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A ROBUST ASYMPTOTIC TRACKING CONTROLLER FOR AN UNCERTAIN 2DOF UNDERACTUATED MECHANICAL SYSTEM MOTIVATED BY A SATELLITE ATTITUDE CONTROL PROBLEM

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The paper is devoted to the theoretical problem of designing a robust asymptotic tracking control system for a rotational motion of a 2DOF underactuated linear mechanical system with parametric uncertainties. The mathematical formulation of the problem is motivated by the attitude control problem of an earth observation satellite with a solar panel. It is assumed that all the parameters of the plant model are uncertain and the plant single input is additively disturbed by an unknown constant torque. By employing the general regulator theory in the state space setup combined with the concept of the structured singular value, we develop a robustly stabilizing and robustly asymptotically tracking error feedback controller. The rotation of the main rigid body of the mechanical system is to asymptotically track a harmonically changing reference signal. The obtained theoretical results are successfully tested on two numerical examples and computations are performed in Matlab.

Keywords: underactuated 2DOF mechanical system, rotational motion control, robust asymptotic tracking, robust error feedback controller.

1. Introduction

The underactuated mechanical systems have established themselves as an important class of mechanical systems with broad applications in engineering; see, e.g., the survey by Liu and Yu (2013) and the references cited therein. Numerous examples of such systems appear in the spacecraft engineering and robotics and generate challenging control problems. In order to deal with the problem of controlling spacecrafts, many approaches have been developed and a good overview is given by Xie *et al*. (2016). In particular, the robust control problems for this class belong to the active field of research within the control community and several interesting control algorithms have been proposed (Almeida *et al*., 2015; Ordaz *et al*., 2024; Mohsenipour *et al*., 2013; Muñoz-Arias, 2019; Ohtani *et al*., 2011; Sumithra and Vadivel, 2021; Wang and Li, 2012; Iannelli *et al*., 2022).

As a motivating example for this paper, we bring the attitude control problem of an Earth observation satellite with an appendage. These satellites are to perform complicated tasks with a demand for high reliability and accuracy (Wang *et al*., 2020). However, the close interference between the flexible structure of elastic appendages like solar panels and the structure of the satellite itself can be a major factor in lowering the accuracy of performance Angeletti *et al.* (2020; 2021). The precise mathematical model of dynamics of a satellite with appendages reveals an infinite number of oscillation modes but in the real life we can always select just a few modes which are excited by the satellite operation (see, e.g., Narkiewicz *et al*., 2020; 2024; Angeletti *et al*., 2021). Usually, an observing satellite, which images a

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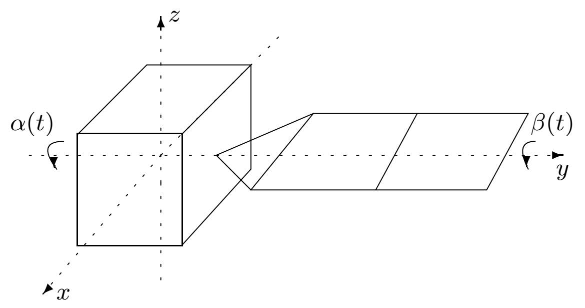

Fig. 1. Simplified model of a satellite with a solar panel.

sequence of adjacent pieces of field, performs a periodical motion around a constant axis. In such a case, only one mode is dominant. In this simplified case the satellite can be interpreted as two rigid bodies with a viscoelastic interconnection, rotating around the y-axis, as shown in Fig. 1.

The motion of this mechanical system, referred to as the *plant*, can be described by two second order ODEs

$$\Sigma_G : \begin{cases} I\ddot{\alpha}(t) = k(\beta(t) - \alpha(t)) \\ \qquad +b(\dot{\beta}(t) - \dot{\alpha}(t)) + u(t)\,, \\ p\ddot{\beta}(t) = -k(\beta(t) - \alpha(t)) \\ \qquad -b(\dot{\beta}(t) - \dot{\alpha}(t))\,, \end{cases} \tag{1}$$

where Σ_G is used to denote the *plant mathematical model*, $(u(t))_{t\geq 0} \subset \mathbb{R}$ is an *input* torque applied to the main body, $(\alpha(t))_{t\geq 0} \subset \mathbb{R}$ is the main body rotation (*measured output*), $(\beta(t))_{t\geq 0} \subset \mathbb{R}$ is the panel rotation, I is the main body rotational inertia, p is the panel rotational inertia, k is the stiffness coefficient of the interconnection, b is the friction coefficient. According to the physical meaning of the parameters we assume that $I > 0$, $p > 0$, $k > 0$ and $b > 0$. At some places we also consider $b = 0$ just to see how the lack of friction influences the properties of the system. Since the input u appears in one equation, from a theoretical point of view the model Σ_G is an example of a 2DOF underactuated mechanical system.

Our aim is to develop a robust control algorithm which makes the orientation α to track asymptotically a periodically changing reference signal α_r, in the presence of significant parametric uncertainties of the plant model Σ_G. The results will be based on the robust general regulator theory in the state space setup (Isidori *et al.*, 2003), which is an extension of the multivariable regulator theory (Francis and Wonham, 1975). The main difference between our approach and the robust control theory based on the μ-synthesis (see, e.g., Zhou and Doyle, 1998; Scherer, 2001), is that the controller, due to its structure, has only to guarantee robust stability and then the robustness of the performance follows. However, in the analysis of the robustness of the stability we will also use of the concept of the structured singular value μ (Scherer, 2001; Zhou and Doyle, 1998). We emphasize here that the exact asymptotic tracking we consider does not fit as a performance criterion in the μ-synthesis problem since it cannot be expressed in terms of $\mathcal{H}_\infty$-norm minimization.

The paper is organized into six sections. Section 1 is an introduction to the subject including the introduction of the plant state space model and a preliminary formulation of the control problem. In Section 2 we make precise in mathematical terms what is the robust control problem we intend to solve. Section 3 is devoted to the characterization of a structure of the robust controller and here the regulator equation and the internal model principle appear. As our original contribution we prove that the regulator equation has a solution and this solution is unique. Moreover, we find this solution explicitly. We show that if the controller is robustly stabilizing, then it also provides robust asymptotic tracking. One such controller, based on the full order state observer, is proposed. In Section 4 we transform the plant model with uncertain parameters to the form involving the upper fractional transformation, which is essential in the considerations to follow. In general, that section is devoted to the robustness analysis of a controller which stabilizes the nominal plant. We show that scaling of the structured singular value allows us to define bounds for uncertain parameters which guarantee the robustness of the internal stability and asymptotic tracking.

Section 5 presents results of numerical computations showing the effectiveness of the obtained theoretical results. The Matlab package with its several toolboxes is used as the computational environment. Some discussion and final remarks are contained in Section 6 which concludes the paper.

Before we start with formal considerations, we need to introduce and explain the basic notation which used in the paper:

- $t \in [0, \infty)$ denotes the *time* variable,
- $\mathbb{R}$: the space of real numbers, $\mathbb{R}^n$: the space of n-dimensional real vectors, $\mathbb{R}^{n\times m}$: the space of $(n \times m)$-dimensional matrices,
- $\mathbb{C}$: the space of complex numbers, $\mathbb{C}^n$ and $\mathbb{C}^{n\times m}$ - analogously as in the real case,
- $\mathbb{C}_-$: the open left half plane, $j\mathbb{R}$: the imaginary axis, $\emptyset$: an empty set,
- $(\alpha(t))_{t\geq 0} \subset \mathbb{R}$: a function of $t \geq 0$ taking values in $\mathbb{R}$, $\dot{\alpha}(t), \ddot{\alpha}(t)$: time derivatives,
- $\det(A)$: the determinant of A, $\sigma(A)$: the spectrum (the set of eigenvalues) of A, $\sigma_{\max}(A)$: the maximum singular value of A,

- for the state space model

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du \end{cases}$$

the matrix

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

is called the state space matrix and

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

is used to denote its transfer function, i.e., $C(sI - A)^{-1}B + D$,

- $[\,\cdot\,]^T$ denotes the transpose of a vector or matrix.

1.1. Basic formulation of the control problem. In practice, all the physical parameters I, p, k and b in the plant model Σ_G, given by (1), cannot be measured exactly, so we assume that they are *uncertain*. More precisely, real values of the parameters I, p, k and b are assumed to belong to *known intervals*, i.e.,

$$\begin{aligned} I &\in (I_{\min}, I_{\max})\,, & p &\in (p_{\min}, p_{\max})\,, \\ k &\in (k_{\min}, k_{\max})\,, & b &\in (b_{\min}, b_{\max})\,, \end{aligned} \tag{2}$$

where

$$\begin{aligned} I_{\max} &> I_{\min} \geq 0, & p_{\max} &> p_{\min} \geq 0, \\ k_{\max} &> k_{\min} \geq 0, & b_{\max} &> b_{\min} \geq 0, \end{aligned}$$

are known. By introducing *nominal* (mean) values $I(0)$, $p(0)$, $k(0)$ and $b(0)$, defined as

$$\begin{aligned} I(0) &= \frac{I_{\min} + I_{\max}}{2}\,, & p(0) &= \frac{p_{\min} + p_{\max}}{2}\,, \\ k(0) &= \frac{k_{\min} + k_{\max}}{2}\,, & b(0) &= \frac{b_{\min} + b_{\max}}{2}\,, \end{aligned} \tag{3}$$

and the *weight coefficients*

$$\begin{aligned} W_I &= \frac{I_{\max} - I_{\min}}{2}\,, & W_p &= \frac{p_{\max} - p_{\min}}{2}\,, \\ W_k &= \frac{k_{\max} - k_{\min}}{2}\,, & W_b &= \frac{b_{\max} - b_{\min}}{2}\,, \end{aligned} \tag{4}$$

we can express the uncertain real parameters in the *additive* forms

$$\begin{aligned} I(\delta_I) &= I(0) + W_I\delta_I\,, & p(\delta_p) &= p(0) + W_p\delta_p\,, \\ k(\delta_k) &= k(0) + W_k\delta_k\,, & b(\delta_b) &= b(0) + W_b\delta_b\,, \end{aligned} \tag{5}$$

where δ_I, δ_p, δ_k and δ_b are *normalized uncertainties*, i.e.,

$$|\delta_I| < 1\,, \quad |\delta_p| < 1\,, \quad |\delta_k| < 1\,, \quad |\delta_b| < 1\,. \tag{6}$$

It is convenient to interpret the nominal values of parameters $I(0)$, $p(0)$, $k(0)$ and $b(0)$ as the results of real measurements or computations. Then, the corresponding weights W_I, W_p, W_k and W_b describe bounds on the errors of these measurements and define the ends of the corresponding intervals (2) as follows:

$$\begin{aligned} I_{\min} &= I(0) - W_I, & p_{\min} &= p(0) - W_p, \\ I_{\max} &= I(0) + W_I, & p_{\max} &= p(0) + W_p, \\ k_{\min} &= k(0) - W_k, & b_{\min} &= b(0) - W_b, \\ k_{\max} &= k(0) + W_k, & b_{\max} &= b(0) + W_b. \end{aligned} \tag{7}$$

For simplicity of notation, we also introduce the joint uncertainty $\delta := (\delta_I, \delta_p, \delta_k, \delta_b)$, and to emphasize that parameters are uncertain, we rewrite the plant (1) in the more explicit form

$$\Sigma_G(\delta) : \begin{cases} I(\delta_I)\ddot{\alpha}(t) = k(\delta_k)(\beta(t) - \alpha(t)) \\ \qquad\qquad + b(\delta_b)(\dot{\beta}(t) - \dot{\alpha}(t)) + u(t)\,, \\ p(\delta_p)\ddot{\beta}(t) = -k(\delta_k)(\beta(t) - \alpha(t)) \\ \qquad\qquad - b(\delta_b)(\dot{\beta}(t) - \dot{\alpha}(t))\,, \end{cases} \tag{8}$$

and refer to $\Sigma_G(\delta)$ as the *uncertain plant model* or *uncertain plant*, for brevity. What is essential, we also assume that the input torque $(u(t))_{t\geq 0} \subset \mathbb{R}$ consists of a *control torque* $(\tau(t))_{t\geq 0} \subset \mathbb{R}$ and an unknown *disturbance torque* $(d(t))_{t\geq 0} \subset \mathbb{R}$, i.e.,

$$u(t) = \tau(t) + d(t)\,, \quad t \geq 0\,, \tag{9}$$

where $d(t) = d_0 = \text{const}$ for $t \geq 0$, with an *unknown* magnitude $d_0 \in \mathbb{R}$.

The only *measured signal* is the rotational displacement α and we want the *plant output* $(\alpha(t))_{t\geq 0}$ to track the *reference signal* $(\alpha_r(t))_{t\geq 0} \subset \mathbb{R}$ of the form

$$\alpha_r(t) = a\sin(\omega_r t + \varphi)\,, \quad t \geq 0\,, \tag{10}$$

where $a \in \mathbb{R}$ and $\varphi \in \mathbb{R}$ are allowed to be *unknown* but $\omega_r > 0$ has to be *known*. If we define the *control error* $(e(t))_{t\geq 0} \subset \mathbb{R}$ as follows:

$$e(t) = \alpha(t) - \alpha_r(t)\,, \quad t \geq 0\,, \tag{11}$$

then we can formulate the *control goal* as *asymptotic tracking* of the reference signal α_r by the *uncertain plant* output α, i.e.,

$$\lim_{t\to\infty} e(t) = 0\,, \tag{12}$$

for all disturbances $d(t) \equiv d_0 \in \mathbb{R}$. We want to achieve the goal (12) by developing the *dynamic error feedback controller*

$$\Sigma_K : \begin{cases} \dot{x}_K = A_K x_K + B_K e\,, \quad x_K(0) = x_{K0}\,, \\ \tau = C_K x_K + D_K e\,, \end{cases} \tag{13}$$

where $(x_K(t))_{t\geq 0} \subset \mathbb{R}^{n_K}$, n_K is the order of the controller and the error e is the only signal available to the controller.

$$\left[\begin{array}{c|c} A(\delta) & B(\delta) \\ \hline C & 0 \end{array}\right] := \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{k(\delta_k)}{I(\delta_I)} & \frac{k(\delta_k)}{I(\delta_I)} & -\frac{b(\delta_b)}{I(\delta_I)} & \frac{b(\delta_b)}{I(\delta_I)} & \frac{1}{I(\delta_I)} \\ \frac{k(\delta_k)}{p(\delta_p)} & -\frac{k(\delta_k)}{p(\delta_p)} & \frac{b(\delta_b)}{p(\delta_p)} & -\frac{b(\delta_b)}{p(\delta_p)} & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \end{array}\right], \tag{17}$$

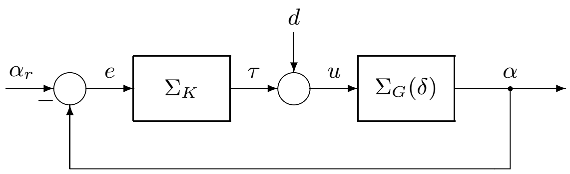

Fig. 2. Error feedback control system.

It immediately follows that the control system we want to design is the *error feedback control system*, shown in Fig. 2, and we also want the controller Σ_K to achieve the control goal (12) for every uncertain plant (8) and every constant disturbance $d_0 \in \mathbb{R}$.

1.2. Plant state space model. We will mainly employ the state space methods, so we start with a state space model of the plant $\Sigma_G(\delta)$. Since the plant is a 2DOF mechanical system, it is convenient to introduce the following state variables with obvious physical meanings:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix}. \tag{14}$$

Then we get the plant *state space model*

$$\Sigma_G(\delta) : \begin{cases} \dot{x}_1 = x_3, \\ \dot{x}_2 = x_4, \\ \dot{x}_3 = -\frac{k(\delta_k)}{I(\delta_I)}x_1 + \frac{k(\delta_k)}{I(\delta_I)}x_2 \\ \qquad -\frac{b(\delta_b)}{I(\delta_I)}x_3 + \frac{b(\delta_b)}{I(\delta_I)}x_4 + \frac{1}{I(\delta_I)}u, \\ \dot{x}_4 = \frac{k(\delta_k)}{p(\delta_p)}x_1 - \frac{k(\delta_k)}{p(\delta_p)}x_2 + \frac{b(\delta_b)}{p(\delta_p)}x_3 \\ \qquad -\frac{b(\delta_b)}{p(\delta_p)}x_4, \\ \alpha = x_1, \end{cases} \tag{15}$$

which can be written in the compact form

$$\Sigma_G(\delta) : \begin{cases} \dot{x} = A(\delta)x + B(\delta)u, \quad x(0) = x_0, \\ \alpha = Cx, \end{cases} \tag{16}$$

where $A(\delta)$, $B(\delta)$ and C are defined in (17) and $\Sigma_G(\delta)$ is again referred to as the *uncertain plant model* or *uncertain plant*, for brevity. We also write

$$\Sigma_G(0) : \begin{cases} \dot{x} = A(0)x + B(0)u, \quad x(0) = x_0, \\ \alpha = Cx, \end{cases} \tag{18}$$

for $\delta = 0$ ($(\delta_k, \delta_I, \delta_p, \delta_b) = (0,0,0,0)$), where $\Sigma_G(0)$ is referred to as the *nominal plant model* or *nominal plant*, for brevity.

For the uncertain plant $\Sigma_G(\delta)$ the *controllability* of $(A(\delta), B(\delta))$ can be checked by means of the *controllability matrix* $W(\delta)$ and the *observability* of $(C, A(\delta))$ - by means of the *observability matrix* $V(\delta)$. Namely,

$$\begin{aligned} \det W(\delta) &= -\frac{k^2(\delta_k)}{I^4(\delta_I)p^2(\delta_p)} \neq 0, \\ \det V(\delta) &= -\frac{k^2(\delta_k)}{I^2(\delta_I)} \neq 0, \end{aligned} \tag{19}$$

for all δ_I, δ_p, δ_k and δ_b satisfying (6), where determinants have been computed by means of the Matlab Symbolic Math Toolbox (MathWorks, 2020c). In particular, the nominal plant $\Sigma_G(0)$ is also controllable and observable.

2. Robust control problem

In order to design a suitable controller we employ the general regulator theory (see, e.g., Saberi *et al.*, 2000). The essential feature of this approach is that we assume the *reference signal* and the *disturbance* are generated by a known dynamical system, called the *exosystem*, which is aggregated with the plant model.

2.1. Exosystem. The *reference signal* of the form $\alpha_r(t) = a\sin(\omega_r t + \varphi)$ is generated by the dynamical system

$$\begin{cases} \dot{r}_1 = r_2, & r_1(0) = a\sin\varphi, \\ \dot{r}_2 = -\omega_r^2 r_1, & r_2(0) = a\omega_r\cos\varphi, \\ \alpha_r = r_1, \end{cases} \tag{20}$$

where $\omega_r > 0$ has to be known, $a \in \mathbb{R}$ and $\varphi \in \mathbb{R}$ may be unknown. In turn, the *disturbance* of the form $d(t) \equiv d_0$ is generated by the dynamical system

$$\begin{cases} \dot{d} = 0 \cdot d, \quad d(0) = d_0, \\ d = 1 \cdot d, \end{cases} \tag{21}$$

where $d_0 \in \mathbb{R}$ is unknown. Combining (20) and (21), we get a dynamical system Σ_S, called the *exosystem*,

$$\Sigma_S : \begin{cases} \dot{r}_1 = r_2 & r_1(0) = a\sin\varphi\,, \\ \dot{r}_2 = -\omega_r^2 r_1\,, & r_2(0) = a\omega_r\cos\varphi\,, \\ \dot{d} = 0\cdot d\,, & d(0) = d_0\,, \\ \alpha_r = r_1\,, \\ d = 1\cdot d\,, \end{cases} \tag{22}$$

i.e.,

$$\Sigma_S : \begin{cases} \dot{w} = Sw\,, \quad w(0) = w_0\,, \\ \alpha_r = T_r w\,, \\ d = T_d w\,, \end{cases} \tag{23}$$

where

$$w = \begin{bmatrix} r_1 \\ r_2 \\ d \end{bmatrix}, \qquad S = \begin{bmatrix} 0 & 1 & 0 \\ -\omega_r^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{24}$$

$$T_r = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad T_d = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

The *characteristic polynomial* of S is given by

$$\det(\lambda I - S) = \lambda(\lambda^2 + \omega_r^2)\,, \tag{25}$$

with eigenvalues (the spectrum)

$$\sigma(S) = \{0, j\omega_r, -j\omega_r\}\,, \tag{26}$$

so that the system Σ_S satisfies $\sigma(S) \cap \mathbb{C}_- = \emptyset$.

2.2. Robust control system. Recall that we consider the error feedback control system shown in Fig. 2, where the uncertain plant $\Sigma_G(\delta)$ is described by (16) and the uncertainties $\delta = (\delta_I, \delta_p, \delta_k, \delta_b)$ satisfy (6). It allows us to define the real *uncertainty matrix*

$$\Delta(\delta) = \begin{bmatrix} \delta_k & 0 & 0 & 0 \\ 0 & \delta_b & 0 & 0 \\ 0 & 0 & \delta_I & 0 \\ 0 & 0 & 0 & \delta_p \end{bmatrix} \in \mathbb{R}^{4\times 4}\,, \tag{27}$$

with the constraints (6), and since we can write

$$\Delta(\delta) = \begin{bmatrix} \delta_k + j0 & 0 & 0 & 0 \\ 0 & \delta_b + j0 & 0 & 0 \\ 0 & 0 & \delta_I + j0 & 0 \\ 0 & 0 & 0 & \delta_p + j0 \end{bmatrix}, \tag{28}$$

we also have that $\Delta(\delta) \in \mathbb{C}^{4\times 4}$. Thus we define the *uncertainty structure set* $\Delta_c \subset \mathbb{C}^{4\times 4}$ as follows:

$$\Delta_c := \{\Delta(\delta) \in \mathbb{C}^{4\times 4} : \sigma_{\max}(\Delta(\delta)) < 1\}\,, \tag{29}$$

where $\sigma_{\max}(\Delta(\delta))$ is the maximum singular value of $\Delta(\delta)$. Moreover, for the plant input u we have

$$u = \tau + d\,, \tag{30}$$

where τ is the control torque and $d = d_0$ is a disturbance torque. The exosystem Σ_S, generating the reference α_r and the disturbance d, is given by (23) and the error feedback controller Σ_K is described by (18) with the error e given by

$$e = \alpha - \alpha_r\,. \tag{31}$$

If we put together Eqns. (16), (30), (18) and (31), then we obtain the basic model of the *error feedback control system*, denoted by $\Sigma_e(\delta)$, with α_r and d as two external signals, and taking the form

$$\Sigma_e(\delta) : \begin{cases} \dot{x} = (A(\delta) + B(\delta)D_K C)x \\ \quad +B(\delta)C_K x_K - B(\delta)D_K\alpha_r \\ \quad +B(\delta)d\,, \quad x(0) = x_0\,, \\ \dot{x}_K = B_K C x + A_K x_K - B_K\alpha_r\,, \\ \quad x_K(0) = x_{K0}\,, \\ e = Cx - \alpha_r\,, \end{cases} \tag{32}$$

where $\Delta(\delta) \in \Delta_c$. The error feedback control system $\Sigma_e(\delta)$ with zero inputs, i.e., $\alpha_r \equiv 0$ and $d \equiv 0$, and without the output equation, is referred to as the *unforced closed loop system* $\Sigma_{uf}(\delta)$, and its description takes the form

$$\Sigma_{uf}(\delta) : \begin{cases} \dot{x} = (A(\delta) + B(\delta)D_K C)x \\ \quad +B(\delta)C_K x_K\,, \quad x(0) = x_0\,, \\ \dot{x}_K = B_K C x + A_K x_K\,, \\ \quad x_K(0) = x_{K0}\,, \end{cases} \tag{33}$$

where $\Delta(\delta) \in \Delta_c$. If we now take into account that the reference α_r and the disturbance d are generated by the exosystem Σ_S and combine Eqns. (23) and (32), then we obtain a complete state space model of the error feedback control system $\Sigma_e(\delta)$, which is referred to as the *closed loop system* $\Sigma_{cl}(\delta)$, and has the form

$$\Sigma_{cl}(\delta) : \begin{cases} \dot{x} = (A(\delta) + B(\delta)D_K C)x \\ \quad +B(\delta)C_K x_K + B(\delta)(T_d - D_K T_r)w\,, \\ \quad x(0) = x_0\,, \\ \dot{x}_K = B_K C x + A_K x_K \\ \quad -B_K T_r w\,, \quad x_K(0) = x_{K0}\,, \\ \dot{w} = Sw\,, \quad w(0) = w_0\,, \\ e = Cx - T_r w\,, \end{cases} \tag{34}$$

where $\Delta(\delta) \in \Delta_c$.

Now we make precise what is an error feedback controller Σ_K we are looking for. We require the controller (18) to guarantee the following two conditions to hold:

RIS: *Robust internal stability*. The error feedback control system $\Sigma_e(\delta)$ is said to be *robustly internally stable* if the unforced closed loop system $\Sigma_{uf}(\delta)$ is asymptotically stable for all $\Delta(\delta) \in \Delta_c$, i.e., for all $x(0) = x_0$ and $x_K(0) = x_{K0}$ we have

$$\lim_{t\to\infty} \begin{bmatrix} x(t) \\ x_K(t) \end{bmatrix} = 0\,, \quad \Delta(\delta) \in \Delta_c\,. \tag{35}$$

RAT: *Robust asymptotic tracking* (or *robust regulation*). The error feedback control system $\Sigma_e(\delta)$ is said to satisfy the *robust asymptotic tracking* condition if for all $w(0) = w_0$, $x(0) = x_0$ and $x_K(0) = x_{K0}$ the closed loop system $\Sigma_{cl}(\delta)$ satisfies

$$\lim_{t\to\infty} e(t) = 0\,, \quad \Delta(\delta) \in \Delta_c\,. \tag{36}$$

Every error feedback controller Σ_K which guarantees RIS and RAT is said to be a *robust controller*. We easily see from (33) that RIS holds if and only if

$$\sigma\left(\begin{bmatrix} A(\delta) + B(\delta)D_K C & B(\delta)C_K \\ B_K C & A_K \end{bmatrix}\right) \subset \mathbb{C}_-\,, \quad \Delta(\delta) \in \Delta_c\,. \tag{37}$$

Examination of RIS is a hard task and this problem will be solved in Section 4. Before that, in Section 3, we show how to deal with RAT under the assumption that RIS is already guaranteed.

3. Characterization of a robust controller

We know from Section 1.2 that for $b \geq 0$ the pair $(A(\delta), B(\delta))$ is controllable and $(C, A(\delta))$ is observable for all $\Delta(\delta) \in \Delta_c$. Hence, for every fixed $\Delta(\delta) \in \Delta_c$ there always exists a controller $(A_K(\delta), B_K(\delta), C_K(\delta), D_K(\delta))$ (possibly, dependent of δ) satisfying

$$\sigma\left(\begin{bmatrix} A(\delta) + B(\delta)D_K(\delta) C & B(\delta)C_K(\delta) \\ B_K(\delta) C & A_K(\delta) \end{bmatrix}\right) \subset \mathbb{C}_-\,. \tag{38}$$

However, if we have a controller (A_K, B_K, C_K, D_K), independent of δ, and such that RIS holds, i.e.,

$$\sigma\left(\begin{bmatrix} A(\delta) + B(\delta)D_K C & B(\delta)C_K \\ B_K C & A_K \end{bmatrix}\right) \subset \mathbb{C}_-\,, \quad \Delta(\delta) \in \Delta_c\,, \tag{39}$$

then from the robust general regulator theory (Isidori *et al.*, 2003) we can derive the following necessary and sufficient conditions for the robust asymptotic tracking RAT.

3.1. Fundamental result.

In order to keep the presentation complete, we provide all the results with proofs.

Theorem 1. *If for a given controller Σ_K the error feedback control system $\Sigma_e(\delta)$ satisfies* RIS*, then* RAT *holds if and only if there exist matrices $\Pi(\delta) \in \mathbb{R}^{4\times3}$, $\Gamma(\delta) \in \mathbb{R}^{1\times3}$ and $\Sigma(\delta) \in \mathbb{R}^{n_K\times3}$ such that*

$$\text{RE}: \begin{cases} A(\delta)\Pi(\delta) - \Pi(\delta)S + B(\delta)\Gamma(\delta) + B(\delta)T_d = 0\,, \\ C\Pi(\delta) - T_r = 0\,, \end{cases} \tag{40}$$

and

$$\text{IMP}: \begin{cases} \Gamma(\delta) = C_K\Sigma(\delta)\,, \\ \Sigma(\delta)S = A_K\Sigma(\delta)\,, \end{cases} \tag{41}$$

for all $\Delta(\delta) \in \Delta_c$. If this is the case, then Σ_K is a robust controller.

Proof. Let (39) hold. For the closed loop system (34) we introduce new state variables

$$\begin{bmatrix} p \\ q \\ w \end{bmatrix} = \begin{bmatrix} I & 0 & -\Pi \\ 0 & I & -\Sigma \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} x \\ x_K \\ w \end{bmatrix}, \tag{42}$$

where $\Pi \in \mathbb{R}^{4\times3}$ and $\Sigma \in \mathbb{R}^{n_K\times3}$ can be arbitrary, and obtain an equivalent state space model

$$\Sigma_{cl}(\delta): \begin{cases} \dot p = (A(\delta) + B(\delta)D_K C)p + B(\delta)C_K q \\ \qquad +(A(\delta)\Pi - \Pi S + B(\delta)C_K\Sigma \\ \qquad +B(\delta)T_d + B(\delta)D_K(C\Pi - T_r))w\,, \\ \qquad p(0) = x_0 - \Pi w_0\,, \\ \dot q = B_K C p + A_K q \\ \qquad +(A_K\Sigma - \Sigma S + B_K(C\Pi - T_r))w\,, \\ \qquad q(0) = x_{K0} - \Sigma w_0\,, \\ \dot w = Sw\,, \quad w(0) = w_0\,, \\ e = Cp + (C\Pi - T_r)w\,, \end{cases} \tag{43}$$

where $\Delta(\delta) \in \Delta_c$. The crucial role in the proof is played by the following system of two matrix equations:

$$\begin{bmatrix} A(\delta) + B(\delta)D_K C & B(\delta)C_K \\ B_K C & A_K \end{bmatrix} \begin{bmatrix} \Pi(\delta) \\ \Sigma(\delta) \end{bmatrix} - \begin{bmatrix} \Pi(\delta) \\ \Sigma(\delta) \end{bmatrix} S = \begin{bmatrix} B(\delta)(T_d - D_K T_r) \\ B_K T_r \end{bmatrix}, \tag{44}$$

and

$$C\Pi(\delta) - T_r = 0\,, \tag{45}$$

where the pair $(\Pi(\delta), \Sigma(\delta))$, with $\Pi(\delta) \in \mathbb{R}^{4\times3}$ and $\Sigma(\delta) \in \mathbb{R}^{n_K\times3}$, denotes any solution to this system. In general, this solution does not have to exist. The important problem of the existence of this solution is dealt with further on in Sections 3.2 and 3.3.

If we now assume that a solution $(\Pi(\delta), \Sigma(\delta))$ exists, then (43) simplifies to the form

$$\Sigma_{cl}(\delta): \begin{cases} \dot p = (A(\delta) + B(\delta)D_K C)p + B(\delta)C_K q\,, \\ \qquad p(0) = x_0 - \Pi(\delta)w_0\,, \\ \dot q = B_K C p + A_K q\,, \\ q(0) = x_{K0} - \Sigma(\delta)w_0\,, \\ \dot w = Sw\,, \quad w(0) = w_0\,, \\ e = Cp\,, \end{cases} \tag{46}$$

and, due to RIS (see (39)),

$$\begin{aligned}&\lim_{t\to\infty} e(t)\\ &= \lim_{t\to\infty} \left[\begin{array}{cc} C & 0 \end{array}\right]\\ &\quad\times \exp(\left[\begin{array}{cc} A(\delta)+B(\delta)D_K C & B(\delta)C_K \\ B_K C & A_K \end{array}\right] t)\\ &\quad\times \left[\begin{array}{c} p(0)\\ q(0)\end{array}\right] = 0\,, \quad \Delta(\delta)\in\Delta_c\,,\end{aligned} \tag{47}$$

which implies RAT. On the other hand, since

$$\sigma(\left[\begin{array}{cc} A(\delta)+B(\delta)D_K C & B(\delta)C_K \\ B_K C & A_K \end{array}\right]) \cap \sigma(S) = \emptyset\,, \tag{48}$$

for every $\Delta(\delta) \in \Delta_c$, for every right hand side the Sylvester equation (44) has a unique solution

$$\left[\begin{array}{c} \Pi(\delta)\\ \Sigma(\delta)\end{array}\right]$$

for all $\Delta(\delta)\in\Delta_c$. This equation and the RAT condition, applied to the system (43), imply that

$$\begin{aligned}&\lim_{t\to\infty} e(t)\\ &= \lim_{t\to\infty} \left[\begin{array}{cc} C & 0 \end{array}\right]\\ &\quad\times \exp(\left[\begin{array}{cc} A(\delta)+B(\delta)D_K C & B(\delta)C_K \\ B_K C & A_K \end{array}\right] t)\\ &\quad\times \left[\begin{array}{c} x_0-\Pi(\delta)w_0\\ x_{K0}-\Sigma(\delta)w_0\end{array}\right]\\ &\quad+ \lim_{t\to\infty} (C\Pi(\delta)-T_r)\exp(St)w_0\\ &= \lim_{t\to\infty} (C\Pi(\delta)-T_r)\exp(St)w_0 = 0\,,\end{aligned} \tag{49}$$

and, since $\sigma(S)\cap\mathbb{C}_- = \emptyset$, we get $C\Pi(\delta)-T_r = 0$. Thus we have got (44) and (45).

Now let us notice that by substituting (45) into (44) we get an equivalent system of equations

$$\left\{\begin{array}{l} A(\delta)\Pi(\delta)-\Pi(\delta)S+B(\delta)C_K\Sigma(\delta)+B(\delta)T_d = 0\,,\\ A_K\Sigma(\delta)-\Sigma(\delta)S = 0\,,\\ C\Pi(\delta)-T_r = 0\,,\end{array}\right. \tag{50}$$

and if we introduce $\Gamma(\delta) = C_K\Sigma(\delta)$, then (50) can be equivalently written as (40) and (41). ■

The relation (40) is referred to as the *regulator equation* and hence we denote it, shortly, as RE. The second relation (41) is referred to as the *internal model principle* and hence we denote it as IMP. The latter relation reflects the fact that the dynamics of the exosystem appears in the controller (Francis and Wonham, 1975). In Section 3.2 we prove that in our case RE has a unique solution $(\Pi(\delta)),\Gamma(\delta))$ and in Section 3.3 we show how to choose a controller (A_K, B_K, C_K, D_K) such that for every $\Gamma(\delta)$ there exists a matrix $\Sigma(\delta)$ satisfying IMP. Let us also notice that the existence of a solution to RE is independent of the existence of a solution to IMC.

3.2. Analysis of RE.

The results proved in the previous subsection can be derived from the general regulator theory (Isidori *et al.*, 2003). However, the usual difficulty is in showing *if* the regulator equation RE admits a solution. As our original contribution we prove that for the uncertain plant $\Sigma_G(\delta)$ and the exosystem Σ_S the regulator equation RE has a solution and this solution is unique. Moreover, we find this solution explicitly.

Theorem 2. *If $b(\delta_b) > 0$ or $b = 0$ and $k(\delta_k) \neq p(\delta_p)\omega_r^2$, then there exists a unique pair $(\Pi(\delta),\Gamma(\delta))$, where $\Pi(\delta)\in\mathbb{R}^{4\times 3}$ and $\Gamma(\delta)\in\mathbb{R}^{1\times 3}$, such that*

$$\text{RE}: \left\{\begin{array}{l} A(\delta)\Pi(\delta)-\Pi(\delta)S+B(\delta)\Gamma(\delta)+B(\delta)T_d = 0\,,\\ C\Pi(\delta)-T_r = 0\,,\end{array}\right. \tag{51}$$

for all $\Delta(\delta)\in\Delta_c$. Moreover, by introducing

$$\begin{aligned}\Pi(\delta) &= \left[\begin{array}{ccc} \pi_{11} & \pi_{12} & \pi_{13}\\ \pi_{21} & \pi_{22} & \pi_{23}\\ \pi_{31} & \pi_{32} & \pi_{33}\\ \pi_{41} & \pi_{42} & \pi_{43}\end{array}\right],\\ \Gamma(\delta) &= \left[\begin{array}{ccc} \gamma_1 & \gamma_2 & \gamma_3\end{array}\right],\end{aligned} \tag{52}$$

and omitting $\delta, \delta_I, \delta_p, \delta_k, \delta_b$ in the notation, we obtain the following explicit expressions:

$$\begin{aligned}\pi_{11} &= 1,\\ \pi_{12} &= 0,\\ \pi_{13} &= 0,\\ \pi_{21} &= \frac{k^2+(b^2-pk)\omega_r^2}{b^2\omega_r^2+(k-p\omega_r^2)^2},\\ \pi_{22} &= \frac{-bp\omega_r^2}{b^2\omega_r^2+(k-p\omega_r^2)^2},\\ \pi_{23} &= 0,\\ \pi_{31} &= 0,\\ \pi_{32} &= 1,\\ \pi_{33} &= 0,\\ \pi_{41} &= \frac{bp\omega_r^4}{b^2\omega_r^2+(k-p\omega_r^2)^2},\\ \pi_{42} &= \frac{k^2+(b^2-pk)\omega_r^2}{b^2\omega_r^2+(k-p\omega_r^2)^2},\\ \pi_{43} &= 0,\end{aligned} \tag{53}$$

and

$$\begin{aligned}\gamma_1 &= -\frac{\omega_r^2}{b^2\omega_r^2 + (k - p\omega_r^2)^2}(b^2\omega_r^2(p + I)\\ &\quad + kp(k - p\omega_r^2) + I(k - p\omega_r^2)^2),\end{aligned} \tag{54}$$

$$\begin{aligned}\gamma_2 &= \frac{bp^2\omega_r^4}{b^2\omega_r^2 + (k - p\omega_r^2)^2},\\ \gamma_3 &= -1.\end{aligned}$$

Proof. We can convert RE to the equivalent system of algebraic equations (more details on such a transformation can be found in the work of Emirsajłow *et al.* (2023))

$$M\mathrm{vec}\,(\Pi, \Gamma) = N\,, \tag{55}$$

where $\mathrm{vec}\,(\Pi, \Gamma)$ denotes the single column matrix build by columns of Π and Γ stacking up on each other. If $I > 0$, $p > 0$, $k > 0$ and $b > 0$, then after tedious computations, supported by the Matlab Symbolic Math Toolbox (MathWorks, 2020c) we obtain

$$\det M = b^2\omega_r^2 + (k - p\omega_r^2)^2 \neq 0\,, \tag{56}$$

and if $b = 0$, then for $k \neq p\omega_r^2$ we still get $\det M \neq 0$. In both cases we can invert M and solve the system (55) to get a unique solution $\mathrm{vec}\,(\Pi, \Gamma) = M^{-1}N$, i.e., (53) and (54). Again, these computations have been supported by the Matlab Symbolic Math Toolbox. ■

3.3. Special structure of a controller and IMP.

In this subsection we show that every controller (A_K, B_K, C_K, D_K), which is independent of δ and has a special structure, guarantees that for every $\Gamma(\delta)$ there exists a matrix $\Sigma(\delta)$ such that IMP holds. We follow the general ideas of Isidori *et al.* (2003).

Recall that the matrix S of the exosystem Σ_S has the form (see (22) and (23))

$$S = \begin{bmatrix} 0 & 1 & 0 \\ -\omega_r^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{57}$$

The characteristic polynomial of S, which is also minimal, is given by

$$\Lambda_S(\lambda) = \lambda^3 + \omega_r^2\lambda\,. \tag{58}$$

Hence, S satisfies the equation

$$\Lambda_S(S) = S^3 + \omega_r^2 S = 0 \tag{59}$$

and, for any matrix $\Gamma \in \mathbb{R}^{1\times 3}$, we have

$$\Gamma S^3 = -\omega_r^2 \Gamma S\,. \tag{60}$$

If we define the following two matrices:

$$P := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\omega_r^2 & 0 \end{bmatrix}, \quad R := \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \tag{61}$$

then one can check that the matrix

$$V(\Gamma) := \begin{bmatrix} \Gamma \\ \Gamma S \\ \Gamma S^2 \end{bmatrix} \in \mathbb{R}^{3\times 3} \tag{62}$$

satisfies the system of the two matrix equations

$$\begin{cases} \Gamma = RV(\Gamma)\,, \\ V(\Gamma)S = PV(\Gamma)\,. \end{cases} \tag{63}$$

In proving (63) we have to use (60). One can also see that if Γ depends on δ so does $V(\Gamma)$. Moreover, the characteristic polynomial of P, which is also minimal, is given by

$$\Lambda_P(\lambda) = \lambda^3 + \omega_r^2\lambda\,, \tag{64}$$

so that $\sigma(P) = \sigma(S)$.

Lemma 1. *Let* $(\Pi(\delta), \Gamma(\delta))$ *be a solution of RE (see* (51)*). For every controller* Σ_K *of the order* n_K*, which is independent of* δ *and has the form*

$$\begin{aligned} A_K &= \begin{bmatrix} P & 0 \\ 0 & A_v \end{bmatrix} \in \mathbb{R}^{n_K\times n_K}, \\ B_K &= \begin{bmatrix} Q \\ B_v \end{bmatrix} \in \mathbb{R}^{n_K\times 1}, \\ C_K &= \begin{bmatrix} R & C_v \end{bmatrix} \in \mathbb{R}^{1\times n_K}, \\ D_K &= D_v \in \mathbb{R}^{1\times 1}, \end{aligned} \tag{65}$$

where $P \in \mathbb{R}^{3\times 3}$, $R \in \mathbb{R}^{1\times 3}$ *are given by* (61) *and* $Q \in \mathbb{R}^{3\times 1}$, $A_v \in \mathbb{R}^{(n_K-3)\times(n_K-3)}$, $B_v \in \mathbb{R}^{(n_K-3)\times 1}$, $C_v \in \mathbb{R}^{1\times(n_K-3)}$, $D_v \in \mathbb{R}^{1\times 1}$ *are arbitrary, there always exists a matrix* $\Sigma(\delta) \in \mathbb{R}^{n_K\times 3}$ *such that IMP holds (see* (41)*).*

Proof. In order to see this, we define

$$\Sigma(\delta) = \begin{bmatrix} V(\Gamma(\delta)) \\ 0 \end{bmatrix} \in \mathbb{R}^{n_K\times 3} \tag{66}$$

and substitute it into (41), which gives

$$\begin{cases} \Gamma(\delta) = \begin{bmatrix} R & C_v \end{bmatrix}\begin{bmatrix} V(\Gamma(\delta)) \\ 0 \end{bmatrix}, \\ \begin{bmatrix} V(\Gamma(\delta)) \\ 0 \end{bmatrix} S = \begin{bmatrix} P & 0 \\ 0 & A_v \end{bmatrix}\begin{bmatrix} V(\Gamma(\delta)) \\ 0 \end{bmatrix}. \end{cases} \tag{67}$$

After simple manipulations we obtain

$$\begin{cases} \Gamma(\delta) = RV(\Gamma(\delta))\,, \\ V(\Gamma(\delta))S = PV(\Gamma(\delta))\,, \end{cases} \tag{68}$$

which, by (63), holds for all $\Delta(\delta) \in \Delta_c$. ■

Summing up, Theorem 2 and Lemma 1 show that RE has a unique solution and if the controller Σ_K has the structure (65), then IMP holds, too. It remains to set the free parameters Q, A_v, B_v, C_v, D_v in (65) such that Σ_K will guarantee RIS. If this is done, then RAT will follow. The development of an appropriate stabilizing controller is accomplished in the next subsection and its robustness is analyzed in Section 4.

3.4. Construction of a stabilizing controller. In this subsection we develop a controller Σ_K based on the full order state observer of some modified plant. We obviously assume that parameters (A_K, B_K, C_K, D_K) of the controller Σ_K are described by the formulas (65) of Lemma 1, where $P \in \mathbb{R}^{3\times 3}$ and $R \in \mathbb{R}^{1\times 3}$ are given by (61) and $Q \in \mathbb{R}^{3\times 1}$, $A_v \in \mathbb{R}^{(n_K-3)\times(n_K-3)}$, $B_v \in \mathbb{R}^{(n_K-3)\times 1}$, $C_v \in \mathbb{R}^{1\times(n_K-3)}$, $D_v \in \mathbb{R}^{1\times 1}$ are to be chosen. If we partition the state x_K of the controller Σ_K as follows:

$$x_K(t) = \begin{bmatrix} w(t) \\ v(t) \end{bmatrix}, \qquad (w(t))_{t\geq 0} \subset \mathbb{R}^3,\ (v(t))_{t\geq 0} \subset \mathbb{R}^{n_K-3}, \quad (69)$$

then

$$\Sigma_K : \begin{cases} \dot w = Pw + Qe\,, \\ \dot v = A_v v + B_v e\,, \\ \tau = Rw + C_v v + D_v e\,. \end{cases} \quad (70)$$

The controller (70) consists of two parallel systems

$$\Sigma_w : \begin{cases} \dot w = Pw + Qe\,, \\ y_w = Rw\,, \end{cases} \quad (71)$$

and

$$\Sigma_v : \begin{cases} \dot v = A_v v + B_v e\,, \\ y_v = C_v v + D_v e\,, \end{cases} \quad (72)$$

with the joint output

$$\tau = y_w + y_v = Rw + C_v v + D_v e\,. \quad (73)$$

The above structure of Σ_K leads to the error feedback control system as shown in Fig. 3.

The matrices P and R are already given and if we are able to find $Q \in \mathbb{R}^{3\times 1}$, $A_v \in \mathbb{R}^{(n_K-3)\times(n_K-3)}$, $B_v \in \mathbb{R}^{(n_K-3)\times 1}$, $C_v \in \mathbb{R}^{1\times(n_K-3)}$, $D_v \in \mathbb{R}^{1\times 1}$ that guarantee RIS, i.e.,

$$\sigma\left(\begin{bmatrix} A(\delta)+B(\delta)D_K C & B(\delta)C_K \\ B_K C & A_K \end{bmatrix}\right) \subset \mathbb{C}_-\,, \qquad \Delta(\delta) \in \Delta_c\,, \quad (74)$$

then RAT will follow and Σ_K will be a robust controller. If we now substitute (65) into (74), then we get

$$\sigma\left(\begin{bmatrix} A(\delta)+B(\delta)D_v C & B(\delta)R & B(\delta)C_v \\ QC & P & 0 \\ B_v C & 0 & A_v \end{bmatrix}\right) \subset \mathbb{C}_-\,, \quad (75)$$

for all $\Delta(\delta) \in \Delta_c$, which is equivalent to say that the unforced closed loop system $\Sigma_{uf}(\delta)$ (the error feedback control system of Fig. 3 with $\alpha_r \equiv 0$, $d \equiv 0$ and no output), i.e.,

$$\Sigma_{uf}(\delta) : \begin{cases} \dot x = (A(\delta)+B(\delta)D_v C)x + B(\delta)Rw \\ \qquad +B(\delta)C_v v\,, \quad x(0) = x_0\,, \\ \dot w = QCx + Pw\,, \quad w(0) = w_0\,, \\ \dot v = B_v Cx + A_v v\,, \quad v(0) = v_0\,, \end{cases} \quad (76)$$

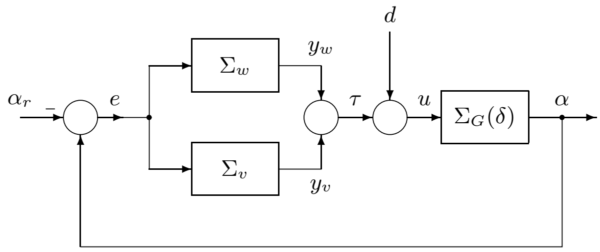

Fig. 3. Error feedback control system.

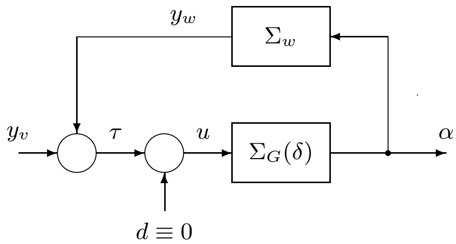

Fig. 4. Uncertain modified plant $\Sigma_m(\delta)$.

is asymptotically stable for all $\Delta(\delta) \in \Delta_c$.

Let us now define a system $\Sigma_m(\delta)$, referred to as the *uncertain modified plant* and shown in Fig. 4.

One can notice that $\Sigma_m(\delta)$ is a part of the unforced closed loop system $\Sigma_{uf}(\delta)$, has order $n_m = 7$, and is described by the equations

$$\Sigma_m(\delta) : \begin{cases} \dot x = A(\delta)x + B(\delta)Rw + B(\delta)y_v\,, \\ \dot w = QCx + Pw\,, \\ \alpha = Cx\,, \end{cases} \quad (77)$$

where y_v is the input and α is the output. Moreover, the unforced closed loop system (76) can be viewed as an interconnection of the uncertain modified plant $\Sigma_m(\delta)$ and the output feedback *subcontroller*

$$\Sigma_v : \begin{cases} \dot v = A_v v + B_v \alpha\,, \\ y_v = C_v v + D_v \alpha\,, \end{cases} \quad (78)$$

of order $n_K - 3$. For simplicity of the notation, we can write $\Sigma_m(\delta)$ as follows:

$$\Sigma_m(\delta) : \begin{cases} \dot\xi = A_m(\delta)\xi + B_m(\delta)y_v\,, \\ \alpha = C_m \xi\,, \end{cases} \quad (79)$$

where

$$\xi = \begin{bmatrix} x \\ w \end{bmatrix},$$

with the state space matrix

$$\left[\begin{array}{c|c} A_m(\delta) & B_m(\delta) \\ \hline C_m & 0 \end{array}\right] = \left[\begin{array}{cc|c} A(\delta) & B(\delta)R & B(\delta) \\ QC & P & 0 \\ \hline C & 0 & 0 \end{array}\right]. \quad (80)$$

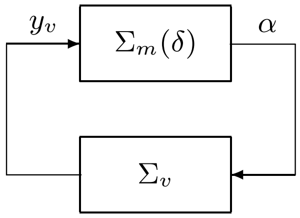

Fig. 5. $\Sigma_{uf}(\delta)$ as an interconnection of $\Sigma_m(\delta)$ and Σ_v

The interconnection of $\Sigma_m(\delta)$ and Σ_v is shown in Fig. 5.

What we have to do now is to choose the subcontroller (78) such that the resulting closed loop system, i.e., the unforced closed loop system $\Sigma_{uf}(\delta)$, described by (76), is asymptotically stable for all $\Delta(\delta) \in \Delta_c$. In order to develop such a controller, we will start with checking that the uncertain modified plant $\Sigma_m(\delta)$ is *controllable* and *observable*. First we choose $Q \in \mathbb{R}^{3\times 1}$ such that the pair (P, Q) is controllable. One possible choice is to take

$$Q = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T, \tag{81}$$

which gives

$$\det \begin{bmatrix} Q & PQ & P^2Q \end{bmatrix} = -(\omega_r^2 + 1)^2 \neq 0, \tag{82}$$

and (81) applies throughout the work. The controllability matrix, computed by means of the Symbolic Math Toolbox of Matlab (MathWorks, 2020c),

$$\begin{aligned} &W_m(\delta) \\ &= \begin{bmatrix} B_m(\delta) & A_m(\delta)B_m(\delta) & \cdots & A_m^6(\delta)B_m(\delta) \end{bmatrix} \end{aligned}$$

gives

$$\begin{aligned} \det(W_m(\delta)) = \frac{k^3(\delta_k)(\omega_r^2+1)^2}{I^7(\delta_I)p^5(\delta_p)} \big(b^2(\delta_b)\omega_r^2 \\ + (k(\delta_k) - p(\delta_p)\omega_r^2)^2\big) \neq 0, \end{aligned} \tag{83}$$

for all $k(\delta_k) > 0$, $I(\delta_I) > 0$, $p(\delta_p) > 0$ and $b(\delta_b) > 0$, which means that the pair $(A_m(\delta), B_m(\delta))$ is controllable for all $\Delta(\delta) \in \Delta_c$. If $b = 0$, then the extra condition $k(\delta_k) \neq p(\delta_p)\omega_r^2$ is required for controllability. Similarly, the observability matrix

$$V_m(\delta) = \begin{bmatrix} C_m & C_mA_m(\delta) & \cdots & C_mA_m^6(\delta) \end{bmatrix}^T$$

satisfies

$$\begin{aligned} \det(V_m(\delta)) = -\frac{k^3(\delta_k)}{I^5(\delta_I)p^3(\delta_p)} \big(b^2(\delta_b)\omega_r^2 \\ + (k(\delta_k) - p(\delta_p)\omega_r^2)^2\big) \neq 0, \end{aligned} \tag{84}$$

for all $k(\delta_k) > 0$, $I(\delta_I) > 0$, $p(\delta_p) > 0$ and $b(\delta_b) > 0$, which means that the pair $(C_m, A_m(\delta))$ is observable for all $\Delta(\delta) \in \Delta_c$. If $b = 0$, then the extra condition $k(\delta_k) \neq p(\delta_p)\omega_r^2$ is required for observability.

We start with the nominal case, i.e., with zero uncertainties, by setting $\delta = 0$, i.e., $(\delta_k, \delta_I, \delta_p, \delta_b) = (0, 0, 0, 0)$. In this case the modified plant is denoted by $\Sigma_m(0)$ and called the *nominal modified plant*. It is described as

$$\Sigma_m(0): \begin{cases} \dot{\xi} = A_m(0)\xi + B_m(0)y_v, \\ \alpha = C_m\xi. \end{cases} \tag{85}$$

For the nominal modified plant $\Sigma_m(0)$ we will construct a stabilizing controller (A_v, B_v, C_v, D_v), based on a full order state observer. It is clear that the final controller Σ_K (given by (65)) will provide the asymptotic stability for the unforced closed loop system $\Sigma_{uf}(0)$ as well as the asymptotic tracking for the closed loop system $\Sigma_{cl}(0)$, or, in other words, the internal stability and asymptotic tracking for the error feedback control system with the nominal plant $\Sigma_G(0)$. Once we have a stabilizing controller Σ_K for the nominal plant, we will analyze its robustness for the uncertain plant $\Sigma_G(\delta)$ with the uncertainty structure set Δ_c, using the structured singular value.

3.4.1. Controller based on the full order observer.

For $\Sigma_m(0)$, with the state space model (85), the full order Luenberger state observer is of the order $n_m = 7$ and has the form (see, e.g., Williams and Lawrence, 2007)

$$\dot{\tilde{\xi}} = (A_m(0) - LC_m)\tilde{\xi} + B_m(0)y_v + L\alpha, \tag{86}$$

with

$$\tilde{\xi} = \begin{bmatrix} \tilde{x} \\ \tilde{w} \end{bmatrix}$$

and the *output injection* gain matrix $L \in \mathbb{R}^{7\times 1}$ such that

$$\sigma(A_m(0) - LC_m) \subset \mathbb{C}_-, \tag{87}$$

where the spectrum can be freely assigned (by observability of $\Sigma_m(0)$). If we now apply the *feedback control law*

$$y_v = -F\tilde{\xi}, \tag{88}$$

with the *state feedback* gain matrix $F \in \mathbb{R}^{1\times 7}$ satisfying

$$\sigma(A_m(0) - B_m(0)F) \subset \mathbb{C}_-, \tag{89}$$

where the spectrum can be freely assigned (by controllability of $\Sigma_m(0)$), then the resulting closed loop system with the nominal plant $\Sigma_m(0)$, the observer (86) and the control law (88) is internally stable (Williams and Lawrence, 2007).

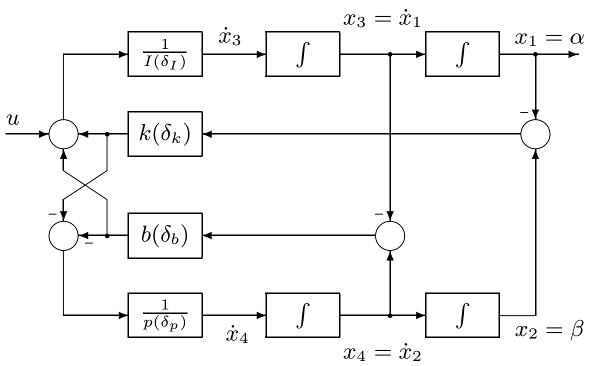

Fig. 6. Block diagram of the plant $\Sigma_G(\delta)$.

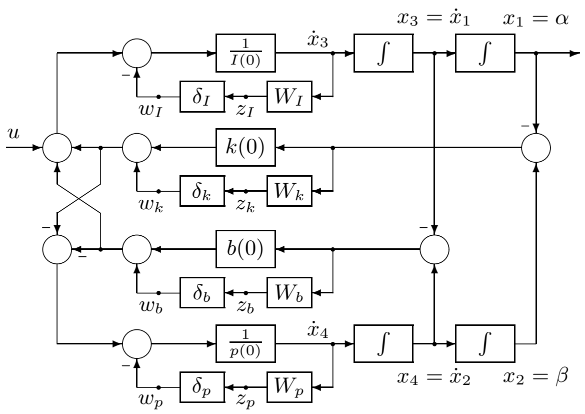

Fig. 7. Block diagram of the plant $\Sigma_G(\delta)$ with normalized parametric uncertainties.

Combining (86) and (88), we obtain the subcontroller Σ_v in the form

$$\Sigma_v : \begin{cases} \dot{\tilde{\xi}} = (A_m(0) - LC_m - B_m(0)F)\tilde{\xi} + L\alpha\,, \\ y_v = -F\tilde{\xi}\,, \end{cases} \tag{90}$$

i.e., $v = \tilde{\xi}$ and

$$\begin{aligned} A_v &= A_m(0) - LC_m - B_m(0)F, \qquad & B_v &= L\,, \\ C_v &= -F, & D_v &= 0\,. \end{aligned} \tag{91}$$

From our considerations it follows that the final controller (A_K, B_K, C_K, D_K) with

$$A_K = \begin{bmatrix} P & 0 \\ 0 & A_v \end{bmatrix} \in \mathbb{R}^{10\times 10},\ B_K = \begin{bmatrix} Q \\ B_v \end{bmatrix} \in \mathbb{R}^{10\times 1}, \\ C_K = \begin{bmatrix} R & C_v \end{bmatrix} \in \mathbb{R}^{1\times 10}, \quad D_K = D_v\,, \tag{92}$$

where matrices P, R are defined by (61), Q is given by (81) and (A_v, B_v, C_v, D_v) are given by (91), guarantees the internal stability and the asymptotic tracking for the error feedback control system with the nominal plant $\Sigma_G(0)$. We have also proved that if this controller satisfies RIS, then it also satisfies RAT.

In Section 4 we will show how to examine if this controller guarantees the internal stability of the control system with the uncertain plant $\Sigma_G(\delta)$ for all $\Delta(\delta) \in \Delta_c$. By exploring the scaling feature of the structured singular value we also develop a useful procedure allowing to characterize the uncertain parameters bounds which guarantee the robustness of this controller.

4. Robust internal stability

In this section we analyze the RIS condition by deriving a test based on the structured singular value as defined by Scherer (2001). For this purpose we will first develop a suitable mathematical model of the uncertain plant $\Sigma_G(\delta)$. This model uses the lower fractional transformation which has been comprehensively described by Zhou and Doyle (1998). Then we analyze the robustness of the internal stability of the error feedback control system by making use of the concept of structured singular value.

It is emphasized that computing structured singular values for uncertain real parameters is a demanding problem. However, an effective computational algorithm of that measure is available within the Matlab Robust Control Toolbox (MathWorks, 2020b).

4.1. Modelling the uncertain plant. The uncertain plant $\Sigma_G(\delta)$ is described by the state space model (15) which corresponds to the diagram shown below in Fig. 6.

Using now the expressions (5) we can transform the diagram from Fig. 6 to the form shown in Fig. 7.

In the latter diagram we have introduced four *fictitious signals* z_k, z_b, z_I, z_p, entering the four corresponding normalized uncertainties δ_k, δ_b, δ_I, δ_p and four *fictitious signals* w_k, w_b, w_I, w_p, leaving uncertainties, respectively. If we now *cut out* all uncertainty blocks, then we obtain a state space model of a system $\Sigma^{\Delta}_{G(0)}$ with inputs w_k, w_b, w_I, w_p, u and outputs z_k, z_b, z_I, z_p, α. The system $\Sigma^{\Delta}_{G(0)}$ is referred to as the *uncertain plant without uncertainties* and simple computations show that its model has the following compact form:

$$\Sigma^{\Delta}_{G(0)} : \begin{cases} \dot{x} = A(0)x + B_1 w_\Delta + B(0)u\,, \\ z_\Delta = WC_1 x + WB_1 w_\Delta + WB(0)u\,, \\ \alpha = Cx\,, \end{cases} \tag{93}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad z_\Delta = \begin{bmatrix} z_k \\ z_b \\ z_I \\ z_p \end{bmatrix}, \quad w_\Delta = \begin{bmatrix} w_k \\ w_b \\ w_I \\ w_p \end{bmatrix}, \tag{94}$$

with explicit formulas

$$B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{I(0)} & \frac{1}{I(0)} & -\frac{1}{I(0)} & 0 \\ -\frac{1}{p(0)} & -\frac{1}{p(0)} & 0 & -\frac{1}{p(0)} \end{bmatrix},$$

$$C_1 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ -\frac{k(0)}{I(0)} & \frac{k(0)}{I(0)} & -\frac{b(0)}{I(0)} & \frac{b(0)}{I(0)} \\ \frac{k(0)}{p(0)} & -\frac{k(0)}{p(0)} & \frac{b(0)}{p(0)} & -\frac{b(0)}{p(0)} \end{bmatrix},$$

and

$$W = \begin{bmatrix} W_k & 0 & 0 & 0 \\ 0 & W_b & 0 & 0 \\ 0 & 0 & W_I & 0 \\ 0 & 0 & 0 & W_p \end{bmatrix}. \tag{95}$$

Recall that in Section 2.2 we introduced the uncertainty matrix $\Delta(\delta) \in \mathbb{C}^{4\times 4}$ (see (27) and (28)) and the uncertainty structure set $\Delta_c \subset \mathbb{C}^{4\times 4}$ (see (29)). Notice that $\Delta_c \subset \mathbb{C}^{4\times 4}$ is a star-shaped set with center at zero. The star-shape property means that

$$\Delta(\delta) \in \Delta_c \quad \Rightarrow \quad \gamma\Delta(\delta) \in \Delta_c\,, \quad \gamma \in [0,\,1]\,. \tag{96}$$

By introducing the *block of uncertainties*

$$\Sigma_{\Delta(\delta)}:\; w_\Delta = \Delta(\delta) z_\Delta\,, \tag{97}$$

we can model the uncertain plant $\Sigma_G(\delta)$ (see (15)), with uncertain parameters transformed to the additive forms (5), as the interconnection shown in Fig. 8.

It is clear that $\Sigma_G(\delta)$ is the upper fractional transformation of $\Sigma^\Delta_{G(0)}$ and $\Sigma_{\Delta(\delta)}$ (Zhou and Doyle, 1998), i.e.,

$$\Sigma_G(\delta) = \mathcal{F}_u(\Sigma^\Delta_{G(0)}, \Sigma_{\Delta(\delta)})\,, \tag{98}$$

and for this interconnection to be well-posed we require the condition

$$\det(I - WB_1\Delta(\delta)) = \Big(1 + \frac{W_I}{I(0)}\delta_I\Big)\Big(1 + \frac{W_p}{p(0)}\delta_p\Big) \neq 0\,,$$

for $|\delta_k| < 1$, $|\delta_b| < 1$, $|\delta_I| < 1$ and $|\delta_p| < 1$, which obviously holds. Briefly,

$$\det(I - WB_1\Delta(\delta)) \neq 0\,, \quad \Delta(\delta) \in \Delta_c\,. \tag{99}$$

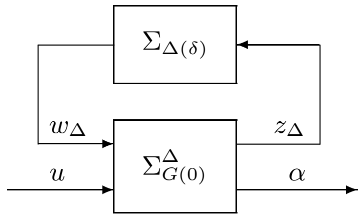

Fig. 8. Model of the uncertain plant $\Sigma_G(\delta)$.

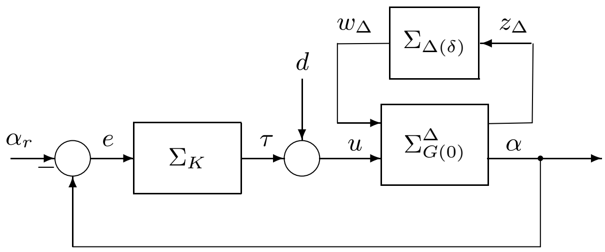

Fig. 9. Model of the error feedback control system with an uncertain plant $\Sigma_G(\delta)$.

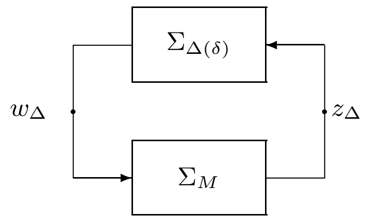

Fig. 10. $\Sigma_{uf}(\delta)$ as an interconnection of Σ_M and $\Sigma_{\Delta(\delta)}$.

4.2. Control system with the uncertain plant. Since the uncertain plant $\Sigma_G(\delta)$ is modelled as in Fig. 8, the error feedback control system can be reshaped as shown in Fig. 9.

Recall that the controller Σ_K has been designed to stabilize the nominal plant $\Sigma_G(0)$ (see Section 3.4), which means that

$$\sigma\left(\begin{bmatrix} A(0) + B(0)D_K C & B(0)C_K \\ B_K C & A_K \end{bmatrix}\right) \in \mathbb{C}_-\,. \tag{100}$$

First of all, let us notice that if in the error feedback control system in Fig. 9 we assume $\alpha_r \equiv 0$ and $d \equiv 0$, then the obtained unforced closed loop system $\Sigma_{uf}(\delta)$ can be viewed as an interconnection of some system Σ_M and the block of uncertainties $\Sigma_{\Delta(\delta)}$ as it is shown in Fig. 10.

Simple computations show that Σ_M is described by the following state space model:

$$\Sigma_M : \begin{cases} \dot{x} &= (A(0)+B(0)D_K C)x \\ &\quad +B(0)C_K x_K + B_1 w_\Delta\,, \quad x(0)=x_0\,, \\ \dot{x}_K &= B_K C x + A_K x_K\,, \quad x_K(0)=x_{K0}\,, \\ z_\Delta &= W(C_1+B(0)D_K C)x \\ &\quad +WB(0)C_K x_K + WB_1 w_\Delta\,. \end{cases} \tag{101}$$

One can check that if we now connect the system Σ_M and the uncertainty block $\Sigma_{\Delta(\delta)}$ by setting $w_\Delta = \Delta(\delta) z_\Delta$, then after some manipulations we arrive at a very complicated state space model, which is rather unsuitable for the internal stability analysis. Although the internal stability is essentially a state space concept, it can be also examined by using *transfer functions* of the systems involved instead of their state space models. However, for such an analysis the state space models have to be stabilizable and detectable.

The transfer function of Σ_M is given by

$$\left(\begin{array}{cc|c} A(0)+B(0)D_K C & B(0)C_K & B_1 \\ B_K C & A_K & 0 \\ \hline WC_1+WB(0)D_K C & WB(0)C_K & WB_1 \end{array}\right)$$

$$= W\left(\begin{array}{cc|c} A(0)+B(0)D_K C & B(0)C_K & B_1 \\ B_K C & A_K & 0 \\ \hline C_1+B(0)D_K C & B(0)C_K & B_1 \end{array}\right)$$

$$= W\widehat{M}_0(s)\,, \tag{102}$$

where W is the matrix of uncertainty weights and $\widehat{M}_0(s)$ is an auxiliary transfer function. It is clear that $W\widehat{M}_0(s)$ is stable (in the BIBO sense). Since $\Delta(\delta)$ is just a static matrix, it is also stable. In turn, $\det W \neq 0$ implies that Σ_M is stabilizable and detectable if and only if so is the state space realization of $\widehat{M}_0(s)$. Actually, the internal stability condition (100) can be used to show that the pair

$$\left(\begin{bmatrix} A(0)+B(0)D_K C & B(0)C_K \\ B_K C & A_K \end{bmatrix}, \begin{bmatrix} B_1 \\ 0 \end{bmatrix}\right) \tag{103}$$

is stabilizable and the pair

$$\left(\begin{bmatrix} C_1+B(0)D_K C & B(0)C_K \end{bmatrix}, \begin{bmatrix} A(0)+B(0)D_K C & B(0)C_K \\ B_K C & A_K \end{bmatrix}\right) \tag{104}$$

is detectable.

Remark 1. For example, in order to show that (103) is stabilizable, we can use the feedback gain matrix

$$-f\begin{bmatrix} B_1 \\ 0 \end{bmatrix}^T, \quad f \in \mathbb{R}\,. \tag{105}$$

Since the spectrum

$$\sigma(f) = \sigma\Big(\begin{bmatrix} A(0)+B(0)D_K C & B(0)C_K \\ B_K C & A_K \end{bmatrix} - f\begin{bmatrix} B_1 \\ 0 \end{bmatrix}\begin{bmatrix} B_1 \\ 0 \end{bmatrix}^T\Big) \in \mathbb{C}^{4+n_K} \tag{106}$$

is a continuous function of f on $\mathbb{R}$, with values in $\mathbb{C}^{4+n_K}$), we have $\lim_{f\to 0}\sigma(f) = \sigma(0) \subset \mathbb{C}_-$. Hence, for sufficiently small $|f| \neq 0$ we have $\sigma(f) \subset \mathbb{C}_-$. Analogously, we prove the detectability of (104). ■

Since $\Delta(\delta)$ and $W\widehat{M}_0(s)$ are proper and stable, it immediately follows from the robust control theory (e.g., Scherer, 2001) that the asymptotic stability of $\Sigma_{uf}(\delta)$ for all $\Delta(\delta) \in \Delta_c$ (see Fig. 8), which by definition means that the error feedback control system $\Sigma_e(\delta)$ satisfies RIS, can be characterized as follows.

Lemma 2. *The error feedback control system $\Sigma_e(\delta)$ satisfies RIS if and only if*

$$(I - W\widehat{M}_0(s)\Delta(\delta))^{-1} \in \mathcal{RH}_\infty\,, \quad \Delta(\delta) \in \Delta_c\,, \tag{107}$$

where $\mathcal{RH}_\infty$ is the space of real, rational, proper and stable matrices.

The condition (107) requires that the transfer matrix $I - W\widehat{M}_0(s)\Delta(\delta)$ does have a proper inverse for all $\Delta(\delta) \in \Delta_c$. This is the case if

$$\det(I - W\widehat{M}_0(\infty)\Delta(\delta)) = \det(I - WB_1\Delta(\delta)) \neq 0\,, \tag{108}$$

for all $\Delta(\delta) \in \Delta_c$, which holds by (99). However, a verification of the stability of the inverse

$$(I - W\widehat{M}_0(s)\Delta(\delta))^{-1} = \frac{\operatorname{adj}(I - W\widehat{M}_0(s)\Delta(\delta))}{\det(I - W\widehat{M}_0(s)\Delta(\delta))}\,, \quad \Delta(\delta) \in \Delta_c\,, \tag{109}$$

is a hard job since we have to check the condition

$$\det(I - W\widehat{M}_0(s)\Delta(\delta)) \neq 0\,, \quad s \in j\mathbb{R} \cup \mathbb{C}_+\,, \quad \Delta(\delta) \in \Delta_c\,, \tag{110}$$

which means that the rational function $\det(I - W\widehat{M}_0(s)\Delta(\delta))$ has no zeros in the closed right half plane for all $\Delta(\delta) \in \Delta_c$. It follows from the proof of Theorem 13 of Scherer (2001) that for static uncertainties Δ the following *equivalent* condition holds and makes life a bit easier.

Lemma 3. *The condition* (107) *holds and, consequently, the error feedback control system $\Sigma_e(\delta)$ satisfies RIS if and only if*

$$\det(I - W\widehat{M}_0(j\omega)\Delta(\delta)) \neq 0\,, \quad \Delta(\delta) \in \Delta_c\,, \quad \omega \in \mathbb{R}\,. \tag{111}$$

The main problem with (111) is that it has to be checked for all matrices $\Delta(\delta) \in \boldsymbol{\Delta}_c$ and all $\omega \in \mathbb{R}$. And here the concept of the structured singular value turns out to be helpful since it allows us to replace (111) by a much more practical but still equivalent condition.

4.3. Structured singular value. Recall that Lemma 3 says that for every $\omega \in \mathbb{R}$ we have to check the condition

$$\det(I - W\widehat{M}_0(j\omega)\Delta(\delta)) \neq 0\,, \quad \Delta(\delta) \in \boldsymbol{\Delta}_c\,, \quad (112)$$

and by (108) we already know that it holds for $\omega = \infty$.

Introducing the set $\gamma\boldsymbol{\Delta}_c \subset \mathbb{C}^{4\times 4}$, where $\gamma > 0$ is a scaling factor, we can modify the above problem as follows: *Find $\gamma^*(\omega)$ such that*

$$\gamma^*(\omega) = (\sup\{\gamma:\ \det(I - W\widehat{M}_0(j\omega)\Delta(\delta)) \neq 0\,, \quad \Delta(\delta) \in \gamma\boldsymbol{\Delta}_c\})^{-1}. \quad (113)$$

If the scaling factor $\gamma > 0$ is decreasing or increasing, then the set $\gamma\boldsymbol{\Delta}_c$ shrinks or becomes larger. For *sufficiently small* γ the condition $\det(I - W\widehat{M}_0(j\omega)\Delta(\delta)) \neq 0$ for all $\Delta(\delta) \in \gamma\boldsymbol{\Delta}_c$, always holds. Increasing γ we may meet a matrix $\Delta(\delta) \in \gamma\boldsymbol{\Delta}_c$ such that $\det(I - W\widehat{M}_0(j\omega)\Delta(\delta)) = 0$. If such a value γ does not exist, we set $\gamma^*(\omega) = \infty$.

Definition 1. (*Scherer, 2001*) Let $\omega \in \mathbb{R}$. The *structured singular value* of a matrix $W\widehat{M}_0(j\omega)$ for the uncertainty structure set $\boldsymbol{\Delta}_c$ is a non-negative number $\mu_{\boldsymbol{\Delta}_c}(W\widehat{M}_0(j\omega))$ defined by the expression

$$\begin{aligned}\mu_{\boldsymbol{\Delta}_c}(W\widehat{M}_0(j\omega)) &:= \frac{1}{\gamma^*(\omega)} \\ &= (\sup\{\gamma:\ \det(I - W\widehat{M}_0(j\omega)\Delta(\delta)) \neq 0\,, \\ &\quad \Delta(\delta) \in \gamma\boldsymbol{\Delta}_c\})^{-1}\,.\end{aligned}$$

It should be emphasized that the structured singular value $\mu_{\boldsymbol{\Delta}_c}(W\widehat{M}_0(j\omega))$ depends on both the matrix $W\widehat{M}_0(j\omega)$ and the uncertainty structure set $\boldsymbol{\Delta}_c$.

Since the structure set $\boldsymbol{\Delta}_c$ is star-shaped, for $0 < \gamma_1 \leq \gamma_2$ we have the inclusion $\gamma_1\boldsymbol{\Delta}_c \subset \gamma_2\boldsymbol{\Delta}_c$. It follows from Definition 1 that for $\gamma \leq \gamma^*(\omega)$ we always have

$$\det(I - W\widehat{M}_0(j\omega)\Delta(\delta)) \neq 0\,, \quad \Delta(\delta) \in \gamma\boldsymbol{\Delta}_c\,. \quad (114)$$

If $\gamma^*(\omega)$ satisfies $1 \leq \gamma^*(\omega)$, then

$$\det(I - W\widehat{M}_0(j\omega)\Delta(\delta)) \neq 0\,, \quad \Delta(\delta) \in \boldsymbol{\Delta}_c\,. \quad (115)$$

If $\gamma^*(\omega)$ satisfies $1 > \gamma^*(\omega)$, then there always exists $\Delta(\delta) \in \boldsymbol{\Delta}_c$ such that

$$\det(I - W\widehat{M}_0(j\omega)\Delta(\delta)) = 0\,, \quad (116)$$

i.e., (115) fails.

Since $1 \leq \gamma^*(\omega)$ is equivalent to

$$1 \geq \frac{1}{\gamma^*(\omega)} = \mu_{\boldsymbol{\Delta}_c}(W\widehat{M}_0(j\omega))\,, \quad (117)$$

and $1 > \gamma^*(\omega)$ is equivalent to

$$1 < \frac{1}{\gamma^*(\omega)} = \mu_{\boldsymbol{\Delta}_c}(W\widehat{M}_0(j\omega))\,, \quad (118)$$

we obtain the following result which relates the structure singular value and the robust internal stability of the error feedback control system.

Theorem 3. *The condition*

$$\det(I - W\widehat{M}_0(j\omega)\Delta(\delta)) \neq 0\,, \quad \Delta(\delta) \in \boldsymbol{\Delta}_c \quad \omega \in \mathbb{R}\,, \quad (119)$$

holds if and only if the structured singular value of the matrix $W\widehat{M}_0(j\omega)$ for the structure set $\boldsymbol{\Delta}_c$ satisfies

$$\mu_{\boldsymbol{\Delta}_c}(W\widehat{M}_0(j\omega)) \leq 1\,, \quad \omega \in \mathbb{R}\,. \quad (120)$$

Consequently, the error feedback control system $\Sigma_e(\delta)$ satisfies the RIS condition if and only if (120) *holds.*

Unfortunately, there is no general method of computing $\mu_{\boldsymbol{\Delta}_c}(W\widehat{M}_0(j\omega))$ exactly and we can only compute its lower and upper bounds.

In this paper we propose to explore the essential feature of the structure singular value following from the fact that

$$\gamma\,\mu_{\boldsymbol{\Delta}_c}(W\widehat{M}_0(j\omega)) = \mu_{\boldsymbol{\Delta}_c}(\gamma W\widehat{M}_0(j\omega))\,, \quad (121)$$

which means that scaling μ by the factor γ is equivalent to scaling the matrix W. In practice, we can always expect that we are in a position to compute some global *upper bound* γ_u of μ, i.e.,

$$\mu_{\boldsymbol{\Delta}_c}(W\widehat{M}_0(j\omega)) \leq \gamma_u\,, \quad \omega \in \mathbb{R}\,. \quad (122)$$

Since (122) can be equivalently written in the form

$$\mu_{\boldsymbol{\Delta}_c}(\gamma_u^{-1}W\widehat{M}_0(j\omega)) \leq 1\,, \quad \omega \in \mathbb{R}\,, \quad (123)$$

(123) is equivalent to the robust internal stability of $\Sigma_e(\delta)$ for the scaled matrix of weights W_γ, where

$$W_\gamma := \gamma_u^{-1}W\,, \quad (124)$$

with the unchanged structure set $\boldsymbol{\Delta}_c$. This new (rescaled) matrix of weights allows us to define new (rescaled) intervals for parameters (see (7)) which guarantee the RIS and RAS conditions.

Remark 2. It is worth mentioning that the Robust Control Toolbox of the Matlab package (MathWorks, 2020b) has a powerful function `mussv` which adaptively selects a finite series of frequencies $(\omega_i)_{i=0}^{i=N} \subset [0, \infty)$ and returns the lower and the upper bounds of the structured singular value

$$\gamma_l(\omega_i) \le \mu_{\Delta_c}(W\widehat{M}_0(j\omega_i)) \le \gamma_u(\omega_i)\,, \qquad i = 0, 1, \ldots, N\,. \tag{125}$$

It allows us to estimate the global upper bound (122) as follows

$$\gamma_u = \max_{i=0,1,\ldots,N} \gamma_u(\omega_i)\,. \tag{126}$$

5. Numerical simulations

In order to test the performance of the obtained robust controller Σ_K, we have computed an example of the error feedback control system with an uncertain plant $\Sigma_G(\delta)$ using Matlab/Simulink. The results are presented for the controller (A_K, B_K, C_K, D_K) with the subcontroller (A_v, B_v, C_v, D_v) based on the full order observer (86) and the control law (88), as described in Section 3.4. The data assumed in computations correspond to the class of microsatellites of weight 50–100 kg.

5.1. Nominal plant $\Sigma_G(0)$. We assume the following data for the nominal plant $\Sigma_G(0)$:

$$\begin{aligned} &k(0) = 750 \left[\frac{\mathrm{N \cdot m}}{\mathrm{rad}}\right], \quad b(0) = 0.01\ [\mathrm{N \cdot m \cdot s}]\,, \\ &I(0) = 1.7\ [\mathrm{kg \cdot m^2}]\,, \quad p(0) = 0.1\ [\mathrm{kg \cdot m^2}]\,, \end{aligned} \tag{127}$$

and for the reference signal $\alpha_r = a\sin(\omega_r t)$ and the disturbance d_0:

$$\begin{aligned} a = 1\ [\mathrm{rad}]\,, \quad \omega_r = 1 \left[\frac{\mathrm{deg}}{\mathrm{s}}\right] = \frac{\pi}{180}\left[\frac{\mathrm{rad}}{\mathrm{s}}\right], \\ d_0 = 0.01\ [\mathrm{N \cdot m}]\,. \end{aligned} \tag{128}$$

The nominal plant $\Sigma_G(0)$ has the state space matrix (129), the exosystem Σ_S has the state space matrix

$$\left[\begin{array}{c} S \\ \hline T_r \\ T_d \end{array}\right] = \left[\begin{array}{ccc} 0 & 1 & 0 \\ -0.0003 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right], \tag{130}$$

with

$$\sigma(S) = \left[\begin{array}{ccc} 0 & j0.0175 & -j0.0175 \end{array}\right]^T, \tag{131}$$

and generates the reference α_r and the disturbance d in the form

$$\begin{aligned} \alpha_r(t) &= \sin 0.0175t\,, \\ d(t) &= 0.01\,. \end{aligned} \tag{132}$$

Moreover, the nominal modified plant $\Sigma_m(0)$ has the state space matrix (133). The real parameters k, b, I and p of the uncertain plant belong to the intervals

$$\begin{aligned} k \in (k_{\min}, k_{\max})\,, \quad b \in (b_{\min}, b_{\max})\,, \\ I \in (I_{\min}, I_{\max})\,, \quad p \in (p_{\min}, p_{\max})\,, \end{aligned} \tag{134}$$

where the bounds are assumed as follows:

$$\begin{aligned} k_{\min} &= 0.8k(0) = 600\,, & k_{\max} &= 1.2k(0) = 900\,, \\ b_{\min} &= 0.7b(0) = 0.007\,, & b_{\max} &= 1.3b(0) = 0.013\,, \\ I_{\min} &= 0.9I(0) = 1.53\,, & I_{\max} &= 1.1I(0) = 1.87\,, \\ p_{\min} &= 0.95p(0) = 0.095\,, & p_{\max} &= 1.05p(0) = 0.105\,, \end{aligned} \tag{135}$$

and obviously agree with the nominal values (127). Hence the weight matrix W is given by

$$\begin{aligned} W &= \begin{bmatrix} W_k & 0 & 0 & 0 \\ 0 & W_b & 0 & 0 \\ 0 & 0 & W_I & 0 \\ 0 & 0 & 0 & W_p \end{bmatrix} \\ &= \begin{bmatrix} 150 & 0 & 0 & 0 \\ 0 & 0.003 & 0 & 0 \\ 0 & 0 & 0.17 & 0 \\ 0 & 0 & 0 & 0.005 \end{bmatrix}, \end{aligned} \tag{136}$$

and the uncertain parameters in the additive form

$$\begin{aligned} k(\delta_k) = 750 + 150\delta_k\,, \quad b(\delta_b) = 0.01 + 0.003\delta_b\,, \\ I(\delta_I) = 1.7 + 0.17\delta_I\,, \quad p(\delta_p) = 0.1 + 0.005\delta_p\,, \end{aligned} \tag{137}$$

where δ_k, δ_b, δ_I and δ_p, are normalized uncertainties. The uncertainty structure set Δ_c has been defined by (27) and (28).

5.2. Controller based on the full order observer. In order to obtain the subcontroller Σ_v (see (90) and (91)) the state feedback gain matrix $F \in \mathbb{R}^{1\times 7}$ is chosen to minimize the quadratic functional

$$J(y_v) = q_0 \int_0^\infty \xi^T(t)\xi(t)\,\mathrm{d}t + u_0 \int_0^\infty y_v^2(t)\,\mathrm{d}t\,, \tag{138}$$

with weights $q_0 = 1$ and $u_0 = 1$, for the nominal modified plant $\Sigma_m(0)$, i.e., $\dot{\xi}(t) = A_m(0)\xi(t) + B_m(0)y_v(t)$ with $y_v(t) = -F\xi(t)$. To solve this optimization, we used the `lqr` procedure from the Control System Toolbox of MATLAB (MathWorks, 2020a). Hence

$$F = \begin{bmatrix} 11.8073 \\ -2.5633 \\ 6.3924 \\ -0.4524 \\ 2.4142 \\ 3.4758 \\ 3.9193 \end{bmatrix}^T, \tag{139}$$

$$\left[\begin{array}{c|c} A(0) & B(0) \\ \hline C & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -441.17 & 441.17 & -0.0059 & 0.0059 & 0.5882 \\ 7500 & -7500 & 0.1 & -0.1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \end{array}\right], \tag{129}$$

$$\left[\begin{array}{c|c} A_m(0) & B_m(0) \\ \hline C_m & 0 \end{array}\right] = \left[\begin{array}{ccccccc|c} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -441.17 & 441.17 & -0.0059 & 0.0059 & 0.5882 & 0 & 0 & 0.5882 \\ 7500 & -7500 & 0.1 & -0.1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -0.0003 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{133}$$

and

$$\sigma(A_m(0) - B_m(0)F) = \begin{bmatrix} -0.2833 + j89.1145 \\ -0.2833 - j89.1145 \\ -1.2608 \\ -0.3849 + j0.8545 \\ -0.3849 - j0.8545 \\ -0.6345 + j0.5541 \\ -0.6345 - j.5541 \end{bmatrix}. \tag{140}$$

The output injection gain matrix $L \in \mathbb{R}^{7\times 1}$ is chosen such that $L^T \in \mathbb{R}^{1\times 7}$ minimizes the quadratic cost functional

$$J(\vartheta) = q_1 \int_0^\infty \xi^T(t)\xi(t)\,\mathrm{d}t + u_1 \int_0^\infty \vartheta^2(t)\,\mathrm{d}t, \tag{141}$$

with weights $q_1 = 1$ and $u_1 = 1$, for the system dual to $\Sigma_m(0)$, i.e. $\dot{\xi}(t) = A_m^T(0)\xi(t) + C_m^T\vartheta(t)$ with $\vartheta(t) = -L^T\xi(t)$. To solve this problem we again used the `lqr` MATLAB procedure. Hence,

$$L = \begin{bmatrix} 3.4156 \\ 2.9474 \\ 5.3332 \\ 4.6864 \\ 9.9954 \\ 6.0420 \\ 2.4117 \end{bmatrix}, \tag{142}$$

and

$$\sigma(A_m(0) - LC_m) = \begin{bmatrix} -0.0659 + j89.1130 \\ -0.0659 - j89.1130 \\ -0.3577 + j0.8413 \\ -0.3577 - j0.8413 \\ -1.2102 \\ -0.7321 + j0.4910 \\ -0.7321 - j0.4910 \end{bmatrix}. \tag{143}$$

The obtained state space matrix of the subcontroller (A_v, B_v, C_v, D_v) is given by (144) (according to (91)) and the final error feedback controller (A_K, B_K, C_K, D_K) is described by (92) with

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -0.0003 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$R = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

5.3. Performance for the nominal plant $\boldsymbol{\Sigma_G(0)}$. The results of numerical simulations showing the performance of the controller Σ_K in the error feedback control system with the nominal plant $\Sigma_G(0)$ are displayed in Figs. 11–13.

5.4. Structured singular value $\boldsymbol{\mu_{\Delta_c}(W\widehat{M}_0)}$. In order to show the robustness of the controller Σ_K in the error feedback control system with an uncertain plant $\Sigma_G(\delta)$, we have used the `mussv` MATLAB procedure to compute the global upper bound γ_u of the structured singular value

$$\sup_{\omega\geq 0} \mu_{\Delta_c}(W\widehat{M}_0(j\omega)) \leq \gamma_u, \tag{145}$$

where $W\widehat{M}_0(j\omega) = W\widehat{M}_0(s)|_{s=j\omega}$ and $W\widehat{M}_0(s)$ is the transfer function of the system Σ_M (see (101) and (102)). We have obtained the maximum

$$\gamma_u = 1.2875 \quad \text{for} \quad \omega_u = 1.276, \tag{146}$$

and

$$1.2845 = \gamma_l \leq \mu_{\Delta_c}(W\widehat{M}_0(j1.276)) \leq \gamma_u = 1.2875, \tag{147}$$

which shows that (147) is a reasonably tight estimate. If we now rescale the weight matrix W by the factor $\gamma^{-1} = 0.7692$, where $\gamma = 1.3 \geq \gamma_u$, then we get

$$\mu_{\Delta_c}(W_\gamma\widehat{M}_0(j1.276)) \leq \frac{\gamma_u}{\gamma} = 0.9905 < 1, \tag{148}$$

$$\left[\begin{array}{c|c} A_v & B_v \\ \hline C_v & D_v \end{array}\right] = \left[\begin{array}{ccccccc|c} -2.8 & 0 & 1 & 0 & 0 & 0 & 0 & 3.4156 \\ -2.9 & 0 & 0 & 1 & 0 & 0 & 0 & 2.9474 \\ -453.5 & 442.7 & -3.8 & 0.272 & -0.8319 & -2.0446 & -2.3055 & 5.3332 \\ 7495.3 & -7500 & 0.1 & -0.1 & 0 & 0 & 0 & 4.6864 \\ -9.0 & 0 & 0 & 0 & 0 & 1 & 0 & 9.9954 \\ -5.0 & 0 & 0 & 0 & 0 & 0 & 1 & 6.0420 \\ -1.4 & 0 & 0 & 0 & 0 & -0.0003 & 0 & 2.4117 \\ \hline -11.8073 & 2.5633 & -6.3924 & 0.4524 & -2.4142 & -3.4758 & -3.9193 & 0 \end{array}\right]. \tag{144}$$

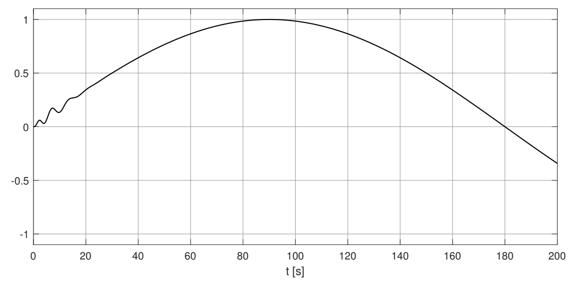

Fig. 11. Output $\alpha(t)$ for $\Sigma_G(0)$.

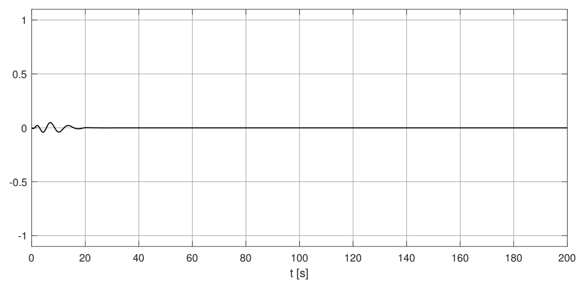

Fig. 12. Error $e(t)$ for $\Sigma_G(0)$.

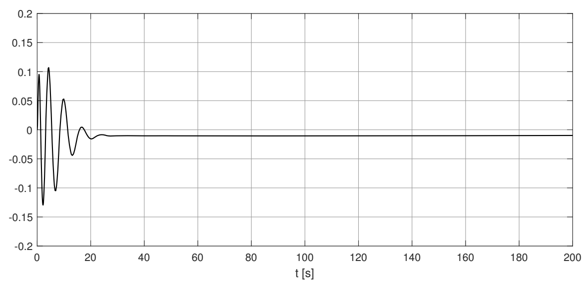

Fig. 13. Control torque $\tau(t)$ for $\Sigma_G(0)$.

and for the new matrix

$$W_\gamma = \gamma^{-1}W$$
$$= \begin{bmatrix} W_{\gamma k} & 0 & 0 & 0 \\ 0 & W_{\gamma b} & 0 & 0 \\ 0 & 0 & W_{\gamma I} & 0 \\ 0 & 0 & 0 & W_{\gamma p} \end{bmatrix}$$

$$= \begin{bmatrix} 115.3846 & 0 & 0 & 0 \\ 0 & 0.0023 & 0 & 0 \\ 0 & 0 & 0.1308 & 0 \\ 0 & 0 & 0 & 0.0038 \end{bmatrix} \tag{149}$$

we have new (rescaled) parameter bounds

$$\begin{aligned} k_{\min} &= k(0) - W_{\gamma k} = 634.6154\,, \\ k_{\max} &= k(0) + W_{\gamma k} = 865.3846\,, \\ b_{\min} &= b(0) - W_{\gamma b} = 0.0077\,, \\ b_{\max} &= b(0) + W_{\gamma b} = 0.0123\,, \\ I_{\min} &= I(0) - W_{\gamma I} = 1.5692\,, \\ I_{\max} &= I(0) + W_{\gamma I} = 1.8308\,, \\ p_{\min} &= p(0) - W_{\gamma p} = 0.0962\,, \\ p_{\max} &= p(0) + W_{\gamma p} = 0.1038\,. \end{aligned} \tag{150}$$

It is guaranteed that the controller Σ_K will robustly stabilize all uncertain plants $\Sigma_G(\delta)$ with real parameters k, b, I and p from these new intervals and, moreover, the robust asymptotic tracking condition will hold. Notice that the weight matrix W_γ is modified but the uncertainty structure set Δ_c has not been changed. The results of numerical simulations showing the performance of the controller for different uncertain plants are given below.

5.5. Performance for two uncertain plants $\Sigma_G(\delta)$.

Example 1. For $\Sigma_G(\delta)$ with the parameters

$$\begin{aligned} k &= k(0) - 0.9W_{\gamma k} = 646.1538\,, \\ b &= b(0) - 0.9W_{\gamma b} = 0.0079\,, \\ I &= I(0) + 0.9W_{\gamma I} = 1.8177\,, \\ p &= p(0) + 0.9W_{\gamma p} = 0.1035\,, \end{aligned} \tag{151}$$

the state space matrix takes the form of (152).

The corresponding results are shown in Figs. 14–16. ♦

Example 2. For the plant $\Sigma_G(\delta)$ with the parameters

$$\begin{aligned} k &= k(0) + 0.9W_{\gamma k} = 853.8462\,, \\ b &= b(0) + 0.9W_{\gamma b} = 0.0121\,, \\ I &= I(0) - 0.9W_{\gamma I} = 1.582\,, \\ p &= p(0) - 0.9W_{\gamma p} = 0.0965, \end{aligned} \tag{153}$$

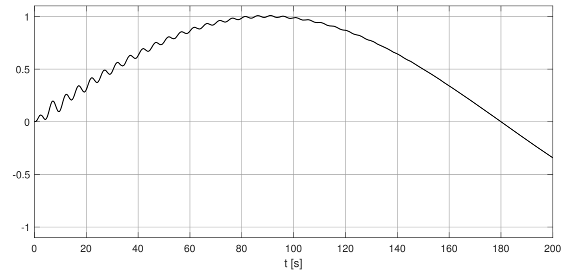

Fig. 14. Output $\alpha(t)$ for $\Sigma_G(\delta)$.

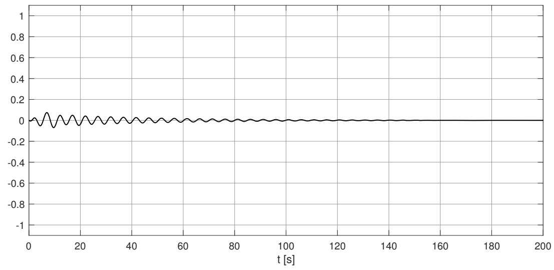

Fig. 15. Error $e(t)$ for $\Sigma_G(\delta)$.

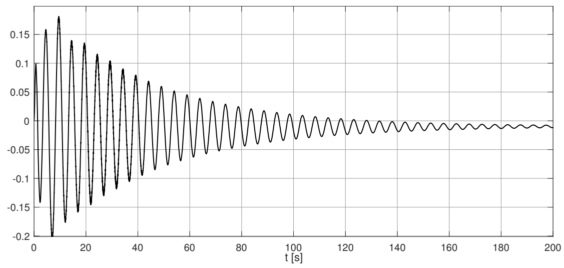

Fig. 16. Control torque $\tau(t)$ for $\Sigma_G(\delta)$.

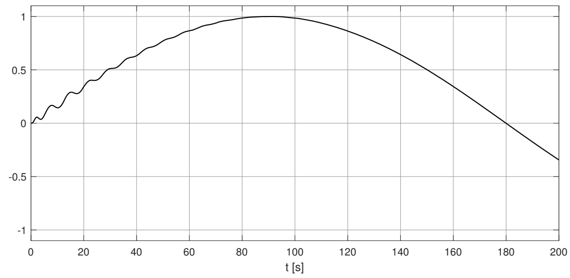

Fig. 17. Output $\alpha(t)$ for $\Sigma_G(\delta)$.

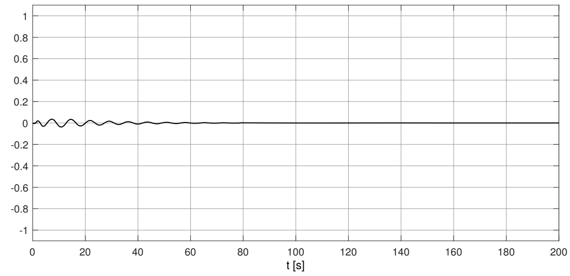

Fig. 18. Error $e(t)$ for $\Sigma_G(\delta)$.

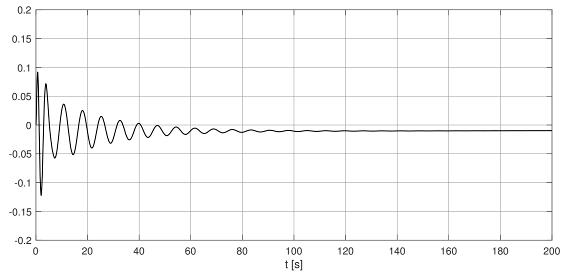

Fig. 19. Control torque $\tau(t)$ for $\Sigma_G(\delta)$.

the state space matrix takes the form of (154).

The corresponding results are shown in Figs. 17–19. ♦

For choosing the feedback gain F and the output injection L we have used the LQR optimization which resulted in a satisfactory behaviour of signals. The scalar weights q_0, u_0 and q_1, u_1 have been introduced to control that behaviour and have been chosen by trial and error. Instead of the simple scalar weights q_0, q_1 in the functionals (138) and (141), one can introduce matrices to weigh the state variables of $\Sigma_m(0)$ selectively.

In both the cases of uncertain plants (151) and (153) we have assumed significant deviations of parameters from their nominal values. The graphs presented in Figs. 14–18 confirm that the controller Σ_K, or rather the subcontroller Σ_v, guarantees robustness of the exact asymptotic tracking.

6. Final remarks

In the paper we have demonstrated that the robust general regulator theory provides an efficient algorithm for a robust feedback error controller which makes the displacement of an underactuated 2DOF mechanical system to asymptotically track a harmonic signal in the presence of significant parametric uncertainties in the mathematical model. As an example, we have considered the attitude control problem of an earth observation microsatellite with a solar panel. The performed numerical computations show that this approach may be seen as a completion to the robust control theory based on the μ-synthesis which is well supported by the Matlab/Simulink computational software. It should be emphasized that the exact asymptotic tracking (36) cannot be fit in the μ-synthesis problem as a performance criterion since it cannot be expressed in terms of $\mathcal{H}_\infty$-norm minimization.

$$\left[\begin{array}{c|c} A(\delta) & B(\delta) \\ \hline C & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -355.5 & 355.5 & -0.0044 & 0.0044 & 0.5501 \\ 6245.4 & -6245.4 & 0.0766 & -0.0766 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \end{array}\right], \tag{152}$$

$$\left[\begin{array}{c|c} A(\delta) & B(\delta) \\ \hline C & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -539.6 & 539.6 & -0.0076 & 0.0076 & 0.6320 \\ 8844.6 & -8844.6 & 0.1251 & -0.1251 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{154}$$

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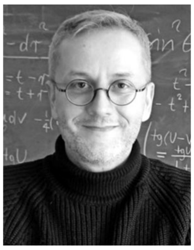

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