Int. J. Appl. Math. Comput. Sci., 2025, Vol. 35, No. 1, 117–128
DOI: 10.61822/amcs-2025-0009

A UNIFIED CONVEX COMBINATION APPROACH TO SWITCHED UNCERTAIN NONLINEAR SYSTEMS

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We address a unified convex combination approach to a class of switched uncertain nonlinear systems, focusing on quadratic stability and $\mathcal{L}_2$ gain. In each subsystem, there are norm-bounded uncertainties in the system matrix and nonlinear terms with quadratic constraints. The proposed convex combination is original and unified in the sense of incorporating not only the nominal subsystem matrices but also uncertainty and quadratic constraints in the same form. When there is no single subsystem having the desired performance but a convex combination of subsystems does, we design a switching law so that the switched system achieves the same performance. Moreover, the discussion is extended to switching state feedback and its application to a boost converter.

Keywords: switched uncertain nonlinear system, convex combination of subsystems, quadratic stability, $\mathcal{L}_2$ gain, state-dependent switching law.

1. Introduction

Switched systems are composed of a set of subsystems and a switching law activating one of the subsystems at each time instant, and have been studied extensively in the last three decades (Liberzon and Morse, 1999; DeCarlo *et al.*, 2000; Shorten *et al.*, 2007; Goebel *et al.*, 2009; Alwan and Liu, 2016; Liu and Li, 2022). When all or part of subsystems are stable, the control problem is to identify the class of switching laws such that the switched system is asymptotically stable. There have been quantities of references working on this problem, and several efficient approaches have been proposed, including the average dwell time approach (Morse, 1996; Hespanha and Morse, 1999; Zhai *et al.*, 2000; Yu and Zhai, 2020), the piecewise/multiple Lyapunov function (Branicky, 1998; Zhai *et al.*, 2000), the conic switching law (Xu and Antsaklis, 2000), etc.

The more interesting and challenging case is that no single subsystem is stable, and the objective is to design a switching law (strategy) such that the resultant switched system is stable (Liberzon and Morse, 1999; Liberzon, 2003). For switched LTI systems, it has been shown (Wicks *et al.*, 1998; Feron, 1996) that if it is possible to obtain a stable convex combination of subsystems (CCS), then we can design a stabilizing switching law which uses the state information at each time instant. Zhai *et al.* (2003) have proposed a quadratic stabilizability condition for a class of switched linear systems that include polytopic uncertainties in subsystem matrices.

Recently, Chang *et al.* (2019) have extended the idea of convex combination (Wicks *et al.*, 1998; Feron, 1996) of subsystem matrices to quadratically stabilizing switched linear systems where there are norm-bounded uncertainties in subsystem matrices. In that context, assuming that no single subsystem is quadratically stable (QS), a new CCS is proposed by incorporating the matrices accounting for uncertainties. If there is a CCS

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which is QS for any uncertainty under consideration, then a switching law depending on the system state can be designed to quadratically stabilize the switched system. Chang *et al*. (2022a) have extended the discussion of Chang *et al*. (2019) to the study of $\mathcal{L}_2$ gain performance for switched systems with norm-bounded uncertainties and disturbance input, and have proposed a new CCS based switching law to guarantee that the switched system exhibits the desired behavior.

Since the above references are mainly on analysis and design of switched linear systems, this paper is aimed at extending the CCS approach to the case where nonlinear terms exist in each subsystem. The motivation behind including nonlinear terms is that almost every real system has nonlinear factors (elements) in its dynamics (Sassano *et al*., 2019), although we may focus on its linear (or linearized) part for simplicity and/or for technical reasons. If the nonlinear term is completely unknown, then it is difficult to perform any quantitative analysis and design for the system. In this paper, we assume that the nonlinear terms are known or unknown but they are upper bounded by a known quadratic function of the system state. Then, we propose a unified convex combination of subsystems which incorporates not only the nominal subsystem matrices but also the uncertainty and the quadratic constraints in the same form. The performance of the system under consideration is characterized by the QS and $\mathcal{L}_2$ gain between disturbance input and controlled output. As before, it is assumed that no single subsystem has the desired performance (QS or $\mathcal{L}_2$ gain). Then, we propose a design strategy to seek a CCS which has the desired performance, and then propose a switching law based on the system state to guarantee that the switched system achieves the same performance. When the design strategy does not work without a control input, we consider the design of switching state feedback so that the CCS approach is applicable, and demonstrate the performance with a boost converter.

The contributions of this paper are summarized as follows:

- Compared with the existing CCS, the CCS in this study presents a major extension in the sense that the convex combinations of subsystem matrices, uncertainty terms, disturbance input matrices, controlled output matrices and quadratic constraint matrices are unified in the same form. Thus, the CCS approach in this paper is both original and practical.

- With the CCS approach, state-dependent switching laws are proposed to guarantee the QS and $\mathcal{L}_2$ gain of the switched uncertain nonlinear system (SUNLS), even if each subsystem may not have the desired property.

This paper is organized as follows. We provide some preliminary lemmas and formulate the control problem in Section 2. Then, Section 3 addresses quadratic stability (QS) analysis for the SUNLS, and Section 4 proceeds to quadratic $\mathcal{L}_2$ gain analysis. Section 5 extends the discussion to the SUNLS with switching state feedback controller design, and presents the application to $\mathcal{L}_2$ gain design for a DC-DC boost converter model. Finally Section 6 provides some concluding remarks.

Throughout this paper, we use $W^\top$ (resp. W^{-1}) to denote the transpose (resp., inverse) of the matrix W. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ stand for the set of real n-dimensional vectors and $n \times m$ matrices, respectively. For a square matrix V, we use $\mathrm{He}\{V\}$ to denote $V + V^\top$ for simplicity, and write $V \succ 0$ (resp. $V \prec 0$) when V is symmetric and positive (resp., negative) definite. For a set of matrices $V_1, \dots, V_n$, we define their convex combination by using the notation $\mathrm{Conv}_\lambda\{V_1, \dots, V_n\}$, which is the set of matrices $\lambda_1 V_1 + \dots + \lambda_n V_n$ with nonnegative scalars λ_i satisfying $\lambda_1 + \dots + \lambda_n = 1$.

2. Some preliminaries and problem formulation

We start with the nonlinear control system

$$\begin{cases} \dot{x} = (A + H\Delta(t)F)\,x + B_1 w + g(t,x), \\ z = Cx, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^r$ and $z \in \mathbb{R}^p$ are the system state, the disturbance input and the controlled output, respectively. $A \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times r}$, $C \in \mathbb{R}^{p\times n}$, $H \in \mathbb{R}^{n\times q}$, $F \in \mathbb{R}^{k\times n}$ are constant matrices denoting the nominal part of the system, and $\Delta(t) \in \mathbb{R}^{q\times k}$ describes the norm-bounded uncertainties and without loss of generality, $\|\Delta(t)\| \le 1$. Here $g(t,x)$ is a nonlinear vector which is known or unknown, and it satisfies the constraint (Siljak and Stipanovic, 2000)

$$g(t,x)^\top g(t,x) \le x^\top G^\top G x\,, \tag{2}$$

where the constant matrix $G \in \mathbb{R}^{s\times n}$ characterizes the nonlinearity. Since the right-hand side is a quadratic function, (2) is called a quadratic constraint (Siljak and Stipanovic, 2000).

Definition 1. (*Petersen, 1987; Khargonekar* et al.*, 1990*) The system (1) with $w(t) \equiv 0$ is said to be *quadratically stable* (QS) if there exist $P \succ 0$ and $\epsilon > 0$ such that the quadratic function $V(x) = x^\top P x$ satisfies

$$\frac{\mathrm{d}V(x)}{\mathrm{d}t} \le -\epsilon V(x) \tag{3}$$

for all solutions $x(t)$ of (1), any $\Delta(t)$ meeting $\|\Delta(t)\| \le 1$ and any $g(t,x)$ satisfying (2).

Lemma 1. (Siljak and Stipanovic, 2000) *When $\Delta(t) \equiv 0$, the system (1) is QS if and only if there exist $P \succ 0$ and $\alpha > 0$ satisfying the LMI (Boyd* et al. *1994)*

$$\begin{bmatrix} He\{PA\} + \alpha G^\top G & P \\ P & -\alpha I_n \end{bmatrix} \prec 0 . \tag{4}$$

When $\Delta(t) \not\equiv 0$, replacing A with $A + H\Delta(t)F$ in the above matrix inequality and then using the Schur complement lemma leads to the following result.

Lemma 2. (Siljak and Stipanovic, 2000; Stankovic *et al.*, 2007; Chang *et al.*, 2023) *The system (1) is QS if and only if there exist $P \succ 0$ and $\alpha > 0$ satisfying the LMI (Boyd* et al. *1994)*

$$\begin{bmatrix} He\{PA\} + \alpha G^\top G + F^\top F & P & PH \\ P & -\alpha I_n & 0 \\ H^\top P & 0 & -I_q \end{bmatrix} \prec 0 . \tag{5}$$

Definition 2. (*Khargonekar* et al.*, 1990; Chang* et al.*, 2022*) Consider the system (1) where $w(t) \in \mathcal{L}_2[0,\infty)$, i.e., $\int_0^\infty w^\top(s)w(s)\,\mathrm{d}s < \infty$. It is said to *have (achieve) quadratic $\mathcal{L}_2$ gain γ* if it is QS and when $x(0) = 0$,

$$\int_0^t z^\top(s)z(s)\,\mathrm{d}s \le \gamma^2 \int_0^t w^\top(s)w(s)\,\mathrm{d}s \tag{6}$$

holds for any $t \ge 0$ and arbitrary $w(t) \in \mathcal{L}_2[0,\infty)$.

Applying the LMI with Kalman–Yakubovich–Popov (KYP) lemma (Rantzer, 1996) to the system (1) with disturbance input, we obtain the following corollary.

Corollary 1. *The system (1) has quadratic $\mathcal{L}_2$ gain γ if and only if there exist $P \succ 0$ and $\alpha > 0$ satisfying the LMI (Boyd* et al. *1994)*

$$\left[\begin{array}{c|ccc} He\{PA\} + \alpha G^\top G + F^\top F + C^\top C & P & PH & PB_1 \\ \hline P & -\alpha I_n & 0 & 0 \\ H^\top P & 0 & -I_q & 0 \\ B_1^\top P & 0 & 0 & -\gamma^2 I_r \end{array}\right] \prec 0 , \tag{7}$$

or equivalently,

$$\left[\begin{array}{c|c} He\{PA\} + \alpha G^\top G + F^\top F + C^\top C + P\left(HH^\top + \gamma^{-2} B_1 B_1^\top\right)P & P \\ \hline P & -\alpha I_n \end{array}\right] \prec 0 . \tag{8}$$

With the above preparation, we now proceed to the problem formulation. In this paper, we deal with the switched uncertain nonlinear system (SUNLS)

$$\begin{cases} \dot{x} = (A_\sigma + H_\sigma \Delta(t) F_\sigma)\, x + B_{1\sigma} w + g_\sigma(t,x), \\ z = C_\sigma x . \end{cases} \tag{9}$$

Here the index function $\sigma(t)$ taking value in the discrete set $S_\mathcal{N} = \{1,\dots,\mathcal{N}\}$ is called the switching law (signal), which will be designed later. In correspondence with the switching law, the matrix A_σ takes values in the set $\{A_1,\dots,A_\mathcal{N}\}$, and $B_{1\sigma}$, C_σ, H_σ, F_σ change with the same rule. In the above, A_i, B_{1i}, C_i, H_i, F_i are constant matrices with compatible dimensions. Similarly to (2), $g_i(t,x)$ in the system (9) is a known or unknown nonlinear vector, which is supposed to satisfy the condition

$$g_i(t,x)^\top g_i(t,x) \le x^\top G_i^\top G_i x . \tag{10}$$

Here, the constant matrix $G_i \in \mathbb{R}^{s\times n}$ characterizes the nonlinearity in the i-th subsystem, and (10) is called a quadratic constraint for the nonlinear part in each subsystem.

In the remaining sections, the following two control problems will be discussed in detail.

Problem 1. *(Quadratic stabilization)* When no subsystem in (9) is QS, design a switching law $\sigma(t)$ under which the resultant SUNLS (9) is QS.

Problem 2. *(Quadratic $\mathcal{L}_2$ gain)* When no subsystem in (9) has quadratic $\mathcal{L}_2$ gain γ, design a switching law $\sigma(t)$ under which the resultant SUNLS (9) has quadratic $\mathcal{L}_2$ gain γ.

It is noted that in Problem 2, some subsystems may not be asymptotically stable, and thus certainly does not achieve the desired $\mathcal{L}_2$ gain. Alternatively, some subsystems may be asymptotically stable but its achievable $\mathcal{L}_2$ gain is larger than the specified scalar γ. In this sense, it covers a large variety of cases. For both Problem 1 and 2, we will try to revise the CCS approach (Wicks *et al.*, 1998; Feron, 1996; Chang *et al.*, 2019; 2022a) so that the switched system has the desired QS or quadratic $\mathcal{L}_2$ gain, respectively.

3. Quadratic stabilization

We first deal with Problem 1 when no single subsystem in (9) is QS. Since the concern here is in stability, we assume $w \equiv 0$ without losing generality. To extend the existing CCS approach of Wicks *et al.* (1998), Feron (1996) and Chang *et al.* (2019; 2022a), we use the scalars $\lambda_i \ge 0$ $(i = 1,\dots,\mathcal{N})$, $\sum_{i=1}^{\mathcal{N}} \lambda_i = 1$ to define the CCS as

$$\dot{x} = (A_\lambda + H_\lambda \Delta(t) F_\lambda)\, x + g_\lambda(t,x) \tag{11}$$

where

$$A_\lambda = \mathrm{Conv}_\lambda\{A_1,\dots,A_\mathcal{N}\} , \tag{12}$$

$H_\lambda \in \mathbb{R}^{n\times q}$, $F_\lambda \in \mathbb{R}^{k\times n}$ are constant matrices satisfying

$$\begin{aligned} H_\lambda H_\lambda^\top &= \mathrm{Conv}_\lambda\{H_1 H_1^\top,\dots,H_\mathcal{N} H_\mathcal{N}^\top\} \\ F_\lambda^\top F_\lambda &= \mathrm{Conv}_\lambda\{F_1^\top F_1,\dots,F_\mathcal{N}^\top F_\mathcal{N}\} \end{aligned} \tag{13}$$

and $g_\lambda(t,x)$ is a nonlinear vector meeting

$$g_\lambda(t,x)^\top g_\lambda(t,x) \le x^\top G_\lambda^\top G_\lambda x\,, \tag{14}$$

where $G_\lambda \in \mathbb{R}^{n\times s}$ is a constant matrix satisfying

$$G_\lambda^\top G_\lambda = \mathrm{Conv}_\lambda\{G_1^\top G_1, \dots, G_\mathcal{N}^\top G_\mathcal{N}\}\,. \tag{15}$$

The key analysis method for QS is described as follows.

Analysis Approach 1. Seek a set of λ_i's ($i=1,\dots,\mathcal{N}$) such that the CCS (11) is QS.

It is emphasized that the specification in Analysis Approach 1 can be called a "convex Hurwitz combination incorporating norm-bounded uncertainty and nonlinear quadratic constraint" since it is an extension of QS for a single uncertain system which includes unit norm-bounded uncertainty, and covers the nonlinearities with quadratic constraints. Note that the convex Hurwitz combination A_λ in (12) has appeared in the literature (Feron, 1996; Wicks *et al*., 1998; Ji *et al*., 2004), while the CCS in (11) with (12), (13) and (15) presents a new concept. Moreover, as in the literature (Wicks *et al*., 1998; Feron, 1996; Chang *et al*., 2019; 2022a), the CCS (11) is not a real system but a differential equation describing an uncertain system with the same norm-bounded uncertainty $\Delta(t)$. The origin is an equilibrium point of (11) due to the constraint (14), and thus we can discuss its quadratic stability.

Lemma 2 indicates that Analysis Approach 1 is to seek $P \succ 0$, $\alpha > 0$ and nonnegative scalars λ_i satisfying

$$\begin{bmatrix} \mathrm{He}\{PA_\lambda\} + \alpha G_\lambda^\top G_\lambda + F_\lambda^\top F_\lambda & P & PH_\lambda \\ P & -\alpha I_n & 0 \\ H_\lambda^\top P & 0 & -I_q \end{bmatrix} \prec 0 \tag{16}$$

or, by the Schur complement,

$$\left[\begin{array}{c|c} \mathrm{He}\{PA_\lambda\} + \alpha G_\lambda^\top G_\lambda + F_\lambda^\top F_\lambda + PH_\lambda H_\lambda^\top P & P \\ \hline P & -\alpha I_n \end{array}\right] \prec 0\,. \tag{17}$$

With the definitions of A_λ, H_λ, F_λ and G_λ in (12), (13), (15), the above inequality is rewritten as

$$\sum_{i=1}^{\mathcal{N}} \lambda_i \left[\begin{array}{c|c} \mathrm{He}\{PA_i\} + \alpha G_i^\top G_i + F_i^\top F_i + PH_i H_i^\top P & P \\ \hline P & -\alpha I_n \end{array}\right] \prec 0\,, \tag{18}$$

which is a convex combination of matrix inequalities for each subsystem to be QS.

Remark 1. For given matrices H_i's, F_i's, G_i's and the scalars λ_i's, the matrices H_λ, F_λ and G_λ in (13), (15) can be computed with the Cholesky decomposition method, and thus numerically tractable by using MATLAB software. Although the decomposition of these matrices (the choice of H_λ, F_λ and G_λ) is not unique, it can be seen from (17) that G_λ, F_λ, H_λ take the product form, and thus the feasibility of the matrix inequality and the switching law will not depend on the choice of the decomposition.

Since we have assumed that no single subsystem is QS, using the necessity part of Lemma 2,

$$\left[\begin{array}{c|c} \mathrm{He}\{PA_i\} + \alpha G_i^\top G_i + F_i^\top F_i + PH_i H_i^\top P & P \\ \hline P & -\alpha I_n \end{array}\right] \prec 0 \tag{19}$$

is NOT satisfied for any $i = 1,\dots,\mathcal{N}$. Therefore, Analysis Approach 1 designs a CCS that is QS when each subsystem is NOT QS. In this sense, this is a significant extension compared with the existing Hurwitz stable combination by Wicks *et al*. (1998) and Zhai *et al*. (2003).

Remark 2. Using the Schur complement for (16), we obtain another equivalent matrix inequality

$$\begin{bmatrix} \mathrm{He}\{PA_\lambda\} & \alpha G_\lambda^\top & P & PH_\lambda & F_\lambda^\top \\ \alpha G_\lambda & -\alpha I_s & 0 & 0 & 0 \\ P & 0 & -\alpha I_n & 0 & 0 \\ H_\lambda^\top P & 0 & 0 & -I_q & 0 \\ F_\lambda & 0 & 0 & 0 & -I_k \end{bmatrix} \prec 0\,. \tag{20}$$

Due to the product of P and λ_i's, the above is not an LMI, and thus it is generally known to be difficult to solve (20). However, since λ_i's are nonnegative scalars satisfying $\sum_{i=1}^n \lambda_i = 1$, it is practical to try a kind of griding method (or traversal search) with respect to λ_i's when the number $\mathcal{N}$ is not very large.

In the following algorithm, we revise the gridding method based algorithm proposed by Chang *et al*. (2022b) so as to solve (20) with respect to λ_i's and $\alpha > 0$, $P \succ 0$. As mentioned by Chang *et al*. (2022b), due to continuity with respect to the scalars λ_i, if the matrix inequality (20) is feasible (there exist solutions to it), the algorithm will succeed when the division integer m is large enough.

Algorithm 1. Solving (20).

Step 1. Set the division number m of the interval $[0,1]$ as a moderate integer—for example, $m = 10$—and define $\mathcal{M} = \{0, \frac{1}{m}, \dots, \frac{m-1}{m}\}$.

Step 2. (i) Choose λ_1 from $\mathcal{M}$ in ascending order; (ii) fix λ_1 and choose λ_2 from $\mathcal{M}$ in ascending order under the constraint $\lambda_1+\lambda_2 \le 1$; (iii) fix λ_1, λ_2 and choose λ_3 from $\mathcal{M}$ in ascending order under the constraint $\lambda_1+\lambda_2+\lambda_3 \le 1$; ... (i) fix $\lambda_1,\dots,\lambda_{i-1}$ and choose

λ_i from $\mathcal{M}$ in ascending order under the constraint $\sum_{j=1}^{i} \lambda_j \leq 1$, and so on, until $\lambda_{\mathcal{N}}$ is chosen.

Step 3. With the λ_i's chosen in Step 2, compute F_λ and H_λ satisfying (13), compute G_λ satisfying (15) and then solve (20). If (20) is feasible, record the solution and end the algorithm. If (20) is not feasible, go back to Step 2 for another set of λ_i's. Or, go back to Step 1 to increase the division integer m.

Here is an example of searching parameters in Step 2. When $\mathcal{N} = 3, m = 3$, we are actually checking the linear matrix inequality (20) in P by fixing the parameters λ_i in sequence as

$$(\lambda_1, \lambda_2, \lambda_3) = \left(0, \frac{1}{3}, \frac{2}{3}\right), \quad \left(0, \frac{2}{3}, \frac{1}{3}\right), \quad \left(\frac{1}{3}, 0, \frac{2}{3}\right), \quad \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \quad \left(\frac{1}{3}, \frac{2}{3}, 0\right), \quad \left(\frac{2}{3}, 0, \frac{1}{3}\right), \quad \left(\frac{2}{3}, \frac{1}{3}, 0\right).$$

The larger m, the more searching parameters in Step 2. Again, since the left-hand side of the matrix inequality (20) is continuous in the parameters λ_i, we can expect to find a feasible solution for (20) when the division integer m is large enough.

With the matrix P obtained above, we now propose the switching law for QS as

$$\mathcal{SW}\mathbf{1}: \quad \sigma(x) = \arg\min_{i \in S_{\mathcal{N}}} f_i(x), \tag{21}$$

$$f_i(x) = x^\top \left(\mathrm{He}\{PA_i\} + \alpha G_i^\top G_i + F_i^\top F_i + PH_iH_i^\top P\right) x\,. \tag{22}$$

Theorem 1. *The SUNLS (9) is QS under the switching law* $\mathcal{SW}1$.

Proof. Since $P \succ 0$ in $\mathcal{SW}1$ satisfies the strict matrix inequality (17), there exists a positive number ϵ such that

$$\left[\begin{array}{c|c} \mathrm{He}\{PA_\lambda\} + \alpha G_\lambda^\top G_\lambda + F_\lambda^\top F_\lambda + PH_\lambda H_\lambda^\top P + \epsilon P & P \\ \hline P & -\alpha I_n \end{array}\right] \prec 0\,. \tag{23}$$

Then, for any nonzero x, $v^\top = \begin{bmatrix} x^\top & g_\sigma^\top \end{bmatrix}$ is nonzero, either, and thus

$$v^\top \left[\begin{array}{c|c} \mathrm{He}\{PA_\lambda\} + \alpha G_\lambda^\top G_\lambda + F_\lambda^\top F_\lambda + PH_\lambda H_\lambda^\top P + \epsilon P & P \\ \hline P & -\alpha I_n \end{array}\right] v < 0\,. \tag{24}$$

On the other hand, the switching law (21) indicates that

$$\begin{aligned} &x^\top \left(\mathrm{He}\{PA_\sigma\} + \alpha G_\sigma^\top G_\sigma + F_\sigma^\top F_\sigma + PH_\sigma H_\sigma^\top P\right) x \\ &\leq x^\top \left(\mathrm{He}\{PA_i\} + \alpha G_i^\top G_i + F_i^\top F_i + PH_iH_i^\top P\right) x \end{aligned} \tag{25}$$

is true for any x and any $i \in S_{\mathcal{N}}$. We multiply the above inequality by $\lambda_i \geq 0$, and then add all the inequalities to obtain

$$\begin{aligned} &x^\top \left(\mathrm{He}\{PA_\sigma\} + \alpha G_\sigma^\top G_\sigma + F_\sigma^\top F_\sigma + PH_\sigma H_\sigma^\top P\right) x \\ &\leq x^\top \left(\mathrm{He}\{PA_\lambda\} + \alpha G_\lambda^\top G_\lambda + F_\lambda^\top F_\lambda + PH_\lambda H_\lambda^\top P\right) x\,. \end{aligned} \tag{26}$$

Using the above inequality in (24), we get

$$v^\top \left[\begin{array}{c|c} \mathrm{He}\{PA_\sigma\} + \alpha G_\sigma^\top G_\sigma + F_\sigma^\top F_\sigma + PH_\sigma H_\sigma^\top P + \epsilon P & P \\ \hline P & -\alpha I_n \end{array}\right] v < 0\,, \tag{27}$$

and thus

$$v^\top \left[\begin{array}{c|c} \mathrm{He}\{PA_\sigma\} + F_\sigma^\top F_\sigma \; PH_\sigma H_\sigma^\top P + \epsilon P & P \\ \hline P & 0 \end{array}\right] v \leq v^\top \left[\begin{array}{c|c} -\alpha G_\sigma^\top G_\sigma & 0 \\ \hline 0 & \alpha I_n \end{array}\right] v\,. \tag{28}$$

Observe that the right-hand side of (28) is $\alpha\left(g_\sigma(t,x)^\top g_\sigma(t,x) - x^\top G_\sigma^\top G_\sigma x\right)$, which is not positive according to the quadratic constraint (10). Thus,

$$x^\top \left(\mathrm{He}\{PA_\sigma\} + PH_\sigma H_\sigma^\top P + F_\sigma^\top F_\sigma + \epsilon P\right) x + x^\top P g_\sigma + g_\sigma^\top P x \leq 0\,, \tag{29}$$

from which the following is obtained:

$$x^\top \left(\mathrm{He}\{PA_\sigma\} + PH_\sigma H_\sigma^\top P + F_\sigma^\top F_\sigma\right) x + x^\top P g_\sigma + g_\sigma^\top P x \leq -\epsilon x^\top P x\,. \tag{30}$$

To check the stability of the switched system, we compute the derivative of $V(x) = x^\top P x$ along the solutions of (9) as

$$\begin{aligned} \dot{V}(x) &= \frac{\mathrm{d}}{\mathrm{d}t} x^\top P x = \mathrm{He}\{x^\top P \dot{x}\} \\ &= x^\top \left(\mathrm{He}\{PA_\sigma\} + \mathrm{He}\{PH_\sigma \Delta F_\sigma\}\right) x \\ &\quad + x^\top P g_\sigma + g_\sigma^\top P x \\ &\leq x^\top \left(\mathrm{He}\{PA_\sigma\} + PH_\sigma H_\sigma^\top P + F_\sigma^\top F_\sigma\right) x \\ &\quad + x^\top P g_\sigma + g_\sigma^\top P x\,. \end{aligned} \tag{31}$$

Combining the above inequality with (30), we have $\dot{V}(x) \leq -\epsilon V(x)$. Since this evaluation is valid for any activated subsystem, and the Lyapunov function $V(x) = x^\top Px$ does not jump at the switching instants, we conclude that the inequality (3) is satisfied for any time t. Therefore, the SUNLS (9) is QS. ■

Remark 3. It is to be noted that the switching law (21) is similar to the minimum (energy) rule (van der Schaft and Schumacher, 2000), and theoretically they may result in the so-called "chattering" or "Zeno" phenomena (switchings occur an infinite number of times on a finite time interval), which are not desired in any real application. The same is true with the switching law (38) in the next section. To avoid the possibility of this trouble, we propose to adopt a kind of hybrid switching rule for our switched system. More precisely, assuming the present activated subsystem index is i_0 ($\sigma(t) = i_0$), for a specified small positive scalar μ, we do not switch to other subsystems until the tolerance bound $f_{i0}(x) < -\mu x^\top Px$ is violated. This is based on the observation that the above inequality holds on a nonzero time interval, since the function $f_i(x)$ is continuous with respect to the state x (and thus time t).

4. Quadratic $\mathcal{L}_2$ gain

Now, we proceed to deal with Problem 2. According to Definition 2, the performance of the quadratic $\mathcal{L}_2$ gain includes QS, and thus we update the CCS (11) as

$$\begin{cases} \dot{x} = (A_\lambda + H_\lambda \Delta(t) F_\lambda)\, x + B_{1\lambda} w + g_\lambda(t,x), \\ z = C_\lambda x\,. \end{cases} \tag{32}$$

In the above, the disturbance input term $B_{1\lambda}w$ and the controlled output term $z = C_\lambda x$ are added, and $B_{1\lambda} \in \mathbb{R}_{n\times r}$, $C_\lambda \in \mathbb{R}_{p\times n}$ are constant matrices satisfying

$$\begin{aligned} B_{1\lambda}B_{1\lambda}^\top &= \mathrm{Conv}_\lambda\{B_{11}B_{11}^\top, \dots, B_{1\mathcal{N}}B_{1\mathcal{N}}^\top\}, \\ C_\lambda^\top C_\lambda &= \mathrm{Conv}_\lambda\{C_1^\top C_1, \dots, C_\mathcal{N}^\top C_\mathcal{N}\}\,. \end{aligned} \tag{33}$$

Obviously, the above convex combinations take the same form as (13) and (15). In this sense, the CCS (32) presents a unified form for convex combinations involving uncertainties, disturbance inputs, controlled outputs and quadratic constraints, and thus is a major extension to the one in the existing literature.

Continuing from Analysis Approach 1 for QS, we now propose the following approach for quadratic $\mathcal{L}_2$ gain:

Analysis Approach 2. Seek a set of λ_i's ($i = 1, \dots, \mathcal{N}$) such that the CCS (32) achieves a quadratic $\mathcal{L}_2$ gain γ.

Corollary 1 indicates that Analysis Approach 2 is to seek $P \succ 0$, $\alpha > 0$ and nonnegative scalars λ_i satisfying

$$\begin{bmatrix} S(\lambda) & P \\ P & -\alpha I_n \end{bmatrix} \prec 0\,, \tag{34}$$

where

$$\begin{aligned} S(\lambda) = {}& \mathrm{He}\{PA_\lambda\} + \alpha G_\lambda^\top G_\lambda + F_\lambda^\top F_\lambda + C_\lambda^\top C_\lambda \\ & + P\left(H_\lambda H_\lambda^\top + \gamma^{-2} B_{1\lambda}B_{1\lambda}^\top\right) P\,. \end{aligned} \tag{35}$$

If (34) is feasible, there exists a positive number ϵ satisfying

$$\begin{bmatrix} S(\lambda) + \epsilon P & P \\ P & -\alpha I_n \end{bmatrix} \prec 0\,. \tag{36}$$

Similarly to the discussion in the previous section, for any nonzero x, $v^\top = \begin{bmatrix} x^\top & g_\sigma^\top \end{bmatrix}$ is nonzero, either. We pre- and postmultiply the matrix inequality (36) by $v^\top$ and v, respectively, to obtain

$$x^\top S(\lambda) x \leq -\epsilon x^\top P x + \alpha g_\sigma^\top g_\sigma - \mathrm{He}\{x^\top P g_\sigma\}\,. \tag{37}$$

The switching law for the quadratic $\mathcal{L}_2$ gain is proposed as follows:

$$\mathcal{SW}\mathbf{2}: \quad \sigma(x) = \arg\min_{i \in S_\mathcal{N}} \hat{f}_i(x), \tag{38}$$

$$\begin{aligned} \hat{f}_i(x) = x^\top \big[& \mathrm{He}\{PA_i\} + \alpha G_i^\top G_i + F_i^\top F_i + C_i^\top C_i \\ & + P\left(H_i H_i^\top + \gamma^{-2} B_{1i}B_{1i}^\top\right) P \big]\, x\,. \end{aligned} \tag{39}$$

Theorem 2. *The SUNLS (9) achieves quadratic $\mathcal{L}_2$ gain γ under the switching law $\mathcal{SW}2$.*

Proof. According to $\mathcal{SW}2$,

$$\begin{aligned} & x^\top \big[\mathrm{He}\{PA_\sigma\} + \alpha G_\sigma^\top G_\sigma + F_\sigma^\top F_\sigma + C_\sigma^\top C_\sigma \\ & \qquad + P\left(H_\sigma H_\sigma^\top + \gamma^{-2} B_{1\sigma}B_{1\sigma}^\top\right) P\big]\, x \\ & \leq x^\top \big[\mathrm{He}\{PA_i\} + \alpha G_i^\top G_i + F_i^\top F_i + C_i^\top C_i \\ & \qquad + P\left(H_i H_i^\top + \gamma^{-2} B_{1i}B_{1i}^\top\right) P\big]\, x \end{aligned} \tag{40}$$

for any x and all $i \in S_\mathcal{N}$. As before, we multiply both the sides of the above inequality by $\lambda_i \geq 0$ and then add all the inequalities to obtain

$$\begin{aligned} & x^\top \big[\mathrm{He}\{PA_\sigma\} + \alpha G_\sigma^\top G_\sigma + F_\sigma^\top F_\sigma + C_\sigma^\top C_\sigma \\ & \qquad + P\left(H_\sigma H_\sigma^\top + \gamma^{-2} B_{1\sigma}B_{1\sigma}^\top\right) P\big]\, x \\ & \leq x^\top \big[\mathrm{He}\{PA_\lambda\} + \alpha G_\lambda^\top G_\lambda + F_\lambda^\top F_\lambda + C_\lambda^\top C_\lambda \\ & \qquad + \left(H_\lambda H_\lambda^\top + \gamma^{-2} B_{1\lambda}B_{1\lambda}^\top\right) P\big]\, x \\ & = x^\top S(\lambda) x\,. \end{aligned} \tag{41}$$

Combining the above inequality with (37) leads to

$$\begin{aligned}&x^\top \left[\mathrm{He}\{PA_\sigma\} + \alpha G_\sigma^\top G_\sigma + F_\sigma^\top F_\sigma + C_\sigma^\top C_\sigma \right.\\&\quad \left. + P\left(H_\sigma H_\sigma^\top + \gamma^{-2} B_{1\sigma} B_{1\sigma}^\top\right) P\right] x + \mathrm{He}\{x^\top P g_\sigma\}\\&= x^\top S(\lambda) x + \mathrm{He}\{x^\top P g_\sigma\}\\&\le -\epsilon x^\top P x + \alpha g_\sigma^\top g_\sigma\,. \end{aligned} \tag{42}$$

Now, we use $V(x) = x^\top Px$ to show QS and estimate the $\mathcal{L}_2$ gain of the SUNLS (9) by computing the derivative of $V(x)$ along the solutions of (9) as

$$\begin{aligned}&\dot V(x)\\&= \frac{\mathrm{d}}{\mathrm{d}t} x^\top P x = \mathrm{He}\{x^\top P \dot x\}\\&= \mathrm{He}\{x^\top P(A_\sigma x + H_\sigma \Delta F_\sigma x + B_{1\sigma} w + g_\sigma)\}\\&= x^\top \left(\mathrm{He}\{PA_\sigma\} + \mathrm{He}\{PH_\sigma \Delta F_\sigma\}\right) x\\&\quad + \mathrm{He}\{x^\top P g_\sigma\} + x^\top P B_{1\sigma} w + w^\top B_{1\sigma}^\top P x\\&\le x^\top \left(\mathrm{He}\{PA_\sigma\} + PH_\sigma H_\sigma^\top P + F_\sigma^\top F_\sigma\right) x\\&\quad - \left(\gamma^{-1} x^\top P B_{1\sigma} - \gamma w^\top\right)\left(\gamma^{-1} x^\top P B_{1\sigma} - \gamma w^\top\right)^\top\\&\quad + \gamma^{-2} x^\top P B_{1\sigma} B_{1\sigma}^\top P x + x^\top C_\sigma^\top C_\sigma x\\&\quad + \mathrm{He}\{x^\top P g_\sigma\} - z^\top z + \gamma^2 w^\top w\\&\le x^\top \left[\mathrm{He}\{PA_\sigma\} + F_\sigma^\top F_\sigma + C_\sigma^\top C_\sigma \right.\\&\quad \left. + P(H_\sigma H_\sigma^\top + \gamma^{-2} B_{1\sigma} B_{1\sigma}^\top) P\right] x\\&\quad + \mathrm{He}\{x^\top P g_\sigma\} - z^\top z + \gamma^2 w^\top w\\&\le -\epsilon V(x) + \alpha(g_\sigma^\top g_\sigma - x^\top G_\sigma^\top G_\sigma x) - z^\top z + \gamma^2 w^\top w\\&\le -\epsilon V(x) - z^\top z + \gamma^2 w^\top w\,. \end{aligned} \tag{43}$$

Firstly, when there is no disturbance ($w \equiv 0$), (3) is obtained immediately from (43), which implies the system (9) is QS.

Secondly, since $V(x) \ge 0$, the inequality (43) implies $\dot V(x(s)) \le -z^\top(s) z(s) + \gamma^2 w^\top(s) w(s)$. Thus, integrating both sides of the inequality from $s = 0$ to $s = t$, we obtain immediately the inequality (6). Therefore, the SUNLS (9) achieves the desired quadratic $\mathcal{L}_2$ gain γ . This completes the proof. ■

Remark 4. With the definitions in (12), (13), (15) and (33), the matrix inequality (34) turns out to be

$$\sum_{i=1}^{\mathcal{N}} \lambda_i \begin{bmatrix} S(i) & P \\ P & -\alpha I_n \end{bmatrix} \prec 0\,, \tag{44}$$

where

$$\begin{aligned} S(i) = {}& \mathrm{He}\{PA_i\} + \alpha G_i^\top G_i + F_i^\top F_i + C_i^\top C_i\\ & + P\left(H_i H_i^\top + \gamma^{-2} B_{1i} B_{1i}^\top\right) P\,. \end{aligned} \tag{45}$$

Therefore, even if each single subsystem does not achieve quadratic $\mathcal{L}_2$ gain γ, Analysis Approach 2 provides us a design tool, by using the convex combination of subsystems, for the switched system to achieve quadratic $\mathcal{L}_2$ gain γ. As mentioned before, this idea is a significant and original extension compared with the stable combination by Wicks *et al*. (1998), Zhai *et al*. (2003) and Chang *et al*. (2019).

Remark 5. Using the Schur complement lemma to the inequality (34) results in

$$\begin{bmatrix} \mathrm{He}\{PA_\lambda\} & \alpha G_\lambda^\top & P & PH_\lambda & F_\lambda^\top & PB_{1\lambda} & C_\lambda^\top \\ \alpha G_\lambda & -\alpha I_s & 0 & 0 & 0 & 0 & 0 \\ P & 0 & -\alpha I_n & 0 & 0 & 0 & 0 \\ H_\lambda^\top P & 0 & 0 & -I_q & 0 & 0 & 0 \\ F_\lambda & 0 & 0 & 0 & -I_k & 0 & 0 \\ B_{1\lambda}^\top P & 0 & 0 & 0 & 0 & -\gamma^2 I_r & 0 \\ C_\lambda & 0 & 0 & 0 & 0 & 0 & -I_p \end{bmatrix} \prec 0\,. \tag{46}$$

Letting $Q = P^{-1}$, $\beta = \alpha^{-1}$ and multipling the above inequality by $\mathrm{diag}\{Q, I, I, I, I, I, I\}$ from both left and right, we obtain

$$\begin{bmatrix} \mathrm{He}\{A_\lambda Q\} & QG_\lambda^\top & \beta I_n & H_\lambda & QF_\lambda^\top & B_{1\lambda} & QC_\lambda^\top \\ G_\lambda Q & -\beta I_s & 0 & 0 & 0 & 0 & 0 \\ \beta I_n & 0 & -\beta I_n & 0 & 0 & 0 & 0 \\ H_\lambda^\top & 0 & 0 & -I_q & 0 & 0 & 0 \\ F_\lambda Q & 0 & 0 & 0 & -I_k & 0 & 0 \\ B_{1\lambda}^\top & 0 & 0 & 0 & 0 & -\gamma^2 I_r & 0 \\ C_\lambda Q & 0 & 0 & 0 & 0 & 0 & -I_p \end{bmatrix} \prec 0\,. \tag{47}$$

This is equivalent to (34), and thus also serves as a design condition for Analysis Approach 2. Furthermore, it will be used later for switching state feedback controller design.

Remark 6. Although the $\mathcal{L}_2$ gain γ has been fixed in the discussion by now, it is easy to extend to minimizing it, which leads to optimal disturbance attenuation in real systems. As described in the previous remark, the matrix inequalities (46) and (47) are linear with respect to γ^2. Thus, the extended specification is reduced to the optimization problem of $\min \gamma^2$, subject to (46) or (47) together with $P \succ 0$ or $Q \succ 0$, which can be dealt with by using the same algorithm combined with the generalized eigenvalue minimization command GEVP in MATLAB Robust Toolbox.

Example 1. Consider the SUNLS (9) with the following coefficient matrices:

$$A_1 = \begin{bmatrix} -9.96 & 17.28 \\ 17.28 & -20.04 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -21.60 & -19.20 \\ -19.20 & -10.40 \end{bmatrix},$$

$$B_{11} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} -1 \\ 0.2 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.5 & 1.5 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -1.0 & 0.5 \end{bmatrix},$$

$$H_1 = \begin{bmatrix} 0.5 & 0.5 \\ 1.5 & 0.5 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0.5 & 2.0 \\ 0.5 & 1.0 \end{bmatrix},$$

$$F_1 = \begin{bmatrix} 0.2 & 0 \\ 0.2 & 0.2 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0.2 & -0.2 \\ 0.2 & 0.4 \end{bmatrix}.$$

The uncertainty term is

$$\Delta(t) = \begin{bmatrix} 0.5\sin t - 0.1\cos t & -0.2\sin t - 0.3\cos t \\ -0.3\sin t - 0.2\cos t & 0.1\sin t - 0.4\cos t \end{bmatrix},$$

satisfying $\|\Delta(t)\| \leq 1$, and the nonlinear terms are

$$g_1(t,x) = \begin{bmatrix} 0.6x_1\sin(x_1^2 - x_2) \\ 0 \end{bmatrix},$$

$$g_2(t,x) = \begin{bmatrix} 0 \\ 0.5x_2\cos(0.5x_1 + x_2^2) \end{bmatrix},$$

satisfying the quadratic constraint with

$$G_1 = \begin{bmatrix} -0.6 & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0 & 0.5 \end{bmatrix}.$$

By computing the eigenvalues of A_1 and A_2, we easily check that both the subsystems are not QS.

When setting $\lambda_1 = \frac{2}{3}, \lambda_2 = \frac{1}{3}$, we find that

$$A_\lambda = \frac{2}{3}A_1 + \frac{1}{3}A_2 = \begin{bmatrix} -13.84 & 5.12 \\ 5.12 & -16.8267 \end{bmatrix}$$

is Hurwitz, whose eigenvalues are $\{-20.6667, -10.0\}$. For given $\gamma = 0.1$ and the same λ_1, λ_2, we solve the design condition (46) with MATLAB Robust Control Toolbox to obtain $\alpha = 2.6635$ and

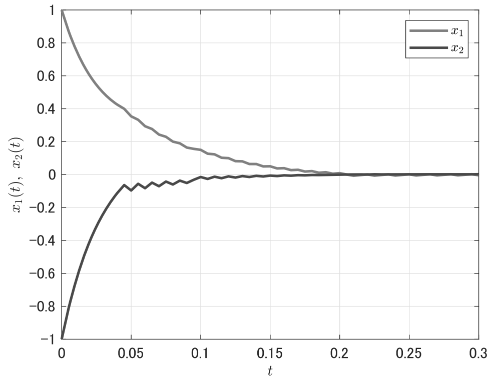

Fig. 1. State trajectories of the SUNLS in Example 1.

$$P = \begin{bmatrix} 0.2248 & 0.0684 \\ 0.0684 & 0.2043 \end{bmatrix},$$

which implies that Analysis Approach 2 is applicable. Other coefficient matrices in the CCS (32) are

$$B_{1\lambda} = \begin{bmatrix} 0.7071 & 0 \\ -0.0943 & 0.0667 \end{bmatrix},$$

$$C_\lambda = \begin{bmatrix} 0.7071 & 0 \\ 0.4714 & 1.1667 \end{bmatrix},$$

$$H_\lambda = \begin{bmatrix} 1.3229 & 0 \\ 1.0709 & 0.9677 \end{bmatrix},$$

$$F_\lambda = \begin{bmatrix} 0.2828 & 0 \\ 0.1414 & 0.2708 \end{bmatrix},$$

$$G_\lambda = \begin{bmatrix} 0.2939 & 0 \\ 0 & 0.1443 \end{bmatrix}.$$

Assume the initial state is $x(0) = \begin{bmatrix} 1.0 & -1.0 \end{bmatrix}^\top$, and the disturbance input is $w(t) = 2e^{-t}\cos 2t$. We apply the switching law $\mathcal{SW}2$ (38) to the SUNLS, and obtain the state trajectories of the SUNLS depicted in Fig. 1. The figure suggests that the states of the system converge to zero quickly. Moreover, the $\mathcal{L}_2$ gain requirement (6) holds for any $t > 0$, which guarantees the desired QS and $\mathcal{L}_2$ gain. ♦

5. Switching state feedback

We deal with the SUNLS, where the control input can be designed together with the switching law. More precisely, we consider

$$\begin{cases} \dot{x} = (A_\sigma + H_\sigma \Delta(t) F_\sigma)\, x + B_{1\sigma} w \\ \qquad +B_{2\sigma} u + g_\sigma(t,x), \\ z = C_\sigma x\,, \end{cases} \tag{48}$$

where $u \in \mathbb{R}^m$ is the control input and B_{2i} is the control input matrix with proper dimensions. All the other vectors and matrices are the same as in (9).

The control objective here is to design a state feedback controller

$$u = K_i x \tag{49}$$

for each subsystem such that the switched closed-loop system achieves the desired quadratic $\mathcal{L}_2$ gain γ. This is nontrivial when the analysis approach by now for the SUNLS (9) without control input does not work well, i.e., the switched system cannot achieve the desired quadratic $\mathcal{L}_2$ gain γ only through switching. In this case, it is practical to consider state feedback for each subsystem so that the whole switched system can have better performance.

The closed-loop system composed of (48) and (49) is

$$\begin{cases} \dot{x} = (A_\sigma + H_\sigma \Delta(t) F_\sigma + B_{2\sigma} K_\sigma)\, x \\ \qquad +B_{1\sigma} w + g_\sigma(t,x), \\ z = C_\sigma x\,. \end{cases} \tag{50}$$

Use the design condition (47) with A_λ replaced by $A_\lambda + \sum_{i=1}^{\mathcal{N}} \lambda_i B_{2i} K_i$, and then define $K_i Q = M_i$ to obtain

$$\begin{bmatrix} \Omega_1 & QG_\lambda^\top & \beta I_n & H_\lambda & QF_\lambda^\top & B_{1\lambda} & QC_\lambda^\top \\ G_\lambda Q & -\beta I_s & 0 & 0 & 0 & 0 & 0 \\ \beta I_n & 0 & -\beta I_n & 0 & 0 & 0 & 0 \\ H_\lambda^\top & 0 & 0 & -I_q & 0 & 0 & 0 \\ F_\lambda Q & 0 & 0 & 0 & -I_k & 0 & 0 \\ B_{1\lambda}^\top & 0 & 0 & 0 & 0 & -\gamma^2 I_r & 0 \\ C_\lambda Q & 0 & 0 & 0 & 0 & 0 & -I_p \end{bmatrix} \prec 0\,, \tag{51}$$

where

$$\Omega_1 = \mathrm{He}\left\{ A_\lambda Q + \sum_{i=1}^{\mathcal{N}} \lambda_i B_{2i} M_i \right\}. \tag{52}$$

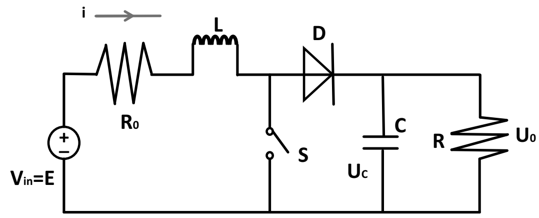

Fig. 2. DC-DC boost converter model.

Theorem 3. *A switching state feedback (49) exists such that the SUNLS (50) under $\mathcal{SW}2$ achieves quadratic $\mathcal{L}_2$ gain γ, if there are $Q \succ 0$, M_i, $\beta > 0$ and nonnegative scalars λ_i satisfying (51).*

When (51) is feasible, the feedback gains are given by $K_i = M_i Q^{-1}$, and the matrix P in the switching law $\mathcal{SW}2$ is given by $P = Q^{-1}$.

Remark 7. Similarly to Remark 2, the matrix inequality (51) is a bilinear matrix inequality in the unknown variables (Q, M_i), β and λ_i's, and thus cannot be solved directly. Observing that (51) is an LMI when λ_i's are fixed, we suggest that the griding method based algorithm proposed in the previous section should be used here to search in the parameter space spanned by $(\lambda_1, \lambda_2, \ldots, \lambda_{\mathcal{N}})$. Moreover, if we expect to obtain minimal γ for greater disturbance attenuation, we need to consider the optimization problem mentioned in Remark 6.

Remark 8. Similarly to Remark 4, the design condition (51) is equivalent to

$$\sum_{i=1}^{\mathcal{N}} \lambda_i \big(\mathrm{He}\{A_i Q + B_{2i} M_i\} + \beta I_n \\ + H_i H_i^\top + \gamma^{-2} B_{1i} B_{1i}^\top \\ + Q\left(\beta^{-1} G_i^\top G_i + F_i^\top F_i + C_i^\top C_i\right) Q \big) \prec 0\,. \tag{53}$$

This is a convex combination of matrix inequalities, where each inequality works for the corresponding subsystem to achieve quadratic $\mathcal{L}_2$ gain γ through a state feedback. In other words, when there is no single subsystem achieving quadratic $\mathcal{L}_2$ gain γ by state feedback, Theorem 3 requires (designs) a CCS that can make it. Therefore, it is natural to regard the condition of Theorem 3 as an extension to Analysis Approach 2.

Example 2. Apply the switching state feedback strategy in Theorem 3 to quadratic stabilization of a DC-DC boost converter (Fig. 2) (Yang *et al.*, 2013), which includes one transistor-diode switch S. Moreover, $i(t)$ is the inductor current, $u_c(t)$ and $u_o(t)$ are the voltage on the capacitance and the resistance.

Define the state variables as $x_1(t) = i(t)$, $x_2(t) = u_c(t)$, and suppose the control input is $u(t) = E(t)$. To consider the $\mathcal{L}_2$ gain, let $w(t)$ be the disturbance input of the input voltage, and let the controlled output $z(t)$ be the capacitance voltage $u_c(t)$. When S is closed, the dynamic equation of the whole circuit is

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -\frac{R_0+\Delta R_0}{L} & 0 \\ 0 & -\frac{1}{(R+\Delta R)C} \end{bmatrix} x(t) \\ \qquad + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} (u(t)+w(t)) - \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} g_1(x_1), \\ z(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t), \end{cases} \tag{54}$$

and when the switch S is open, it is

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -\frac{R_0+\Delta R_0}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{(R+\Delta R)C} \end{bmatrix} x(t) \\ \qquad + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} (u(t)+w(t)) - \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} g_2(x_1), \\ z(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t). \end{cases} \tag{55}$$

In the above equations, ΔR_0 and ΔR denote uncertain perturbation from the nominal resistance R_0 and R, which are bounded by $|\Delta R_0| \le \Delta R_{0\max}$ and $|\Delta R| \le \Delta R_{\max}$, respectively. Moreover, the function $g_i(x_1)$ denotes the diode voltage loss satisfying $|g_i(x_1)| \le \zeta_i |x_1|$, $i = 1, 2$.

After using the first-order Taylor expansion to separate the uncertainty from the term $\frac{1}{(R+\Delta R)C}$, the above two subsystems can be linearized, and thus the switched system turns out to take the form (48) with

$$A_1 = \begin{bmatrix} -\frac{R_0}{L} & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -\frac{R_0}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix},$$

$$B_{11} = B_{12} = B_{12} = B_{12} = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix},$$

$$C_1 = C_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad H_1 = H_2 = I_2,$$

$$F_1 = F_2 = \begin{bmatrix} \frac{\Delta R_{0\max}}{L} & 0 \\ 0 & \frac{\Delta R_{\max}}{R^2 C} \end{bmatrix},$$

and $\Delta_i(t)^\top \Delta_i(t) \preceq I_2$. The nonlinear parts in (54) and (55) are evaluated by (10) with $G_i = \mathrm{diag}\{\zeta_i^2, 0\}$.

To carry out the simulation, we set the parameters and the uncertainty in the dynamical equations as $L = 10^2$mH, $C = 10^5\mu$F, $R = 100\Omega$, $R_0 = 0.1\Omega$, $\zeta_1 = 0.1, \zeta_2 = 0.11$ and the uncertainties are $\Delta R_0 = 0.05\sin(100t)R_0$, $\Delta R = 0.1\cos(100t)R$. We obtain

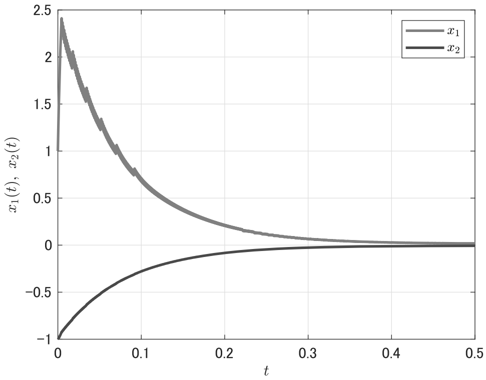

Fig. 3. State trajectories of the SUNLS in Example 2.

$$A_1 = \begin{bmatrix} -1.0 & 0 \\ 0 & -0.1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1.0 & -10.0 \\ 10.0 & -0.1 \end{bmatrix},$$

$$B_{11} = B_{12} = B_{21} = B_{22} = \begin{bmatrix} 10 \\ 0 \end{bmatrix},$$

$$C_1 = C_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad H_1 = H_2 = I_2,$$

$$F_1 = F_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.0001 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} 0.01 & 0 \\ 0 & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0.0121 & 0 \\ 0 & 0 \end{bmatrix}.$$

Here, we set $\gamma = 0.3$, and use the gridding method mentioned in Remark 5 to solve (51) with MATLAB Robust Control Toolbox. It turns out the matrix inequality is feasible when $\lambda_1 = 0.75$ (and thus $\lambda_2 = 0.25$) and thus

$$A_\lambda = \begin{bmatrix} -1.0 & -2.5 \\ 2.5 & -0.1 \end{bmatrix}.$$

The solutions of (51) are

$$Q = \begin{bmatrix} 7.1760 & -2.2942 \\ -2.2942 & 2.3351 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} -85.3135 & 32.3970 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 28.1195 & -99.6660 \end{bmatrix}.$$

Thus, the state feedback gain matrices are

$$K_1 = M_1 Q^{-1} = \begin{bmatrix} -10.8663 & 3.1981 \end{bmatrix},$$

$$K_2 = M_2 Q^{-1} = \begin{bmatrix} -14.1814 & -56.6152 \end{bmatrix}.$$

Let the initial value be $x(0) = \begin{bmatrix} 1 & -1 \end{bmatrix}^\top$ and the disturbance input be $w(t) = e^{-2t}\sin t$, which is a signal in $\mathcal{L}_2[0,\infty)$. Using the switching law $\mathcal{SW}2$ with $P = Q^{-1}$, we obtain the state trajectories of x_1, x_2 of the switched system in Fig. 3. It is observed that good convergence is obtained and the inequality (6) can be confirmed for all $t > 0$, which implies that the desired $\mathcal{L}_2$ gain has been achieved. ♦

6. Conclusion

We have addressed quadratic stabilization (QS) and $\mathcal{L}_2$ gain for a class of switched systems that are composed of nonlinear subsystems with norm-bounded uncertainties in linear parts and quadratic constraints for nonlinear parts. In the case where there is no subsystem which is QS, we have designed an analysis approach seeking a new CCS which is QS, and then have proposed a switching law under which the switched system is QS. Moreover, in the case where no subsystem has quadratic $\mathcal{L}_2$ gain γ, by seeking a new CCS that can make it, we have proposed another switching law under which the switched system achieves the same $\mathcal{L}_2$ gain γ. Both the CCSs involve a combination of linear subsystems and quadratic constraint matrices for nonlinearities, and they present a major extension to the existing approach with the convex combination of subsystems. The discussion has also been extended to the discussion of switching state feedback when the control input can be designed.

It is observed from the existing references and the present results that the convex combination approach is useful both theoretically and practically, especially in the cases where neither subsystem in the switched system has the desired performance. Our future research will extend the approach to more complicated problems such as output dependent switching (Egidio and Deaecto, 2021), robust $\mathcal{H}_\infty$ filter design, fault tolerant control (Sánchez and Bernal, 2019; Yang *et al.*, 2012) for more general or mixed type of switched nonlinear systems (Leth and Wisniewski, 2014; Xiao *et al.*, 2020).

Acknowledgment

This work has been supported by the National Natural Science Foundation of China under the grant 62473133.

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Received: 2 July 2024
Revised: 5 October 2024
Re-revised: 11 October 2024
Accepted: 11 October 2024