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MULTIPLE–WAREHOUSE SLIDING MODE CONTROL WITH A PREDEFINED DEMAND TRAJECTORY PROFILE

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The paper discusses the inventory management problem with a single product stored in two warehouses, where each has its unique suppliers with certain lead times. Moreover, one of the warehouses may act as a backup supplier for the other. In other words, product exchange between two different warehouses within one company is allowed. The first warehouse operates under an *a priori* known time-variant contractual demand and a bounded random one. Its secondary goal is to accumulate emergency stock that can be delivered to the second warehouse within one time period. For this warehouse we use a desired trajectory generator to shape the required stock level and then utilize a trajectory following control law. The demand in the second warehouse is unknown but bounded, and its suppliers have limited delivery capacity. The challenge is to fulfill the customers' needs, although they might exceed the order limit. Therefore, occasional backup supplies from the first warehouse are necessary. For the control of the second warehouse, a simple sliding mode (SM) scheme is applied. The paper proves that, with appropriate compensation of the emergency deliveries in the first warehouse, our proposed control scheme ensures full demand satisfaction in both warehouses despite the second one's control limit.

Keywords: control design, discrete-time systems, inventory control, model reference control, sliding mode control.

1. Introduction

Sliding mode control (SMC) is the most popular branch of variable structure control. It has been widely studied in the literature for decades (Utkin, 1984; 1992; Drakunov and Utkin, 1992; Hung *et al.*, 1993; Steinberger *et al.*, 2020). SM is designed so that the system's trajectory becomes bounded to a preselected sliding manifold. Therefore, first the system is driven from any initial position towards the sliding hypersurface. The distance of the system's representative point from this hypersurface is described with a sliding variable, which combines the system's states, i.e., $s(x) = f[x(t)]$. Once the system's representative point belongs to the sliding manifold, the sliding variable becomes zero, $s(x) = 0$. Then the second stage of the control process occurs i.e., when the control action must maintain the system's trajectory on the sliding manifold. This is achieved with a characteristic switching type control law. Whenever the system's representative point leaves the sliding manifold, the control action is switched based on the current sign of the sliding variable. Consequently, the system's representative point moves along the sliding manifold until it reaches a steady state. As a result, in the sliding phase the system becomes insensitive to matched external disturbances and modelling uncertainties (Draženović, 1969). However, this benefit is only guaranteed assuming no delays in the switching process occur. When applied through sample and hold devices, SMC may result in chattering around the sliding hypersurface introduced by sampling in the control and output channels (Utkin and Lee, 2006; Boiko *et al.*, 2008).

With the development of digital control systems, the discontinuous nature of the SMC law has led to the definition of quasi-sliding mode (QSM), obtained in discrete-time systems (Drakunov and Utkin, 1989; Gao *et al.*, 1995; Bartoszewicz, 1998; Golo and Milosavljević, 2000). As in the discrete-time domain the system's states exist only at certain time instants, it is assumed that the representative point may not be kept on the sliding hypersurface but in its vicinity called the QSM band.

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The width of this vicinity depends on the disturbance impact directly and therefore becomes the measure of the system's robustness (Kaynak and Denker, 1993). An important feature of the QSM is also comparably low computational complexity. SMC is commonly applied in fields such as power electronics and robotics (Edwards and Spurgeon, 1998; Utkin *et al.*, 1999; Liu and Wang, 2011; Ordaz *et al.*, 2024; Hamdi *et al.*, 2021). However, due to their discrete time nature and usually high problem order, inventory systems are often discussed in the context of potential QSMC applications.

The focus of this paper is application of a reference trajectory based SMC for a two-warehouse inventory system. With the recent rapid development of online sales and shortened product launching time, the logistics business is nowadays booming. Inventory systems are required to become more flexible and react faster to changing market demands. Moreover, growing logistics centers dealing with a variety of products, which may be found on the outskirts of each bigger city, not to mention a metropolis, are becoming more and more difficult to manage and supply. Therefore, the problem of inventory systems management has become an important topic for the control engineering society in both modelling (Framinan, 2022) and the control design context (White, 1999; Zipkin, 2000). Various studies considered control methods such as optimal control (Cao and Xie, 2016; Ignaciuk and Bartoszewicz, 2010a), H_∞ control (Boccadoro *et al.*, 2008), the neural networks based approach (Chołodowicz and Orłowski, 2023) and the model predictive approach (Alessandri *et al.*, 2011). Among others some successful attempts of SMC applications were proposed (Ignaciuk and Bartoszewicz, 2010b; Bartoszewicz and Leśniewski, 2014).

This study presents a new approach to the inventory management problem. The manuscript considers an inventory system consisting of two warehouses, storing one product. For the sake of clarity, they will be denoted with α and β. Potentially, this pictures a situation of two warehouses in different locations owned by one company. Each of the warehouses has its unique suppliers, who need certain time to deliver the goods. Moreover, the capabilities of warehouse β suppliers are limited, which results in an order limit. Warehouse β is subject to the conventionally considered random market demand and is controlled according to a simple sliding mode control law. However, for warehouse α, a novel demand definition is introduced. The customer needs are divided into the *a priori* known contractual demand part and unknown but bounded random demand part. Having the partial knowledge of the demand, we propose to generate a desired trajectory profile in advance and apply a trajectory following SMC scheme, as proposed by Bartoszewicz and Adamiak (2018; 2020). Moreover, product exchange between the warehouses is allowed. As warehouse β has limited deliveries, it may not be able to fully satisfy its demand. Therefore, warehouse α may act as an emergency supplier, which delivers the necessary product in one control step. Our study shows that, with appropriate random demand and emergency deliveries compensation, full demand satisfaction in both warehouses in ensured.

2. System presentation

In this paper, we consider an inventory system seen from the perspective of an owner of multiple warehouses. It stands to reason that, in a situation when one warehouse does not have enough stock to fulfill the customers' demand, the owner would prefer to move the product from their nearby facility rather than initiate an often time consuming and costly operation of ordering more goods. The inventory management system in the article is simplified to one product, multiple suppliers and two warehouse systems. Each warehouse has its own suppliers, which require a certain lead time to deliver products to their respective warehouse. The demand profile differs between the warehouses. Therefore, for the sake of clarity, we denote the warehouses as α and β. This section introduces the notation and describes the systems in the state space.

In general, the inventory system may have several suppliers with different delivery times. Let the longest lead time required by those suppliers be denoted as n. Consequently, not later than after $n + 1$ time instants will all the deliveries have arrived at the warehouse. Taking that into account, the lead times of the suppliers in the system are in the range of

$$i = 1, 2, \dots, n. \tag{1}$$

Suppliers with equal lead time are considered as one. For every i, we define the a_i parameter, which is the part of the order placed by the controller that was allocated to the supplier with i periods of lead time. If a supplier with lead time i does not exist, the matching a_i parameter equals zero. The aforementioned a_i parameters satisfy

$$0 \le a_i \le 1 \quad \text{and} \quad \sum_{i=1}^{n} a_i = 1. \tag{2}$$

The system is subject to periodic reviews, with the review period denoted as T. The controller generates orders at regular time instants $k\mathrm{T}$ with $k = 0, 1, 2, \dots$. From now on, the time instants $k\mathrm{T}$ will be referred to as k for simplicity. The inventory system is described with the following state space representation:

$$\begin{aligned} \boldsymbol{x}(k+1) &= \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{b}u(k) - \boldsymbol{f}h(k), \\ y(k) &= x_1(k). \end{aligned} \tag{3}$$

The state vector $\boldsymbol{x}(k)$ represents the amount of product both contained in the warehouse and in various stages of transit. Therefore, the order of the dynamical problem depends on the maximum lead time n of the system and is equal to $n+1$ as the state vector has $n+1$ elements, defined as

$$\boldsymbol{x}(k) = \begin{bmatrix} x_1(k) & x_2(k) & \dots & x_n(k) & x_{n+1}(k) \end{bmatrix}^{\mathrm{T}}. \tag{4}$$

The first state variable of the system, $x_1(k)$, represents the amount of goods present in the warehouse at the time instant k, before any of the demand has been fulfilled. The rest of the state variables define the product that has been ordered and is currently in transit. They are delayed orders generated by the controller, with $x_{n+1}(k)$ being the amount of goods ordered at the previous time instant, i.e., $k-1$, $x_n(k)$ denoting two time instants ago, i.e., $k-2$ etc. Matrix $\boldsymbol{A}$ is the state matrix of the inventory system, defined as

$$\boldsymbol{A} = \begin{bmatrix} 1 & a_n & a_{n-1} & \dots & a_2 & a_1 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}. \tag{5}$$

The a_i parameters present in the state matrix refer to the a_i parts of the order assigned to suppliers with different lead times. Therefore, those products will be delivered to the warehouse at the consecutive delayed time instants. The input vector $\boldsymbol{b}$ has $n+1$ elements and is defined as

$$\boldsymbol{b} = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix}^{\mathrm{T}}. \tag{6}$$

The orders generated by the controller are defined as $u(k)$. We use the scalar output $y(k)$ to denote the amount of product in the warehouse. The initial stock level is defined as $y_0 = x_1(0)$. Customer demand is most commonly considered random and denoted with $d(k)$, where $d(k) \geq 0$. Continuing, we define the vector $\boldsymbol{f}$, which represents the amount of goods that leave the warehouse. This vector has $n+1$ elements and

$$\boldsymbol{f} = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \end{bmatrix}^{\mathrm{T}}. \tag{7}$$

The amount of product successfully sold to the customers is represented by $h(k)$.

Next, by analogy, we will present the specifications of the warehouses α and β, considered in this paper.

2.1. Warehouse α.

The first warehouse is denoted with α. Therefore, its maximum lead time is n_α and the system order becomes $n_\alpha + 1$. As a result, the system's state equation becomes

$$\begin{aligned} \boldsymbol{x}_\alpha(k+1) &= \boldsymbol{A}_\alpha \boldsymbol{x}_\alpha(k) + \boldsymbol{b} u_\alpha(k) - \boldsymbol{f} h_\alpha(k), \\ y_\alpha(k) &= x_{\alpha 1}(k). \end{aligned} \tag{8}$$

The state vector $\boldsymbol{x}_\alpha(k)$ has elements $x_{\alpha 1}, x_{\alpha 2}, \dots, x_{\alpha n}, x_{\alpha n_\alpha + 1}$, as presented in (4). As follows from (5), the system's state matrix is $\boldsymbol{A}_\alpha$ with first row elements denoted as $a_{\alpha 1}, a_{\alpha 2}, \dots, a_{\alpha n_\alpha}$. The systems control signal is $u_\alpha(k)$ and the input vector is $\boldsymbol{b}$, as shown in (6). The sales from the warehouse are denoted by $h_\alpha(k)$ and vector $\boldsymbol{f}$ remains as defined in (7). The current stock level is expressed with $y_\alpha(k)$, with the initial condition $y_{\alpha 0} = x_{\alpha 1}(0)$.

The warehouse is subject to two kinds of customer demand. We define the *a priori* known contractual demand part $d_c(k)$ and the random demand part $d_r(k)$. Moreover, the warehouse's secondary goal is to provide some additional product in stock reserved for emergency needs of the second warehouse. This reserve will be denoted by $d_e(k)$ and it satisfies

$$0 \leq d_e(k) \leq d_{e\,\max}. \tag{9}$$

Therefore, the total demand in the system is

$$d_\alpha(k) = d_c(k) + d_e(k) + d_r(k). \tag{10}$$

The demand's main part, $d_c(k)$, represents the contractual obligations of the warehouse. This demand changes in time and its expected values are known in advance—we denote them as $\widetilde{d}_c(k)$. If the customers fulfill their part of the contract in every review period, then

$$d_c(k) = \widetilde{d}_c(k). \tag{11}$$

However, to include the possibility that some of the contracted goods are not purchased, we extend that to

$$0 \leq d_c(k) \leq \widetilde{d}_c(k). \tag{12}$$

Fulfilling the contractual sales is the warehouse's priority, but its secondary purpose is to keep an emergency stock for warehouse β. The emergency demand is only generated when the warehouse β reaches the limit of its ordering capabilities. If the value of the demand in the second warehouse remains under the maximum order value, then $d_e(k) = 0$. We complete the definition with the random demand $d_r(k)$. This term of the demand in the system is bounded by

$$0 \leq d_r(k) \leq d_{r\,\max}, \tag{13}$$

and the warehouse fulfills it from product leftover after the contractual sales and the emergency deliveries.

As the warehouse has different priority levels for each demand, we split the sales into three parts as well:

$$h_\alpha(k) = h_c(k) + h_e(k) + h_r(k). \tag{14}$$

Those sales are upper bounded by the demand as follows:

$$\begin{aligned} h_c(k) &\leq d_c(k), \\ h_e(k) &\leq d_e(k), \\ h_r(k) &\leq d_r(k). \end{aligned} \tag{15}$$

The sales in the warehouse begin with fulfilling the contractual demand, so

$$h_c(k) = \min[y_\alpha(k),\ d_c(k)], \tag{16}$$

and then, from the leftover product $y_{\alpha_r 1}(k) = y_\alpha(k) - h_c(k)$, we can calculate the amount of goods sent to warehouse β:

$$h_e(k) = \min[y_{\alpha r1}(k),\ d_e(k)]. \tag{17}$$

Finally, after delivering the product to warehouse β, the remaining stock becomes $y_{\alpha r2}(k) = y_{\alpha r1}(k) - h_e(k)$, which allows us to satisfy the random demand:

$$h_r(k) = \min[y_{\alpha r2}(k),\ d_r(k)]. \tag{18}$$

Considering (8), the stock level after all sales commence for any $k \le n_\alpha$ may be obtained as

$$y_\alpha(k) = y_{\alpha 0} + \sum_{i=1}^{n_\alpha-1} a_{\alpha i} \sum_{j=0}^{k-i-1} u_\alpha(j) - \sum_{i=0}^{k-1} h_\alpha(i), \tag{19}$$

and for any $k \le n_\alpha + 1$ as

$$\begin{aligned} y_\alpha(k) = y_{\alpha 0} &+ \sum_{i=0}^{k-n_\alpha-1} u_\alpha(i) \\ &+ \sum_{i=1}^{n_\alpha-1} a_{\alpha i} \sum_{j=k-n_\alpha}^{k-i-1} u_\alpha(j) - \sum_{i=0}^{k-1} h_\alpha(i). \end{aligned} \tag{20}$$

Finally, after all the sales have concluded, the product left in the warehouse is

$$y_{\alpha r}(k) = y_\alpha(k) - h_\alpha(k). \tag{21}$$

2.2. Warehouse β. Warehouse β is characterized by the maximum lead time n_β, which results in the order of the system equal to $n_\beta + 1$. Its dynamics are described as

$$\begin{aligned} \boldsymbol{x}_\beta(k+1) &= \boldsymbol{A}_\beta \boldsymbol{x}_\beta(k) + \boldsymbol{b} u_\beta(k) \\ &\quad - \boldsymbol{f} h_\beta(k) + \boldsymbol{f} h_e(k), \\ y_\beta(k) &= x_{\beta 1}(k). \end{aligned} \tag{22}$$

By analogy to (4), $\boldsymbol{x}_\beta(k)$ contains $n_\beta + 1$ elements, marked from $x_{\beta 1}$ to $x_{\beta n_\beta+1}$. As a result, the system's state matrix $\boldsymbol{A}_\beta$ is constructed as shown in (5), with the first row elements denoted from $a_{\beta 1}$ to $a_{\beta n_\beta}$. The system's control signal is $u_\beta(k)$ and the input vector is $\boldsymbol{b}$, as shown in (6). The sales are represented by $h_\beta(k)$ and vector $\boldsymbol{f}$ remains as defined in (7). The initial stock is represented by $y_{\beta 0} = x_{\beta 1}(0)$ and the current stock level is $y_\beta(k)$.

The demand profile of warehouse β, $d_\beta(k)$, is random but bounded:

$$d_{\beta\,\min} \le d_\beta(k) \le d_{\beta\,\max}, \tag{23}$$

with $d_{\beta\,\min}$ and $d_{\beta\,\max}$ being the lower and upper bounds of the demand, respectively. Moreover, the suppliers of warehouse β have limited delivery capacity, which results in a maximum order value denoted as $u_{\beta\,\max}$. Therefore,

$$u_\beta(k) \le u_{\beta\,\max} \tag{24}$$

for any $k \ge 0$. The goal is to fulfill the demand, which may, at times, exceed the order limit, i.e.,

$$u_{\beta\,\max} < d_{\beta\,\max}. \tag{25}$$

To overcome this problem, additional deliveries from warehouse α are necessary. These situations will be considered emergencies and the appropriate variables are denoted with the suffix e. The term $h_e(k)$ represents emergency deliveries from warehouse α. The purpose of the term $+\boldsymbol{f} h_e(k)$, in (22), is to allow the deliveries from warehouse α, $h_e(k)$, to enter the stock of warehouse β as fast as possible, i.e., after one discrete time period only.

The sales in the system satisfy

$$h_\beta(k) \le d_\beta(k) \tag{26}$$

to account for the possibility of the warehouse being unable to fully satisfy the demand. The actual sales value is calculated as

$$h_\beta(k) = \min[y_\beta(k), d_\beta(k)]. \tag{27}$$

From (22), the stock level, at any $k \le n_\beta$, is obtained as

$$\begin{aligned} y_\beta(k) = y_{\beta 0} &+ \sum_{i=0}^{k-1} h_e(i) \\ &+ \sum_{i=1}^{n\beta-1} a_{\beta i} \sum_{j=0}^{k-i-1} u_\beta(j) - \sum_{i=0}^{k-1} h_\beta(i), \end{aligned} \tag{28}$$

and for any $k \ge n_\beta + 1$ as

$$\begin{aligned} y_\beta(k) = y_{\beta 0} &+ \sum_{i=0}^{k-1} h_e(i) + \sum_{i=0}^{k-n_\beta-1} u_\beta(i) \\ &+ \sum_{i=1}^{n\beta-1} a_{\beta i} \sum_{j=k-n_\beta}^{k-i-1} u_\beta(j) - \sum_{i=0}^{k-1} h_\beta(i). \end{aligned} \tag{29}$$

Finally, after all sales have been accounted for, the remaining stock may be obtained from

$$y_{\beta r}(k) = y_\beta(k) - h_\beta(k). \tag{30}$$

3. Control strategy

In this section, two separate SMC based ordering strategies are designed—one with a desired trajectory generator for warehouse α and a simple order up to control scheme for warehouse β. The controller's goal is to ensure full demand satisfaction in both warehouses under the assumption that the unknown demand in the second one exceeds its ordering capabilities. In other words, the control process is designed so that no sale losses occur, even in the worst case scenario, i.e., if the random demand is continuously at its maximum admissible value.

In general, SMC is based on a predefined sliding surface, which contains the demand position. The distance between the system's representative point and the sliding surface is described by the sliding variable, denoted by $s(k)$, where

$$s(k) = \boldsymbol{c}\boldsymbol{x}(k). \tag{31}$$

When the system's trajectory belongs to the sliding surface, then $s(k) = 0$. The choice of vector $\boldsymbol{c}$, introduced in (31), is crucial to ensure stable system performance. Therefore, in our work, we choose $\boldsymbol{c}$ so that all eigenvalues z_i of the inventory system are placed at the origin of the complex plane. This is ensured when $\boldsymbol{c}$ satisfies

$$\boldsymbol{c} = \left[1 \quad a_n \quad \sum_{i=n-1}^{n} a_i \quad \dots \quad \sum_{i=2}^{n} a_i \quad \sum_{i=1}^{n} a_i\right], \tag{32}$$

as proposed by Bartoszewicz and Leśniewski (2014). This definition of vector $\boldsymbol{c}$ results in additional desirable features of the control system.

Lemma 1. *If vector* $\boldsymbol{c}$ *for the system (3) is defined according to (32), then*

$$\boldsymbol{c}\boldsymbol{A} = \boldsymbol{c}, \quad \boldsymbol{c}\boldsymbol{b} = 1, \quad \boldsymbol{c}\boldsymbol{f} = 1. \tag{33}$$

Proof. The above may be obtained with a straightforward multiplication, considering that vector $\boldsymbol{c}$ has 1 as the first element, (2) holds, and matrix $\boldsymbol{A}$, vectors $\boldsymbol{b}$ and $\boldsymbol{f}$ are defined according to (5), (6) and (7), respectively. ■

Next, we present the control strategies designed for the specific needs of warehouses α and β. It is also worth mentioning that the SMC laws presented in the following sections do not introduce any chattering to the inventory system due to their non-switching nature.

3.1. Warehouse α.

The first warehouse operates under an *a priori* known contractual demand and a random but bounded demand. Moreover, in the emergency situations, warehouse α acts as an additional supplier for warehouse β. The warehouse's goal is to prioritize the known contractual buyers, while retaining some stock for the aforementioned emergency and random buyers. Therefore, we propose to employ a desired trajectory generator based on the known contractual demand profile and an SM controller to fulfill its contract.

The warehouse is described in Section 2.1., with its dynamics defined by (8). The control signal $u_\alpha(k)$ is a scalar amount of goods ordered by the controller at the time instant k. This value will enter the state vector in the time instant $k+1$ as its last state variable $\boldsymbol{x}_{\alpha n_\alpha+1}(k+1)$, which can be inferred from the system's state matrix and its input vector. The vector $\boldsymbol{f}$ and the value $h_\alpha(k)$ represent the sales in the system, with $h_\alpha(k)$ being the sum of the fulfilled contractual obligations $h_c(k)$, the occasional emergency deliveries to warehouse β, $h_e(k)$, and the random sales $h_r(k)$. The control strategy for warehouse α uses a desired trajectory generator to achieve its goals. Hence, we continue with the generation of a trajectory that fulfills the contractual sales in the warehouse. We already know the future values of the contract, so the desired trajectory $s_{\alpha d}(k)$ will contain the sum of the values of the contract for all the $n_\alpha + 1$ future time instants. In other words,

$$s_{\alpha d}(k) = \sum_{l=k+1}^{k+n_\alpha+1} \widetilde{d}_c(l), \tag{34}$$

where, for $k = 0$,

$$\begin{aligned} &s_{\alpha d}(0) \\ &= \widetilde{d}_c(1) + \widetilde{d}_c(2) + \dots + \widetilde{d}_c(n_\alpha) + \widetilde{d}_c(n_\alpha + 1). \end{aligned} \tag{35}$$

The sliding plane for warehouse α is defined as $s_\alpha(k) = 0$, and the position of the representative point of the inventory system relative to the desired trajectory is denoted with the sliding variable $s_\alpha(k)$. It is assumed that at the time instant $k = 0$ the sliding variable equals zero, so the initial condition of the system $y_{\alpha 0}$ is

$$y_{\alpha 0} = x_{\alpha 1}(0) = s_{\alpha d}(0), \tag{36}$$

with $s_\alpha(0) = 0$. Hence, the initial stock present in the warehouse at the time instant $k = 0$ must be enough to satisfy the contractual demand up to the time instant $n_\alpha + 1$ when the slowest supplier delivers the product ordered at $k = 0$. This value must be further modified due to the possibility of the emergency sales $h_e(k)$ and random sales $h_r(k)$ appearing from the very beginning of the control process. The reserve of the product for both demand terms will be added to the initial value with a compensation term. The trajectory of the system $\boldsymbol{c}_\alpha \boldsymbol{x}_\alpha(k)$ shall follow the generated reference trajectory $s_{\alpha d}(k)$. We achieve that with

$$s_\alpha(k+1) = s_{\alpha d}(k+1) - \boldsymbol{c}_\alpha \boldsymbol{x}_\alpha(k+1) = 0, \tag{37}$$

where the control vector $\boldsymbol{c}_\alpha$ is chosen according to (32). Inserting the state equation (8) into the above reaching law yields the control law for the warehouse:

$$u_\alpha(k) = (\boldsymbol{c}_\alpha \boldsymbol{b})^{-1}[s_{\alpha d}(k+1) - \boldsymbol{c}_\alpha \boldsymbol{A}_\alpha \boldsymbol{x}_\alpha(k) + \boldsymbol{c}_\alpha \boldsymbol{f} h_\alpha(k)]. \tag{38}$$

The controller does not know the value of sales $h_\alpha(k)$ at the moment of calculating the control signal

$u_\alpha(k)$. However, the aim of the controller is to provide a sufficient amount of goods to satisfy all demand terms at any time instant k. Therefore, the worst possible scenario must be considered when both emergency and random demands assume their maximum admissible values. Considering the expected value of the contractual demand and the upper bounds of both the emergency demand from warehouse β and a random demand, we substitute $h_\alpha(k)$ with its maximum possible value at time instant k:

$$h_\alpha(k) = \widetilde{d}_c(k) + d_{\beta\max} - u_{\beta\max} + d_{r\max}, \tag{39}$$

where $d_{\beta\max}$ and $u_{\beta\max}$ are the upper bounds of the random demand and the maximum order quantity in warehouse β. The difference between those two values is the maximum possible amount of product needed by warehouse β at a single time instant k, denoted by $d_{\beta\max} - u_{\beta\max} = d_{e\max}$. Additionally, we must secure a reserve of product for both warehouse β and the random sales, bounded by $d_{r\max}$. Therefore, we must ensure that both values are always present in warehouse α after its contractual obligations are fulfilled. We define the sum of those maximum values as $d_{e\max} + d_{r\max} = d_{er\max}$ to simplify the notation. Moreover, in order to avoid any sale losses, the delivery delay in the system must be considered. Thus, the reserve products must be ordered in advance. The goods ordered by the controller at instant k arrive in the warehouse in full at the time instant $k + n_\alpha + 1$, so a compensation term for the whole period of $n_\alpha + 1$ time instants is necessary. Therefore, we define the compensation vector $\boldsymbol{D_{er}}$ with

$$\boldsymbol{D_{er}} = \begin{bmatrix}1 & 1 & \dots & 1 & 1\end{bmatrix}^T_{n_\alpha+1} d_{er\max}. \tag{40}$$

Upon including the compensation term in the control law, the control signal becomes

$$\begin{aligned} u_\alpha(k) = (\boldsymbol{c}_\alpha \boldsymbol{b})^{-1}\{ & s_{\alpha d}(k+1) - \boldsymbol{c}_\alpha \boldsymbol{A}_\alpha \boldsymbol{x}_\alpha(k) \\ & + \boldsymbol{c}_\alpha \boldsymbol{f}[\widetilde{d}_c(k) + d_{er\max}] + \boldsymbol{c}_\alpha \boldsymbol{D_{er}}\}. \end{aligned} \tag{41}$$

We also modify the initial conditions of the system to allow both the emergency deliveries and random demand to be fulfilled at the time instants $k = 0, 1, 2, \dots, n_\alpha$. Therefore,

$$y_{\alpha 0} = x_{\alpha 1}(0) = s_{\alpha d}(0) + \boldsymbol{c}_\alpha \boldsymbol{D_{er}}. \tag{42}$$

Additionally, the control law above keeps the representative point of the system in the vicinity of the sliding surface for any $k \geq 0$, with

$$\begin{aligned} & |s_\alpha(k)| \\ & = |s_{\alpha d}(k) - \boldsymbol{c}_\alpha \boldsymbol{x}_\alpha(k)| \leq \boldsymbol{c}_\alpha \boldsymbol{f} d_{er\max} + \boldsymbol{c}_\alpha \boldsymbol{D_{er}}. \end{aligned} \tag{43}$$

We continue with the proof that neither random sales nor emergency deliveries prevent warehouse α from fulfilling its contractual obligations.

Theorem 1. *Applying the control law (41) to warehouse α with the initial condition (42) ensures that, for any $k \geq 0$, the control signal*

$$u_\alpha(k) \leq \widetilde{d}_c(k + n_\alpha + 1) + d_{er\max} \tag{44}$$

and the stock level

$$y_\alpha(k) \geq \widetilde{d}_c(k) + d_{er\max}. \tag{45}$$

Hence

$$h_c(k) = d_c(k) \quad \textit{and} \quad y_{\alpha r1}(k) \geq d_{er\max}. \tag{46}$$

Therefore, the warehouse fully satisfies all the demand terms present in the system.

Proof. Consider the worst case scenario for the demand in the warehouse, with both the emergency and random demands at their upper bounds $d_e(k) = d_{e\max}$ and $d_r(k) = d_{r\max}$. Taking Lemma 1 into consideration, the control (41) becomes

$$\begin{aligned} u_\alpha(k) = & \, s_{\alpha d}(k+1) - \boldsymbol{c}_\alpha \boldsymbol{x}_\alpha(k) \\ & + \widetilde{d}_c(k) + d_{er\max} + \boldsymbol{c}_\alpha \boldsymbol{D_{er}}, \end{aligned} \tag{47}$$

with the compensation term $\boldsymbol{c}_\alpha \boldsymbol{D_{er}}$ being

$$\begin{aligned} \boldsymbol{c}_\alpha \boldsymbol{D_{er}} = & \, d_{er\max} + a_{\alpha n_\alpha} d_{er\max} d_{er\max} \\ & + \sum_{i=n_\alpha-1}^{n_\alpha} a_{\alpha i} + \dots + \sum_{i=2}^{n_\alpha} a_{\alpha i} d_{er\max} \\ & + d_{er\max} \geq 2 d_{er\max}, \end{aligned} \tag{48}$$

after considering both (32) and (40). The initial condition (42) results in the following control signal at $k = 0$:

$$\begin{aligned} u_\alpha(0) = & \sum_{i=1}^{n_\alpha+1} \widetilde{d}_c(i) - \sum_{i=0}^{n_\alpha} \widetilde{d}_c(i) - \boldsymbol{c}_\alpha \boldsymbol{D_{er}} \\ & + \widetilde{d}_c(0) + d_{er\max} + \boldsymbol{c}_\alpha \boldsymbol{D_{er}} \\ = & \, \widetilde{d}_c(n_\alpha + 1) + d_{er\max}. \end{aligned} \tag{49}$$

In the case considered, the demand is always at its maximum, so $\widetilde{d}_c(k) - d_c(k) = 0$ and the sales satisfy $h_c(0) = \widetilde{d}_c(0)$. The initial condition provides enough product to satisfy the demand at $k = 0$, so the state vector at $k = 1$ becomes

$$\boldsymbol{x}_\alpha(1) = \begin{bmatrix} \sum_{i=1}^{n_\alpha} \widetilde{d}_c(i) + \boldsymbol{c}_\alpha \boldsymbol{D_{er}} - d_{er\max} \\ \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} n_\alpha - 1 \\ \widetilde{d}_c(n_\alpha + 1) + d_{er\max} \end{bmatrix}_{n_\alpha+1} \tag{50}$$

with

$$\begin{aligned} &\boldsymbol{c}_\alpha \boldsymbol{D_{er}} - d_{er\max} \\ &= a_{\alpha n_\alpha} d_{er\max} + \sum_{i=n_\alpha-1}^{n_\alpha} a_{\alpha i} d_{er\max} + \dots \\ &\quad + \sum_{i=2}^{n_\alpha} a_{\alpha i} d_{er\max} + d_{\alpha er\max} \geq d_{er\max}. \end{aligned} \tag{51}$$

We continue with the control for $k = 1$:

$$\begin{aligned} u_\alpha(1) = &\sum_{i=2}^{n_\alpha+2} \widetilde{d}_c(i) - \sum_{i=1}^{n_\alpha} \widetilde{d}_c(i) - \boldsymbol{c}_\alpha \boldsymbol{D_{er}} \\ &+ d_{er\max} - \widetilde{d}_c(n_\alpha+1) - d_{er\max} \\ &+ \widetilde{d}_c(1) + d_{er\max} + \boldsymbol{c}_\alpha \boldsymbol{D_{er}} \\ = &\widetilde{d}_c(1) - \widetilde{d}_c(1) + \widetilde{d}_c(n_\alpha+2) \\ &+ d_{er\max}. \end{aligned} \tag{52}$$

Finally,

$$u_\alpha(1) = \widetilde{d}_c(n_\alpha+2) + d_{er\max}. \tag{53}$$

The stock once again satisfies all the demand, so $h_c(1) = \widetilde{d}_c(1)$ and

$$\boldsymbol{x}_\alpha(2) = \begin{bmatrix} y_\alpha(2) \\ \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} n_\alpha - 2 \\ \widetilde{d}_c(n_\alpha+1) + d_{er\max} \\ \widetilde{d}_c(n_\alpha+2) + d_{er\max} \end{bmatrix}_{n_\alpha+1}, \tag{54}$$

where

$$\begin{aligned} y_\alpha(2) = &\sum_{i=2}^{n_\alpha} \widetilde{d}_c(i) + \boldsymbol{c}_\alpha \boldsymbol{D_{er}} - 2d_{er\max} \\ &+ a_{\alpha 1}\widetilde{d}_c(n_\alpha+1) + a_{\alpha 1} d_{er\max}. \end{aligned} \tag{55}$$

We will use $g(k)$ to denote the terms compensating for both the emergency and random demand, with

$$\begin{aligned} g(2) &= \boldsymbol{c}_\alpha \boldsymbol{D_{er}} - 2d_{er\max} + a_{\alpha 1} d_{er\max} \\ &= a_{\alpha n_\alpha} d_{er\max} + \sum_{i=n_\alpha-1}^{n_\alpha} a_{\alpha i} d_{er\max} \\ &\quad + \dots + \sum_{i=2}^{n_\alpha} a_{\alpha i} d_{er\max} + a_{\alpha 1} d_{er\max}. \end{aligned} \tag{56}$$

With (2), we obtain

$$\begin{aligned} g(2) = &a_{\alpha n_\alpha} d_{er\max} + \sum_{i=n_\alpha-1}^{n_\alpha} a_{\alpha i} d_{er\max} + \dots \\ &+ \sum_{i=3}^{n_\alpha} a_{\alpha i} d_{er\max} \\ &+ \underbrace{\sum_{i=2}^{n_\alpha} a_{\alpha i} d_{er\max} + a_{\alpha 1} d_{er\max}}_{d_{er\max}} \\ \geq &\, d_{er\max}, \end{aligned} \tag{57}$$

which means that the stock is sufficient, so $h_c(2) = \widetilde{d}_c(2)$. Next, the control for $k = 2$ is

$$\begin{aligned} u_\alpha(2) = &\sum_{i=3}^{n_\alpha+3} \widetilde{d}_c(i) - \sum_{i=2}^{n_\alpha} \widetilde{d}_c(i) + 2d_{er\max} \\ &- a_{\alpha 1}[\widetilde{d}_c(n_\alpha+1) + d_{er\max}] \\ &- \sum_{i=2}^{n_\alpha} a_{\alpha i}[\widetilde{d}_c(n_\alpha+1) + d_{er\max}] \\ &- \widetilde{d}_c(n_\alpha+2) + \widetilde{d}_c(2), \end{aligned} \tag{58}$$

which results in

$$u_\alpha(2) = \widetilde{d}_c(n_\alpha+3) + d_{er\max}. \tag{59}$$

The state vector becomes

$$\boldsymbol{x}_\alpha(3) = \begin{bmatrix} y_\alpha(3) \\ \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} n_\alpha - 3 \\ \widetilde{d}_c(n_\alpha+1) + d_{er\max} \\ \widetilde{d}_c(n_\alpha+2) + d_{er\max} \\ \widetilde{d}_c(n_\alpha+3) + d_{er\max} \end{bmatrix}_{n_\alpha+1}, \tag{60}$$

where

$$\begin{aligned} y_\alpha(3) = &\sum_{i=3}^{n_\alpha} \widetilde{d}_c(i) + \sum_{i=1}^{2} a_{\alpha i}\widetilde{d}_c(n_\alpha+1) \\ &+ a_{\alpha 1}\widetilde{d}_c(n_\alpha+2) + g(3), \end{aligned} \tag{61}$$

and

$$\begin{aligned} g(3) &= g(2) - d_{er\max} + \sum_{i=1}^{2} a_{\alpha i} d_{er\max} \\ &= a_{\alpha n_\alpha} d_{er\max} + \sum_{i=n_\alpha-1}^{n_\alpha} a_{\alpha i} d_{er\max} + \dots \\ &\quad + \sum_{i=4}^{n_\alpha} a_{\alpha i} d_{er\max} + \underbrace{\sum_{i=3}^{n_\alpha} a_{\alpha i} d_{er\max} + \sum_{i=1}^{2} a_{\alpha i} d_{er\max}}_{d_{er\max}} \\ &\geq d_{er\max}, \end{aligned} \tag{62}$$

so the stock is sufficient to satisfy all the demand. Considering all the above, we can say that the state vector at any moment $0 \le k \le n_\alpha$ becomes

$$\boldsymbol{x}_\alpha(k) = \begin{bmatrix} y_\alpha(k) \\ \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} n_\alpha - k \\ \widetilde{d}_c(n_\alpha+1)+d_{er\max} \\ \vdots \\ \widetilde{d}_c(n_\alpha+k)+d_{er\max} \end{bmatrix}_{n_\alpha+1}, \tag{63}$$

and the control signal for any $k \ge 0$ is

$$u_\alpha(k) = \widetilde{d}_c(k+n_\alpha+1)+d_{er\max}. \tag{64}$$

For $k = n_\alpha$, the state vector is

$$\boldsymbol{x}_\alpha(n_\alpha) = \begin{bmatrix} y_\alpha(n_\alpha) \\ \widetilde{d}_c(n_\alpha+1)+d_{er\max} \\ \vdots \\ \widetilde{d}_c(n_\alpha+n_\alpha)+d_{er\max} \end{bmatrix}_{n_\alpha+1}, \tag{65}$$

where

$$y_\alpha(n_\alpha) = \widetilde{d}_c(n_\alpha) + \sum_{i=1}^{n_\alpha-1} a_{\alpha i} \sum_{j=n_\alpha+1}^{n_\alpha+n_\alpha-i} \widetilde{d}_c(j) + g(n_\alpha), \tag{66}$$

and

$$g(n_\alpha) = a_{\alpha n_\alpha} d_{er\max} + \sum_{i=1}^{n_\alpha-1} a_{\alpha i} d_{er\max} = d_{er\max}. \tag{67}$$

Considering the above, the stock level for any $0 \le k \le n_\alpha$ is

$$y_\alpha(k) = \sum_{i=k}^{n_\alpha} \widetilde{d}_c(i) + \sum_{i=1}^{n_\alpha-1} a_{\alpha i} \sum_{j=n_\alpha+1}^{k+n_\alpha-i} \widetilde{d}_c(j) + g(k), \tag{68}$$

where

$$\begin{aligned} g(k) &= \underbrace{\sum_{i=k}^{n_\alpha} a_{\alpha i} d_{er\max} + \sum_{i=1}^{k-1} a_{\alpha i} d_{er\max}}_{d_{er\max}} \\ &\quad + \sum_{i=k+1}^{n_\alpha} a_{\alpha i} d_{er\max} + \sum_{i=k+2}^{n_\alpha} a_{\alpha i} d_{er\max} + \dots \\ &\quad + \sum_{i=n_\alpha-1}^{n_\alpha} a_{\alpha i} d_{er\max} + a_{\alpha n_\alpha} d_{er\max} \\ &\ge d_{er\max}. \end{aligned} \tag{69}$$

We conclude that all the demand in warehouse α is fully satisfied for $0 \le k \le n_\alpha$. Thus

$$u_\alpha(n_\alpha+1) = \widetilde{d}_c(n_\alpha+n_\alpha+1)+d_{er\max}, \tag{70}$$

and the state vector is

$$\boldsymbol{x}_\alpha(n_\alpha+1) = \begin{bmatrix} y_\alpha(n_\alpha+1) \\ \widetilde{d}_c(n_\alpha+2)+d_{er\max} \\ \vdots \\ \widetilde{d}_c(n_\alpha+n_\alpha+1)+d_{er\max} \end{bmatrix}_{n_\alpha+1} \tag{71}$$

for $k = n_\alpha+1$, where

$$\begin{aligned} y_\alpha(n_\alpha+1) &= \sum_{i=1}^{n_\alpha} a_{\alpha i} \widetilde{d}_c(n_\alpha+1) \\ &\quad + \sum_{i=1}^{n_\alpha-1} a_{\alpha i} \sum_{j=n_\alpha+2}^{n_\alpha+n_\alpha+1-i} \widetilde{d}_c(j) + g(n_\alpha+1), \end{aligned} \tag{72}$$

and

$$g(n_\alpha+1) = \sum_{i=1}^{n_\alpha} a_{\alpha i} d_{er\max}. \tag{73}$$

As (2) holds, the stock level for $k = n_\alpha+1$ is

$$\begin{aligned} y_\alpha(n_\alpha+1) &= \widetilde{d}_c(n_\alpha+1) \\ &\quad + \sum_{i=1}^{n_\alpha-1} a_{\alpha i} \sum_{j=n_\alpha+2}^{n_\alpha+n_\alpha+1-i} \widetilde{d}_c(j) \\ &\quad + d_{er\max}, \end{aligned} \tag{74}$$

and, for any $k \ge n_\alpha+1$,

$$y_\alpha(k) = \widetilde{d}_c(k) + \sum_{i=1}^{n_\alpha-1} a_{\alpha i} \sum_{j=k+1}^{k+n_\alpha-i} \widetilde{d}_c(j) + d_{er\max}. \tag{75}$$

As the control signal $u_\alpha(k)$ satisfies (64), the stock level is

$$y_\alpha(k) \ge \widetilde{d}_c(k) + d_{er\max}, \tag{76}$$

where $d_{er\max} = d_{e\max} + d_{r\max}$, which concludes the proof. ■

From (76) it is clear that all the ingredients of the demand ($d_c(k)$, $d_e(k)$ and $d_r(k)$) are fully satisfied within their ranges for any $k \ge 0$.

However, the amount of goods in stock, (75), shows a slight surplus. That is caused by the fact that some of the suppliers conclude their deliveries earlier than after $n_\alpha+1$ steps. Those early deliveries may be used to satisfy the contract at current steps. Taking that into account, we introduce a compensation term γ intended to reduce the necessary stock level to minimum, without compromising the contract, such that

$$\gamma = \min_{k\to\infty} \left[s_{\alpha d}(k) - \boldsymbol{c}_\alpha \boldsymbol{x}_{\alpha d}(k)\right], \tag{77}$$

where $\boldsymbol{x}_{\alpha d}(k)$ is comprised of the subsequent contract values:

$$\boldsymbol{x}_{\alpha d}(k) = \left[\widetilde{d}_c(k) \;\; \widetilde{d}_c(k+1) \;\; \dots \;\; \widetilde{d}_c(k+n_\alpha)\right]^{\mathrm{T}}. \tag{78}$$

Utilizing (77), the control signal (41) becomes

$$\begin{aligned} u_\alpha(k) = (\boldsymbol{c}_\alpha \boldsymbol{b})^{-1}\{&s_{\alpha d}(k+1) - \boldsymbol{c}_\alpha \boldsymbol{A}_\alpha \boldsymbol{x}_\alpha(k) \\ &+ \boldsymbol{c}_\alpha \boldsymbol{f}[\widetilde{d}_c(k) + d_{er\max}] + \boldsymbol{c}_\alpha \boldsymbol{D_{er}} - \gamma\}. \end{aligned} \tag{79}$$

In the same manner, the system's initial condition may be reduced to

$$y_{\alpha 0} = x_{\alpha 1}(0) = s_{\alpha d}(0) + \boldsymbol{c}_\alpha \boldsymbol{D_{er}} - \gamma. \tag{80}$$

The above compensation term allows reducing the amount of product stored, which minimizes the warehouse space and increases its efficiency.

3.2. Warehouse β.

The dynamics of warehouse β are presented in (22). The inventory system is subject to bounded market demand $d_\beta(k)$, as shown in (23). Moreover, there is a control limit $u_{\beta\max}$ imposed on the system, caused by limited capabilities of its suppliers, as shown in (24). According to (25), there may be instants when the demand exceeds the maximum order value. In that case, the emergency demand $d_e(k)$ will be generated and sent as a request to warehouse α for an emergency delivery, denoted by $h_e(k)$. Therefore, the second warehouse will be controlled with a simple order up to control strategy, bolstered by the emergency deliveries.

First, we define the desired stock level in warehouse β as $y_{\beta d}$. This value is a positive constant, derived further in this section. The controller's goal is to provide the $y_{\beta d}$ amount of products in stock at any $k \geq 0$. Therefore, the demand state vector becomes

$$\boldsymbol{x}_{\beta d} = \left[y_{\beta d} \quad 0 \quad \dots \quad 0 \quad 0\right]^{\mathrm{T}}. \tag{81}$$

Next, we design a simple discrete-time SM controller. We define the sliding hyperplane as

$$s_\beta(k) = \boldsymbol{c}_\beta \left[\boldsymbol{x}_{\beta d} - \boldsymbol{x}_\beta(k)\right] = 0, \tag{82}$$

where vector $\boldsymbol{c}_\beta$ is defined according to (32) and $\boldsymbol{c}_\beta \boldsymbol{x}_{\beta d} = y_{\beta d}$. The current value of $s_\beta(k)$ describes the distance between the system's representative point and the switching surface at any time. We assume that the initial stock of warehouse β exactly matches the demand value. Therefore,

$$y_\beta(0) = y_{\beta 0} = y_{\beta d}, \tag{83}$$

and the representative point belongs to the sliding hyperplane (82), i.e., $s_\beta(0) = 0$. We propose to derive an SM equivalent control law that would keep the system on the sliding hyperplane with a simple nonswitching reaching law of Drakunov and Utkin (1989), by assigning

$$s_\beta(k+1) = y_{\beta d} - \boldsymbol{c}_\beta \boldsymbol{x}_\beta(k+1) = 0. \tag{84}$$

From the state equation (22) and the above reaching law (84), we obtain the following equivalent control $u_{\beta eq}(k)$:

$$\begin{aligned} u_{\beta eq}(k) = (\boldsymbol{c}_\beta \boldsymbol{b})^{-1}&\left[y_{\beta d} - \boldsymbol{c}_\beta \boldsymbol{A}_\beta \boldsymbol{x}_\beta(k)\right. \\ &\left.+\boldsymbol{c}_\beta \boldsymbol{f} h_\beta(k) - \boldsymbol{c}_\beta \boldsymbol{f} h_e(k)\right]. \end{aligned} \tag{85}$$

As the demand is *a priori* unknown and bounded by (23), we consider the worst possible scenario, when $h_\beta(k) = d_{\beta\max}$. Furthermore, we assume that in the ideal case emergency deliveries do not exist, i.e., $h_e(k) = 0$. Consequently, considering Lemma 1, the equivalent control (85) simplifies to

$$u_{\beta eq}(k) = y_{\beta d} - \boldsymbol{c}_\beta \boldsymbol{x}_\beta(k) + d_{\beta\max}. \tag{86}$$

However, in the system considered, the order volume generated by the controller is upper bounded by the maximum suppliers' capabilities, as defined in (24). Thus, there may exist such time instants when $u_{\beta eq}(k) > u_{\beta\max}$. At those instants $u_{\beta eq}(k)$ is not applicable and the emergency demand $d_e(k)$ will be generated. Therefore, for warehouse β we define the control law as follows:

$$u_\beta(k) = \begin{cases} u_{\beta eq}(k) & \text{for } u_{\beta eq}(k) \leq u_{\beta\max}, \\ u_{\beta\max} & \text{for } u_{\beta eq}(k) > u_{\beta\max}. \end{cases} \tag{87}$$

Furthermore, the controller will generate the emergency demand $d_e(k)$ according to the following rule

$$d_e(k) = \begin{cases} 0 & \text{when } u_\beta(k) \leq u_{\beta\max}, \\ u_{\beta eq}(k) - u_{\beta\max} & \text{when } u_\beta(k) > u_{\beta\max}. \end{cases} \tag{88}$$

From (25) and (86), it follows that the emergency demand is upper bounded by

$$d_e(k) \leq d_{e\max} = d_{\beta\max} - u_{\beta\max}. \tag{89}$$

The emergency signal $d_e(k)$ is sent as a request for lacking good delivery to warehouse α. If the stock of warehouse α is sufficient to fulfill the demand, then the emergency delivery in the volume of $h_e(k)$ arrives at warehouse β at one time instant, as defined in the state equation (22). Next, we prove that the proposed control strategy ensures full demand satisfaction, even in the presence of control limitation.

Theorem 2. *If the demand stock level for the system (22) satisfies*

$$\begin{aligned} y_{\beta d} &= (n_\beta + 1)d_{\beta\max} + n_\beta d_{e\max} \\ &\quad - \sum_{i=1}^{n_\beta-1} a_{\beta i}u_{\beta\max} - \sum_{i=1}^{n_\beta-2} a_{\beta i}u_{\beta\max} \\ &\quad - \cdots - \sum_{i=1}^{2} a_{\beta i}u_{\beta\max} - a_{\beta 1}u_{\beta\max}, \end{aligned} \tag{90}$$

then the control law (87), with the emergency demand generated according to (88) and matching deliveries $h_e(k)$, guarantees that

$$y_\beta(k) \geq d_{\beta\max} \tag{91}$$

for any $k \geq 0$.

Proof. We begin by arguing that warehouse α is able to fulfill the emergency demand $d_e(k)$, within its maximum range $d_{e\max}$ (89), at any $k \geq 0$, as shown in (76). Next, we will carry out the proof assuming the worst admissible conditions, i.e., $d_\beta(k) = d_{\beta\max}$ at each instant and, consequently, $d_\beta(k) > u_{\beta\max}$.

Assuming that the system's initial condition satisfies (83), the state vector at $k = 0$ becomes

$$\boldsymbol{x}_\beta(0) = \begin{bmatrix} y_{\beta d} & 0 & \dots & 0 & 0 \end{bmatrix}^{\mathrm{T}}_{n_\beta+1}. \tag{92}$$

We calculate the equivalent control at $k = 0$:

$$u_{\beta eq}(0) = y_{\beta d} - \underbrace{\boldsymbol{c}_\beta \boldsymbol{x}_\beta(0)}_{y_{\beta d}=y_{\beta 0}} + d_{\beta\max} = d_{\beta\max}. \tag{93}$$

Therefore, according to (87),

$$u_\beta(0) = u_{\beta\max}, \tag{94}$$

and the emergency demand is generated according to (88), that is,

$$d_e(0) = d_{e\max}. \tag{95}$$

As the worst case is considered, $d_\beta(0) = d_{\beta\max}$. Taking into account (83), $y_\beta(0) > d_{\beta\max}$, so the stock is sufficient to satisfy the demand. Therefore,

$$h_\beta(0) = d_{\beta\max}. \tag{96}$$

Moreover, as shown in the previous subsection, warehouse α is able to satisfy $d_e(k)$ at any $k \geq 0$. Consequently, at $k = 1$, warehouse β receives an emergency delivery:

$$h_e(0) = d_{e\max}. \tag{97}$$

Therefore, at $k = 1$, the state vector becomes

$$\boldsymbol{x}_\beta(1) = \begin{bmatrix} y_{\beta d} - d_{\beta\max} + d_{e\max} \\ \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} n_\beta - 1 \\ u_{\beta\max} \end{bmatrix}_{n_\beta+1}. \tag{98}$$

Next, we calculate the equivalent control at $k = 1$:

$$\begin{aligned} u_{\beta eq}(1) &= y_{\beta d} - \boldsymbol{c}_\beta \boldsymbol{x}_\beta(1) + d_{\beta\max} \\ &= y_{\beta d} - y_{\beta d} + d_{\beta\max} \\ &\quad \underbrace{-d_{e\max} - u_{\beta\max}}_{-d_{\beta\max}} + d_{\beta\max} = d_{\beta\max}. \end{aligned} \tag{99}$$

The same situation as for $k = 0$ occurs. The control $u_\beta(1) = u_{\beta\max}$ and $d_e(1) = d_{e\max}$. From (83), (90) and (98), we obtain the current stock level as

$$\begin{aligned} y_\beta(1) &= n_\beta d_{\beta\max} + (n_\beta + 1)d_{e\max} \\ &\quad - \sum_{i=1}^{n_\beta-1} a_{\beta i}u_{\beta\max} - \sum_{i=1}^{n_\beta-2} a_{\beta i}u_{\beta\max} - \dots \\ &\quad - \sum_{i=1}^{2} a_{\beta i}u_{\beta\max} - a_{\beta 1}u_{\beta\max}. \end{aligned} \tag{100}$$

Therefore, the stock satisfies $y_\beta(1) > d_{\beta\max}$. Consequently, $h_\beta(1) = d_{\beta\max}$. After the sales, the state vector at $k = 2$ becomes

$$\boldsymbol{x}_\beta(2) = \begin{bmatrix} y_{\beta d} - 2d_{\beta\max} + 2d_{e\max} + a_{\beta 1}u_{\beta\max} \\ \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} n_\beta - 2 \\ u_{\beta\max} \\ u_{\beta\max} \end{bmatrix}_{n_\beta+1}. \tag{101}$$

Substituting $y_{\beta d}$ from (90), we obtain the stock level as

$$\begin{aligned} y_\beta(2) &= (n_\beta - 1)d_{\beta\max} + (n_\beta + 2)d_{e\max} \\ &\quad - \sum_{i=1}^{n_\beta-1} a_{\beta i}u_{\beta\max} - \sum_{i=1}^{n_\beta-2} a_{\beta i}u_{\beta\max} \\ &\quad - \cdots - \sum_{i=1}^{2} a_{\beta i}u_{\beta\max}. \end{aligned} \tag{102}$$

We continue with $u_{\beta eq}(2)$,

$$\begin{aligned} u_{\beta eq}(2) &= y_{\beta d} - \boldsymbol{c}_\beta \boldsymbol{x}_\beta(2) + d_{\beta\max} \\ &= y_{\beta d} - y_{\beta d} + 2d_{\beta\max} - 2d_{e\max} - u_{\beta\max} \\ &\quad \underbrace{-a_{\beta 1}u_{\beta\max} - \sum_{2}^{n_\beta} a_{\beta i}u_{\beta\max} + d_{\beta\max}}_{-u_{\beta\max}} \\ &= d_{\beta\max}. \end{aligned} \tag{103}$$

We obtain $u_\beta(2) = u_{\beta\max}$ and $d_e(2) = d_{e\max}$. The stock level satisfies $y_\beta(2) \geq d_{\beta\max}$. Thus, $h_\beta(2) = d_{\beta\max}$.

Next, the state vector at $k = 3$ becomes

$$\boldsymbol{x}_\beta(3) = \begin{bmatrix} y_\beta(3) \\ \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} n_\beta - 3 \\ u_{\beta\max} \\ u_{\beta\max} \\ u_{\beta\max} \end{bmatrix}_{n_\beta+1}, \tag{104}$$

where

$$\begin{aligned} y_\beta(3) &= y_{\beta d} - 3d_{\beta\max} + 3d_{e\max} \\ &\quad + \sum_{i=1}^{2} a_{\beta i}u_{\beta\max} + a_{\beta 1}u_{\beta\max} \\ &= (n_\beta - 2)d_{\beta\max} + (n_\beta + 3)d_{e\max} \\ &\quad - \sum_{i=1}^{n_\beta-1} a_{\beta i}u_{\beta\max} - \sum_{i=1}^{n_\beta-2} a_{\beta i}u_{\beta\max} \\ &\quad - \dots - \sum_{i=1}^{3} a_{\beta i}u_{\beta\max}. \end{aligned} \tag{105}$$

The equivalent control for $k = 3$ becomes

$$\begin{aligned} u_{\beta eq}(3) &= y_{\beta d} - \boldsymbol{c}_\beta \boldsymbol{x}_\beta(3) + d_{\beta\max} \\ &= y_{\beta d} - y_{\beta d} + 3d_{\beta\max} \\ &\quad - 3d_{e\max} - u_{\beta\max} \\ &\quad \underbrace{- \sum_{i=1}^{2} a_{\beta i}u_{\beta\max} - \sum_{3}^{n_\beta} a_{\beta i}u_{\beta\max}}_{-u_{\beta\max}} \\ &\quad \underbrace{- a_{\beta 1}u_{\beta\max} - \sum_{2}^{n_\beta} a_{\beta i}u_{\beta\max}}_{-u_{\beta\max}} + d_{\beta\max} \\ &= d_{\beta\max}. \end{aligned} \tag{106}$$

Consequently, $u_\beta(3) = u_{\beta\max}$ and $d_e(3) = d_{e\max}$. The stock level satisfies $y_\beta(3) \geq d_{\beta\max}$. Therefore, $h_\beta(3) = d_{\beta\max}$. We continue in the same manner for the following steps up to $k = n_\beta$, when

$$\boldsymbol{x}_\beta(n_\beta) = \begin{bmatrix} y_\beta(n_\beta) \\ u_{\beta\max} \\ \vdots \\ u_{\beta\max} \end{bmatrix}_{n_\beta+1} \tag{107}$$

and the current stock level $y_\beta(n_\beta)$ satisfies

$$\begin{aligned} y_\beta(n_\beta) &= y_{\beta d} - n_\beta d_{\beta\max} + n_\beta d_{e\max} \\ &\quad + \sum_{i=1}^{n-1} a_{\beta i}u_{\beta\max} + \sum_{i=1}^{n-2} a_{\beta i}u_{\beta\max} + \dots \\ &\quad + \sum_{i=1}^{2} a_{\beta i}u_{\beta\max} + a_{\beta 1}u_{\beta\max}. \end{aligned} \tag{108}$$

Substituting the value of $y_{\beta d}$ from (90), we obtain

$$y_\beta(n_\beta) = d_{\beta\max}. \tag{109}$$

The sales at $k = n_\beta$ are $h_\beta(n_\beta) = d_{\beta\max}$ and the control satisfies $u_\beta(n_\beta) = u_{\beta\max}$ and $d_e(n_\beta) = d_{e\max}$. Finally, for any $k \geq n_\beta + 1$, the state vector becomes

$$\boldsymbol{x}_\beta(k) = \begin{bmatrix} y_\beta(k) \\ u_{\beta\max} \\ \vdots \\ u_{\beta\max} \end{bmatrix}_{n_\beta+1}, \tag{110}$$

where

$$y_\beta(k) = d_{\beta\max}, \tag{111}$$

which ends the proof.

■

As shown in (111), the proposed control strategy ensures that the demand in warehouse β is fully satisfied for any $k \geq 0$, despite the control limit.

4. Simulation example

In this section, we will verify the control properties presented in the paper. We consider two warehouses selling a single product, measured in pieces [pcs].

The first warehouse, α, is a 7th order inventory system with four suppliers:

- Supplier 1 with six time instants of lead time that delivers 40% of the goods,
- Supplier 2 with five time instants of lead time that delivers 30% of the goods,
- Supplier 3 with four time instants of lead time that delivers 20% of the goods,
- Supplier 4 with two time instants of lead time that delivers 10% of the goods.

Therefore, $n_\alpha = 6$ and the $a_{\alpha i}$ parameters are $a_{\alpha 6} = 0.4$, $a_{\alpha 5} = 0.3$, $a_{\alpha 4} = 0.2$ and $a_{\alpha 2} = 0.1$, with the remaining ones equal to zero. The dynamics of the system are described in Section 2.1. The demand in the warehouse is comprised of three terms. The contractual

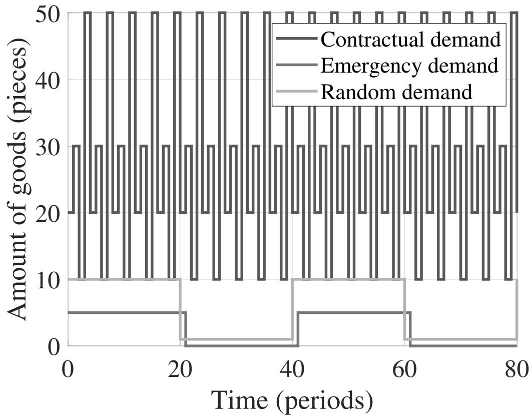

Fig. 1. Demand terms in warehouse α.

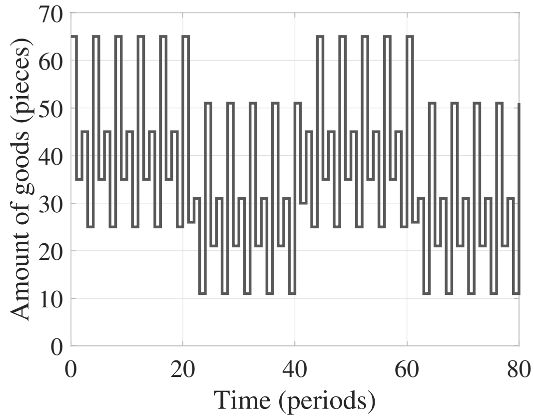

Fig. 3. Control signal $u_\alpha(k)$.

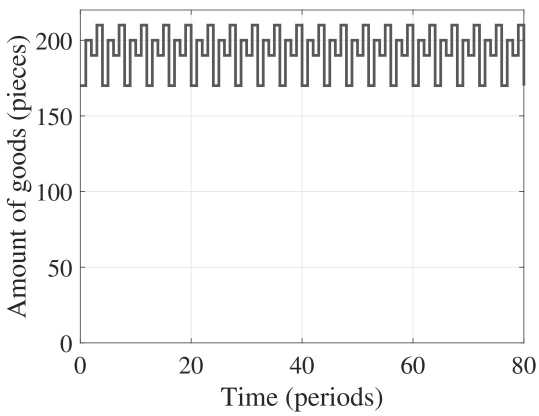

Fig. 2. Desired trajectory $s_{\alpha d}(k)$.

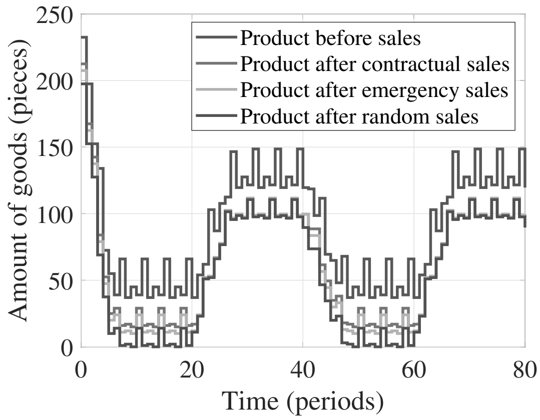

Fig. 4. Amount of product in warehouse α.

demand vector is a repeating sequence of four values $d_c(k) = 20, 30, 10, 50$, presented in Fig. 1. The emergency demand $d_e(k)$ is a signal generated by the controller of warehouse β, as a request for additional deliveries. It may be considered as random and bounded by (89). As depicted in Fig. 1, the emergency demand will change between zero and $d_{e\max}$ every 20 time instants. The random demand $d_r(k)$ changes between its bounds $d_{r\min} = 1$ and $d_{r\max} = 10$ every 20 time instants, as shown in Fig. 1. The control vector of warehouse α, i.e., $\boldsymbol{c}_\alpha$, is chosen according to (32) and $\boldsymbol{c}_\alpha = \begin{bmatrix} 1 & 0.4 & 0.7 & 0.9 & 0.9 & 1 & 1 \end{bmatrix}$.

Considering the contractual demand of warehouse α shown in Fig. 1, we generate the desired trajectory $s_{\alpha d}(k)$. As seen from (34), the trajectory contains the sum of the contractual values for all the $n_\alpha + 1 = 7$ future time instants and it is depicted in Fig. 2. The initial amount of stock required in the warehouse is, as defined in (36), $y_{\alpha 0} = 170$. This amount fulfills the contractual obligations of the warehouse for $k = [0, 6]$.

Using the desired trajectory $s_{\alpha d}(k)$ we design the control for warehouse α. Its representative will shall follow $s_{\alpha d}(k)$ according to the reaching law (37). The emergency deliveries and random sales present in the system are a disturbance and must be compensated for. As described by (40), the compensation vector is $\boldsymbol{D_{er}} = \begin{bmatrix} 15 & 15 & 15 & 15 & 15 & 15 & 15 \end{bmatrix}$.

We also compare the trajectory of warehouse α, $\boldsymbol{c}_\alpha \boldsymbol{x}_{\alpha d}(k)$, with the generated desired trajectory $s_{\alpha d}(k)$ and lower the desired trajectory according to (80), with $\gamma = 26$, to minimize the storage space required by the warehouse α. With both considered, the initial stock required in warehouse $y_{\alpha 0} = 170 + c_\alpha \boldsymbol{D_{er}} - \gamma = 232.5$. The resulting control signal $u_\alpha(k)$ is depicted in Fig. 3. It is clear that the amount of ordered goods closely follows the known contractual demand, with values increased by the need to compensate for the two random terms. When both the emergency and random demand are in their upper bounds, the control signal is at its maximum, and when they decrease, the controller reacts accordingly. It is worth mentioning that the generated control signal must then be distributed between the suppliers. However, in real problems each supplier may only deliver a finite integer number of products, depending on the type of the goods measured in pieces, packs, kilograms, litres, etc. Therefore, the control value for a specific supplier may need to be rounded up, and always up in order to avoid any sale losses.

The amount of goods in the warehouse $y_\alpha(k)$ resulting from the controller's effort can be seen in Fig. 4. The effects of increased emergency and random demands result in the lowered value of $y_\alpha(k)$. Additionally, the figure also presents the stock of warehouse α after each step of sales in the warehouse—the contractual obligations, the emergency deliveries to warehouse α, and the random sales. It can be noticed that the compensations terms result in a system with the smallest possible amount of goods stored in the warehouse, as the stock occasionally falls to zero in situations where the demands are in their upper bounds.

The second warehouse, denoted with β, is a 6th order system with three suppliers:

- Supplier 1 with five time instants of lead time that delivers 40% of the goods,
- Supplier 2 with four time instants of lead time that delivers 30% of the goods,
- Supplier 3 with two time instants of lead time that delivers 30% of the goods.

From the above, we can calculate that $n_\beta = 5$. The maximum order quantity of the warehouse is $u_{\beta\,\max} = 10$. The $a_{\beta i}$ parameters of the warehouse are $a_{\beta 5} = 0.4$ and $a_{\beta 4} = a_{\beta 2} = 0.3$, with the remaining ones equal to zero. The dynamics of the warehouse are described in Section 2.2. The demand in warehouse β, displayed in Fig. 5, is random and changes between its bounds $d_{\beta\min} = 5$ and $d_{\beta\max} = 15$ every 20 time instants.

The warehouse's control vector, i.e., $\boldsymbol{c}_\beta$, is chosen according to (32) and is $\boldsymbol{c}_\beta = \begin{bmatrix} 1 & 0.4 & 0.7 & 0.7 & 1 & 1 \end{bmatrix}$. The initial stock $y_{\beta 0}$ and the desired value $y_{\beta d}$ are calculated according to Theorem 2 and are $y_{\beta 0} = y_{\beta d} = 53$. We continue by applying the control law (87) to warehouse β, presented in Fig. 6. It can be seen that, in situations where the random demand exceeds the maximum order quantity, $u_{\beta eq}(k)$ is higher than what is possible to order from the suppliers. As a result, the control signal $u_\beta(k)$ reaches its upper bounds $u_{\beta\max}$ and the warehouse α is notified about the need for additional deliveries by $d_e(k)$.

These control efforts result in the stock of warehouse β, $y_\beta(k)$, as depicted in Fig. 7. Once again, higher values of the demand result in lower amounts of goods inside the warehouse. We can also see the warehouse's stock after the random sales. With the demand in its upper bound, the amount of goods falls to zero, which means that the smallest possible values of $y_\beta(0)$ and $y_{\beta d}$ were chosen.

Finally, the presented figures show that no sale losses were encountered. We achieved full demand satisfaction with the stock levels of both warehouses at their lowest possible values for the chosen control schemes.

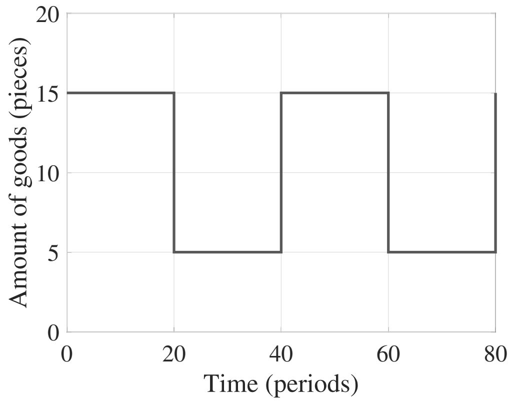

Fig. 5. Demand in warehouse β.

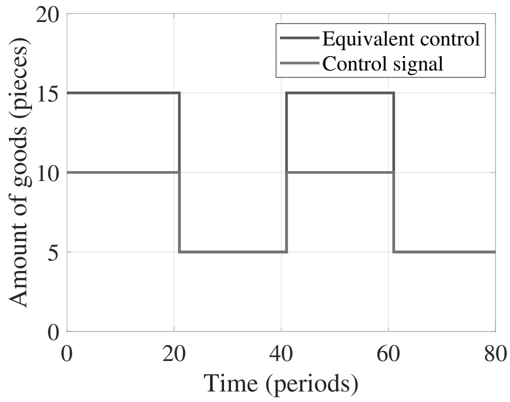

Fig. 6. Equivalent control $u_{\beta eq}(k)$ and control signal $u_\beta(k)$.

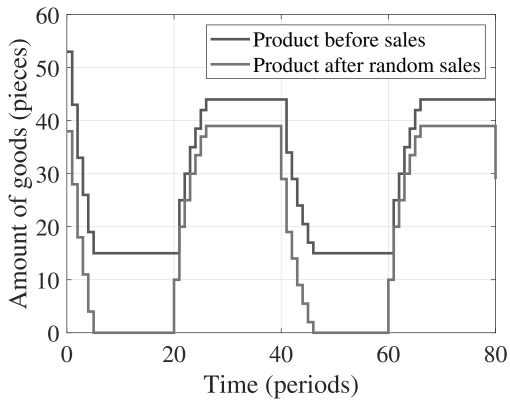

Fig. 7. Amount of product in warehouse β.

5. Conclusions

In this study we considered an inventory system consisting of two periodic review warehouses owned by the same company. Both warehouses store one and same product, and each of the warehouses is characterised by its specific suppliers with different lead times and different customer demand profiles. Warehouse β operates under random customer demand and is controlled according to a simple order-up-to control strategy. Moreover, system β is subject to delivery limitations, which at times may result in an inability to fulfill the demand. To tackle such emergency situations, we propose to transfer some of the leftover stock from warehouse α and use it to satisfy β's clients. On the other hand, warehouse α is subject to market demand of two sorts. We defined an *a priori* known contractual demand part and a random demand term. Based on the contractual demand knowledge, we generated a reference trajectory profile in advance. Next, we designed a reference trajectory following the SMC law, which ensures sufficient stock level to fulfill the contract at any step k. The secondary goal of system α is to provide supplementary deliveries to warehouse β in case of emergencies. Therefore, the proposed control law was enriched with random demand and emergency deliveries compensation. The paper proves that the suggested control strategy ensures full demand satisfaction in both warehouses, while retaining the minimum necessary stock level. We believe that the idea presented in this manuscript may be successfully extended to multiple warehouse systems, which will greatly simplify the control of large logistics centres.

It is worth pointing out that the proposed control strategy is much more flexible than commonly used inventory resupply strategies, such as the *EOQ* method. Unlike the *EOQ* strategy, the proposed control scheme takes into account changing market conditions. The contractual demand profile considered changes over time and will be known with the advance of $n + 1$ time instants only. Therefore, the control may be easily adjusted when the demand changes, whereas *EOQ* provides an optimal order number with constant market demand. Moreover, the proposed strategy provides much better time coordination of the process. It considers delivery delays and minimizes storing costs as a certain amount of goods arrives at the warehouse at the exact time instants they are needed for contracted buyers.

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