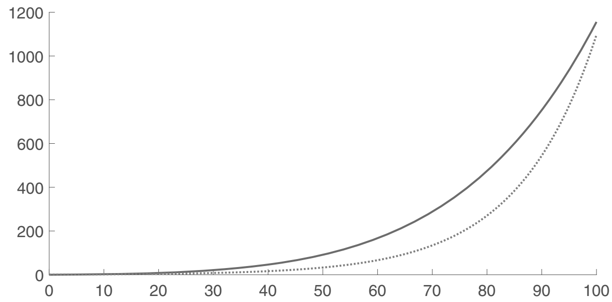

Fig. 1. Graphs of $F(T)$ for $\mu_0 = 1$ and $\lambda = 1/2$ (top) and $\exp(7\,T/100)$ (bottom, dotted).

Corollary 1 implies that $A_*(T)$ decays faster than at any polynomial rate. Due to the fast growth of C_n compared with $n!$, the decay is slower than exponential.

For $\mu_0 = 1$ and $\lambda = 1/2$ we obtain

$$F(T) = \sum_{n=1}^{\infty} \frac{1}{2^{\frac{n^2+5\,n}{2}}} T^n.$$

Figure 3 shows the graph of F for this case. For comparison, it also shows the graph of an exponential function. Note that the exponential function finally grows faster than F.

Remark 4. Note that for all $k \ge 0$ we have $g_k \to 0$ as $T \to \infty$. Moreover, there exist constants $D_k > 0$ such that for every $T > 1$ we have

$$|g_k| \le \frac{D_k}{T}.$$

Remark 5. An analogous result as in Theorem 2 can be shown for any interval of the form $[\frac{T}{N}, T]$, where $N \in \{2, 3, \dots\}$ is a natural number: The contribution of the integral from T/N to T in the objective function is of the order $1/T^n$, where the natural number n can be chosen arbitrarily large.

Now we present the proof of Theorem 2.

Proof. (Theorem 2) For $n = 0$ we have already shown (23) in the inequality (21). To be precise, we have shown that for $n = 0$ there exist

$$t_{n+1}^{+} \in \left((1-\lambda^n)\, \tfrac{T}{2},\, (1-\lambda^{n+1})\, \tfrac{T}{2} \right) \tag{25}$$

such that

$$A_n \le \int_{t_{n+1}^{+}}^{T} v(s)\,\mathrm{d}s \le \hat{v}(t_{n+1}^{+},\, T,\, \hat{y}(0, T,\, y_0)(t_{n+1}^{+})) + S(\hat{y}(0, T,\, y_0)(t_{n+1}^{+})) - S(\hat{y}(0, T,\, y_0)(T))$$

$$\leq \frac{\mu_0^{n+2}}{\prod_{k=0}^{n} \mu(I_k(\lambda, T))} \alpha\left(\|y_0 - y^{(\sigma)}\|_{\mathcal{Y}}\right) + \varepsilon_0(g_n + 1). \tag{26}$$

In addition, due to Lemma 1, the restriction of the control–state pair $(\hat{u}(0,T,y_0), \hat{y}(0,T,y_0))$ to the interval (t_{n+1}^+, T) is an optimal control–state pair for the optimal control problem

$$P(t_{n+1}^+, T, \hat{y}(0,T,y_0)(t_{n+1}^+)).$$

Now we continue inductively. Assume that for some $n \geq 0$ there exists $t_{n+1}^+ \in I_n(\lambda, T)$ such that the chain of inequalities (26) holds and $(\hat{u}(0,T,y_0), \hat{y}(0,T,y_0))$ is optimal for

$$P(t_{n+1}^+, T, \hat{y}(0,T,y_0)(t_{n+1}^+)). \tag{27}$$

Due to Lemma 2, there exists a point

$$t_{n+2}^+ \in \left((1-\lambda^{n+1})\tfrac{T}{2},\, (1-\lambda^{n+2})\tfrac{T}{2}\right) \tag{28}$$

such that

$$\upsilon(t_{n+2}^+) \leq \frac{Z_n}{U_n} \tag{29}$$

with

$$Z_n = \hat{v}(t_{n+1}^+, T, \hat{y}(0,T,y_0)(t_{n+1}^+)) + S(\hat{y}(0,T,y_0)(t_1^+)) - S(\hat{y}(0,T,y_0)(T))$$

and $U_n = \mu(I_{n+1}(\lambda, T))$.

Then we have $t_{n+2}^+ \geq t_{n+1}^+$.

Due to Lemma 1 and our assumption on (27), the control–state pair $(\hat{u}(0,T,y_0), \hat{y}(0,T,y_0))$ is optimal for

$$P(t_{n+2}^+, T, \hat{y}(0,T,y_0)(t_{n+2}^+)). \tag{30}$$

Then due to the strict dissipativity and the cheap control assumption (19) we have

$$\begin{aligned} A_{n+1} &= \int_{(1-\lambda^{n+2})\,T/2}^{T} \upsilon(s)\,\mathrm{d}s \\ &\leq \int_{t_{n+2}^+}^{T} \upsilon(s)\,\mathrm{d}s \\ &\leq S(\hat{y}(0,T,y_0)(t_{n+2}^+)) - S(\hat{y}(0,T,y_0)(T)) \\ &\quad + \int_{t_{n+2}^+}^{T} \omega(\hat{y}(0,T,y_0)(t), \hat{u}(0,T,y_0)(t))\,\mathrm{d}t \\ &\leq \mu_0\, \alpha(\|\hat{y}(0,T,y_0)(t_{n+2}^+) - y^{(\sigma)}\|_{\mathcal{Y}}) + \varepsilon_0. \end{aligned} \tag{31}$$

Together with (29) this yields the inequality

$$A_{n+1} \leq \mu_0 \tfrac{Z_n}{U_n} + \varepsilon_0.$$

Finally, our induction assumption (26) implies

$$A_{n+1} \leq \frac{\mu_0}{\mu(I_{n+1}(\lambda, T))} \times \left(\frac{\mu_0^{n+2}}{\prod_{k=0}^{n} \mu(I_k(\lambda, T))} \alpha\left(\|y_0 - y^{(\sigma)}\|_{\mathcal{Y}}\right) + \varepsilon_0(g_n+1)\right) + \varepsilon_0$$

from which the assertion follow.

4. Examples

To illustrate our results, we present examples of optimal control problems where Theorem 2 is applicable.

Example 2. Let y_0, y_1 in $\mathbb{R}^n$ be given and $f: \mathbb{R}^n \to \mathbb{R}^n$, $g: \mathbb{R}^n \to \mathbb{R}^{n\times m}$ be C^2 maps with $f(0) = 0$. Let $C \in \mathbb{R}^{n\times n}$ be regular and define the Hilbert space Y with the norm

$$\|z\|_{\mathcal{Y}} = \left(z^\top C^\top C z\right)^{1/2}.$$

In the work of Sakamoto and Zuazua (2021) the following problem with a system that is governed by an ordinary differential equation is considered with the additional terminal condition $y(T) = y_1$:

$$(\mathrm{OCP})_T \begin{cases} \min\limits_{u\in L^\infty(0,T)} \int_0^T \|y(t)\|_{\mathcal{Y}}^2 + \|u(t)\|_{\mathbb{R}^m}^2 \,\mathrm{d}t \\ \text{subject to} \\ y(0) = y_0,\ y'(t) = f(y(t)) + g(y(t))\,u(t). \end{cases}$$

Here the turnpike is zero, that is, $y^{(\sigma)} = 0$ and $u^{(\sigma)} = 0$. In this case, the cheap control assumption (19) requires that there exist constants $\mu_0 > 0$ and $\varepsilon_0 \geq 0$ such that for all $y_0 \in Y$ we have

$$\hat{v}(a, t_0, y_0) \leq \mu_0 \|y_0\|_{\mathcal{Y}}^2 + \varepsilon_0. \tag{32}$$

If the system is linear and satisfies Kalman's controllability rank condition, it is exactly null controllable (see, e.g., Sontag, 1991; Rabah *et al.*, 2017). This implies the cheap control inequality. As in Example 1, the strict dissipativity follows with $S = 0$. Here we have $y^{(\sigma)} = 0$ and $u^{(\sigma)} = 0$.

Our results show that also with additional constraints, for example, with pointwise control constraints as in the set

$$U = \{u \in L^\infty(0,T) : |u(t)| \leq 1 \text{ almost everywhere}\},$$

the solution of $(\mathrm{OCP})_T$ has a turnpike structure as pointed out in Theorem 2. ♦

Example 3. Let $y_0(x) \in L^2(0, L)$ and $u_d \in \mathbb{R}$ be given. Consider the optimal control problem

$$\min_{u\in L^2(0,T)} \int_0^T |u(t) - u_d|^2 + \int_0^L |y(t, x) - u_d|^2 \, dx \, dt$$

subject to

$$\begin{aligned} y(0, x) &= y_0(x), & x &\in (0, L),\\ y(t, 0) &= u(t), & t &\in (0, T) \end{aligned}$$

and for $(t, x) \in (0, T) \times (0, L)$:

$$y_t(t, x) = -\exp(x)\, y_x(t, x).$$

Then the state has the form $y(t,x) = H(t + \exp(-x))$ with $H(t + 1) = u(t)$, $t \in (0,T)$ and $H(\exp(-x)) = y_0(x)$, $x \in (0, L)$.

For the optimal control we obtain $u(t) = u_d$ for all $t \in (0, T)$. Therefore, here the state is controlled exactly on the turnpike for all (t, x) with $t + \exp(-x) > 1$. This implies that all our turnpike estimates apply. ♦

Example 4. Now we study a problem with quasilinear dynamics. Let $y_0(x) \in C^1(0, L)$ and $u_d \in \mathbb{R}$, $u_d \geq 0$ be given. Assume that $y_0(0) \geq u_d$ and that y_0 is increasing. Consider the optimal control problem

$$\min_{u\in H^2(0,T)} \int_0^T |u(t) - u_d|^2 + |\partial_t u(t)|^2 + |\partial_{tt} u(t)|^2 + \int_0^L |y(t, x) - u_d|^2 \, dx \, dt$$

subject to the following constraints:

$$\begin{aligned} \min_{t\in[0,T]} u(t) &\geq 0, & y(0, x) &= y_0(x),\\ x &\in (0, L), & y(t, 0) &= u(t),\\ t &\in (0, T), & u(0) &= y_0(0), \end{aligned}$$

where the control $u(t)$ is decreasing and

$$y_t(t, x) = -(1 + y(t, x))\, y_x(t, x)$$

for $(t, x) \in (0, T) \times (0, L)$. Then the solution y is constant along the characteristic curves that are straight lines. For the characteristic curves that start at x for $t = 0$ we have $\xi^x(s) = x + (1 + y_0(x))\, s$ for $s \geq 0$. For the characteristic curves that start at $x = 0$ for $t \in (0,T]$ we have $\xi^t(s) = 0 + (1 + u(t))(s - t)$ for $s \geq t$. The assumption that y_0 is increasing, implies that the curves ξ^x do not intersect.

As an alternative to the assumption that y_0 is increasing, we can assume that

$$\max_{x\in[0,L]} |y_0'(x)| < \frac{\min_{x\in[0,\, L]} |1 + y_0(x)|}{L}$$

to guarantee that the curves ξ^x do not intersect with an intersection point in $[0, L]$.

The constraint that the control $u(t)$ is decreasing implies that the curves ξ^t do not intersect.

As an alternative to the assumption that u is decreasing, we can assume that

$$\max_{t\in[0,T]} \{|u(t)|, |u'(t)|\} \leq \varepsilon,$$

where $\varepsilon > 0$ satisfies the inequality $(1 + 2T)\, \varepsilon < 1$ and thus

$$\frac{\varepsilon}{1 - (1 + T)\, \varepsilon} < \frac{1}{T}$$

to guarantee that the curves ξ^t do not intersect with an intersection point for $s \in [0, T]$.

The compatibility condition $y_0(0) = u(0)$ implies that $\xi_{t=0} = \xi_{x=0}$ which means that the characteristic curves that start at $x = 0$ and $t = 0$ coincide. Then the state can be written in the form $y(s, \xi^x(s)) = y_0(x)$ $(s > 0)$, and $y(s, \xi^t(s)) = u(t)$ respectively.

If $y_0(0) = u_d$, for the optimal control we obtain $u(t) = u_d$ for all $t \in (0, T)$. Therefore, here the state is controlled exactly on the turnpike for all (t, x) with $x < (1 + u_d)t$.

If $y_0(0) > u_d$, we can apply our main result. Since the system is exactly controllable to u_d in finite time, the cheap control assumption (19) is satisfied with $\varepsilon_0 = 0$ for all sufficiently large time horizons. Moreover, we have $M_S = 0$. Therefore, the optimal control satisfies the estimate (23) with $\alpha(z) = \frac{1}{2} z^2$ for all sufficiently large time horizons. ♦

Example 5. Let a length $L > 0$, the wave speed $c > 0$ and a time $T_0 > 2L/c$, $k \in \{1, 2, \ldots\}$ and $T = k\, T_0$ be given. Define the T_0-periodic weight function $w(t)$ with $w(t) = T_0 - t$ for $t \in [0, T_0]$. For $y \in C((0,T), H^1(0, L)) \cap C^1((0,T), L^2(0, L))$ and $t \in [0,T]$ let $E(t)$ denote the energy,

$$E(t) = \frac{1}{2} \int_0^L (y_x(t, x))^2 + \frac{1}{c^2} (y_t(t, x))^2 \, dx. \quad (33)$$

For a parameter $\gamma \geq 0$, an initial position $y_0 \in H^1(0, L)$ with $y_0(0) = 0$ and an initial velocity $y_1 \in L^2(0, L)$, we consider the following problem of optimal Neumann control for the wave equation:

$$\begin{cases} \min_{u\in L^2(0,T)} \int_0^T E(t) + \frac{\gamma}{2}\, w(t)\, u^2(t)\, dt \\ \text{subject to} \\ y(0,x) = y_0(x),\ y_t(0,x) = y_1(x),\ x \in (0, L),\\ y(t,0) = 0,\ y_x(t,L) = u(t),\ t \in (0,T),\\ y_{tt}(t,x) = c^2\, y_{xx}(t,x),\ (t,x) \in (0,T) \times (0, L). \end{cases}$$

The exact controllability of the above system is well established (e.g., Tucsnak and Weiss, 2009, Ex. 11.2.6).

Thus, it can be steered to the desired final state of energy zero in finite time, which in turn implies the cheap control assumption (19) with $\varepsilon_0 = 0$ for all sufficiently large time horizons. Moreover, we have $M_S = 0$. Therefore, the optimal control satisfies the estimate (23) with $\alpha(z) = \frac{1}{2} z^2$. In fact, here for all $\gamma \geq 0$ an exponential turnpike property occurs (see Gugat, 2022) and for $\gamma = 0$ we even have a finite-time turnpike property. ♦

5. Conclusion

We have shown a turnpike theorem for a problem of optimal control in a general framework that allows for systems with nonlinear evolution. For strictly dissipative systems that are cheaply controllable, the optimal control problems enjoy the turnpike property with interior decay. In contrast to the previous results on the turnpike property with interior decay, in this paper we have studied problems without a terminal condition. Our cheap controllability assumption implies that in this case the optimal dynamic state and the optimal dynamic control approach the optimal static state and the optimal static control also at the end of the time interval for sufficiently large time horizons.

The turnpike result states that for large time horizons T after an initial transient period of length $T/2$ the integral in the objective function on the remaining part of the time interval that is generated by the optimal controls decays at least with an order of $1/T$. In fact, by applying the result inductively, we obtain a decay with an order $1/T^n$, where $n \in \{1, 2, \dots\}$ can be chosen arbitrarily large.

In the case of parabolic or hyperbolic PDEs, the static problem is an elliptic problem. In 2D and 3D these steady state optimal control problems can be solved very efficiently.

It is interesting to study the connection with delays in the implementation of the optimal controls (see, e.g., Gugat and Leugering, 2017). It is well known that even small delays in the implementation of control laws can have a destabilizing effect. Since close to a steady control this effect becomes smaller, there is a natural link to the turnpike property. Turnpike theorems for an infinite horizon optimal control problem with time delay have been studied by Mammadov (2014). The investigation of the connection between delay and the turnpike phenomenon for large finite horizon optimal control remains an interesting open problem.

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