

## MINIMUM ENERGY CONTROL OF FRACTIONAL POSITIVE CONTINUOUS–TIME LINEAR SYSTEMS WITH BOUNDED INPUTS

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A minimum energy control problem for fractional positive continuous-time linear systems with bounded inputs is formulated and solved. Sufficient conditions for the existence of a solution to the problem are established. A procedure for solving the problem is proposed and illustrated with a numerical example.

**Keywords:** fractional systems, positive systems, minimum energy control, bounded inputs.

### 1. Introduction

Mathematical fundamentals of fractional calculus are given in the monographs of Oldham and Spanier (1974), Ostalczyk (2008) as well as Podlubny (1999). An overview of the state of the art in fractional systems theory is presented by Kaczorek (1992). Some recent interesting results in fractional systems theory and its applications can be found in the works of Dzieliński *et al.* (2009), Kaczorek (2011a; 2008c), Radwan *et al.* (2009), Solteiro Pires *et al.* (2006) or Vinagre (2002).

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs (Farina and Rinaldi, 2000; Kaczorek, 2001). A variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

Positive fractional linear systems were investigated by Kaczorek (2008a; 2011b; 2011c; 2012), while the stability of fractional linear 1D discrete-time and continuous-time systems was discussed by Busłowicz (2008), Dzieliński and Sierociuk (2008) as well as Kaczorek (2012), who also studied the stability of 2D fractional positive linear systems (Kaczorek, 2009). The notion of practical stability of positive fractional discrete-time linear systems was introduced by Kaczorek (2008b), who also addressed the positivity of descriptor linear systems by the use of the shuffle algorithm (Kaczorek, 2011d). The minimum energy control problem for standard linear systems was formulated and solved

by Klamka (1991; 1983; 1976a), and for 2D linear systems with variable coefficients by Kaczorek and Klamka (1986). The controllability and minimum energy control problem of fractional discrete-time linear systems was investigated by Klamka (2010), while minimum energy control of fractional positive continuous-time linear systems was addressed by Kaczorek (2013b; 2013c; 2014; 1992), along with its counterpart for descriptor positive discrete-time linear systems (Kaczorek, 2014a; 2013c; 2014b; 1992). Minimum energy control of positive continuous-time systems with bounded inputs was studied by the same author (Kaczorek, 2013d).

In this paper the minimum energy control problem for fractional positive continuous-time linear systems with bounded inputs will be formulated and solved. The paper is organized as follows. In Section 2, basic definitions and theorems of fractional positive continuous-time linear systems are recalled, and the necessary and sufficient conditions for the reachability of the systems are given. The main result of the paper is given in Section 3, where the minimum energy control problem is formulated and sufficient conditions for its solution are established. A procedure and an illustrating numerical example are presented in Section 4. Concluding remarks are given in Section 5.

The following notation will be used:  $\mathbb{R}$ , the set of real numbers;  $\mathbb{R}^{n \times m}$ , the set of  $n \times m$  real matrices;  $\mathbb{R}_+^{n \times m}$ , the set of  $n \times m$  matrices with nonnegative entries and  $\mathbb{R}_+^n = \mathbb{R}_+^{n \times 1}$ ;  $M_n$ , the set of  $n \times n$  Metzler matrices (real

matrices with nonnegative off-diagonal entries);  $I_n$ , the  $n \times n$  identity matrix.

## 2. Preliminaries

The following Caputo definition of the fractional derivative will be used (Kaczorek, 2012; Oldham and Spanier, 1974; Ostalczyk, 2008; Podlubny, 1999):

$$D^\alpha f(t) = \frac{d^\alpha}{dt^\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \quad n-1 < \alpha \leq n \in \mathbb{N} = \{1, 2, \dots\}, \quad (1)$$

where  $\alpha \in \mathbb{R}$  is the order of fractional derivative and

$$f^{(n)}(\tau) = \frac{d^n f(\tau)}{d\tau^n},$$

while

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$$

is the gamma function.

Consider a continuous-time fractional linear system described by the state equation

$$D^\alpha x(t) = Ax(t) + Bu(t), \quad 0 < \alpha \leq 1, \quad (2)$$

where  $x(t) \in \mathbb{R}^n$ ,  $u(t) \in \mathbb{R}^m$  are the state and input vectors and  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ .

**Theorem 1.** *The solution of Eqn. (2) is given by*

$$x(t) = \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau) d\tau, \quad x(0) = x_0, \quad (3)$$

where

$$\Phi_0(t) = E_\alpha(At^\alpha) = \sum_{k=0}^\infty \frac{A^k t^{k\alpha}}{\Gamma(k\alpha + 1)}, \quad (4)$$

$$\Phi(t) = \sum_{k=0}^\infty \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \quad (5)$$

and  $E_\alpha(At^\alpha)$  is the Mittag-Leffler matrix function (Kaczorek, 2012).

**Definition 1.** (Kaczorek, 2012) The fractional system (2) is called (internally) *positive fractional* if and only if  $x(t) \in \mathbb{R}_+^n$  for  $t \geq 0$  for any initial conditions  $x_0 \in \mathbb{R}_+^n$  and all inputs  $u(t) \in \mathbb{R}_+^m$ ,  $t \geq 0$ .

**Theorem 2.** (Kaczorek, 2012) *The continuous-time fractional system (2) is (internally) positive if and only if the matrix  $A$  is a Metzler matrix and*

$$A \in M_n, \quad B \in \mathbb{R}_+^{n \times m}. \quad (6)$$

**Definition 2.** The state  $x_f \in \mathbb{R}_+^n$  of the fractional system (2) is called *reachable in time  $t_f$*  if there exists an input  $u(t) \in \mathbb{R}_+^m$ ,  $t \in [0, t_f]$ , which steers the state of the system (2) from zero initial state  $x_0 = 0$  to the state  $x_f$ .

A real square matrix is called *monomial* if each its row and each its column contains only one positive entry and the remaining entries are zero.

**Theorem 3.** *The positive fractional system (2) is reachable in time  $t \in [0, t_f]$  if and only if the matrix  $A \in M_n$  is diagonal and the matrix  $B \in \mathbb{R}_+^{n \times m}$  is monomial.*

*Proof.* (Sufficiency) It is well known (Kaczorek, 2001; 2012) that if  $A \in M_n$  is diagonal then  $\Phi(t) \in \mathbb{R}_+^{n \times n}$  is also diagonal, and if  $B \in \mathbb{R}_+^{n \times m}$  is monomial then  $BB^T \in \mathbb{R}_+^{n \times n}$  is also monomial. In this case the matrix

$$R_f = \int_0^{t_f} \Phi(\tau)BB^T\Phi^T(\tau) d\tau \in \mathbb{R}_+^{n \times n} \quad (7)$$

is also monomial and  $R_f^{-1} \in \mathbb{R}_+^{n \times n}$ . The input

$$u(t) = B^T\Phi^T(t_f-t)R_f^{-1}x_f \quad (8)$$

steers the state of the system (2) from  $x_0 = 0$  to  $x_f$  since, using (3) for  $x_0 = 0$  and (5), we obtain

$$\begin{aligned} x(t_f) &= \int_0^{t_f} \Phi(t_f-\tau)Bu(\tau) d\tau \\ &= \int_0^{t_f} \Phi(t_f-\tau)BB^T\Phi^T(t_f-\tau) d\tau R_f^{-1}x_f \\ &= \int_0^{t_f} \Phi(\tau)BB^T\Phi^T(\tau) d\tau R_f^{-1}x_f = x_f. \end{aligned} \quad (9)$$

The proof of necessity is given by Kaczorek (2013a). ■

## 3. Problem formulation and its solution

Consider the fractional positive system (2) with diagonal  $A \in M_n$  and monomial  $B \in \mathbb{R}_+^{n \times m}$ . If the system is reachable in time  $t \in [0, t_f]$ , then usually there exist many different inputs  $u(t) \in \mathbb{R}_+^m$  that steer the state of the system from  $x_0 = 0$  to  $x_f \in \mathbb{R}_+^n$ . Among these inputs we are looking for an input  $u(t) \in \mathbb{R}_+^m$ ,  $t \in [0, t_f]$  satisfying the condition

$$u(t) \leq U \in \mathbb{R}_+^m, \quad t \in [0, t_f] \quad (10)$$

that minimizes the performance index

$$I(u) = \int_0^{t_f} u^T(\tau)Qu(\tau) d\tau, \quad (11)$$

where  $Q \in \mathbb{R}_+^{m \times m}$  is a symmetric positive-definite matrix and  $Q^{-1} \in \mathbb{R}_+^{m \times m}$ .

The minimum energy control problem for the fractional positive continuous-time linear system (2) can be stated as follows: Given matrices  $A \in M_n$ ,  $B \in \mathbb{R}_+^{n \times m}$ ,  $\alpha$ ,  $U \in \mathbb{R}_+^n$  and  $Q \in \mathbb{R}_+^{n \times n}$  of the performance index (11),  $x_f \in \mathbb{R}_+^n$  and  $t_f > 0$ , find an input  $u(t) \in \mathbb{R}_+^m$  for  $t \in [0, t_f]$  satisfying (10) that steers the state vector of the system from  $x_0 = 0$  to  $x_f \in \mathbb{R}_+^n$  and minimizes the performance index (11).

To solve the problem, we define the matrix

$$W(t_f) = \int_0^{t_f} \Phi(t_f - \tau) B Q^{-1} B^T \Phi^T(t_f - \tau) d\tau, \quad (12)$$

where  $\Phi(t)$  is given by (5). From (12) and Theorem 2 it follows that the matrix (12) is monomial if and only if the fractional positive system (2) is reachable in time  $[0, t_f]$ . In this case, we may define the input

$$\hat{u}(t) = Q^{-1} B^T \Phi^T(t_f - t) W^{-1}(t_f) x_f \quad (13)$$

for  $t \in [0, t_f]$ .

Note that the input (13) satisfies the condition  $u(t) \in \mathbb{R}_+^m$  for  $t \in [0, t_f]$  if

$$Q^{-1} \in \mathbb{R}_+^{n \times n} \quad (14a)$$

and

$$W^{-1}(t_f) \in \mathbb{R}_+^{n \times n}. \quad (14b)$$

**Theorem 4.** Let  $\bar{u}(t) \in \mathbb{R}_+^m$  for  $t \in [0, t_f]$  be an input satisfying (10) that steers the state of the fractional positive system (2) from  $x_0 = 0$  to  $x_f \in \mathbb{R}_+^n$ . Then the input (13) satisfying (10) also steers the state of the system from  $x_0 = 0$  to  $x_f \in \mathbb{R}_+^n$  and minimizes the performance index (11), i.e.,

$$I(\hat{u}) \leq I(\bar{u}).$$

The minimal value of the performance index (11) is equal to

$$I(\hat{u}) = x_f^T W^{-1}(t_f) x_f. \quad (15)$$

*Proof.* If the conditions (14a) are met, then the input (13) is well defined and  $\hat{u}(t) \in \mathbb{R}_+^m$  for  $t \in [0, t_f]$ . We shall show that the input steers the state of the system from  $x_0 = 0$  to  $x_f \in \mathbb{R}_+^n$ . Substitution of (13) into (3) for  $t = t_f$  and  $x_0 = 0$  yields

$$x(t_f) = \int_0^{t_f} \Phi(t_f - \tau) B \hat{u}(\tau) d\tau$$

$$= \int_0^{t_f} \Phi(t_f - \tau) B Q^{-1} B^T \Phi^T(t_f - \tau) d\tau W_f^{-1} x_f = x_f \quad (16a)$$

since (12) holds. By assumption, the inputs  $\bar{u}(t)$  and  $\hat{u}(t)$ ,  $t \in [0, t_f]$ , steer the state of the system from  $x_0 = 0$  to  $x_f \in \mathbb{R}_+^n$ , i.e.,

$$x_f = \int_0^{t_f} \Phi(t_f - \tau) B \bar{u}(\tau) d\tau = \int_0^{t_f} \Phi(t_f - \tau) B \hat{u}(\tau) d\tau \quad (16b)$$

and

$$\int_0^{t_f} \Phi(t_f - \tau) B [\bar{u}(\tau) - \hat{u}(\tau)] d\tau = 0. \quad (16c)$$

By transposition of (16) and postmultiplication by  $W^{-1}(t_f) x_f$ , we obtain

$$\int_0^{t_f} [\bar{u}(\tau) - \hat{u}(\tau)]^T B^T \Phi^T(t_f - \tau) d\tau W^{-1}(t_f) x_f = 0. \quad (17)$$

Substitution of (13) into (17) yields

$$\int_0^{t_f} [\bar{u}(\tau) - \hat{u}(\tau)]^T B^T \Phi^T(t_f - \tau) d\tau W^{-1} x_f = \int_0^{t_f} [\bar{u}(\tau) - \hat{u}(\tau)]^T Q \hat{u}(\tau) d\tau = 0. \quad (18)$$

Using (18), it is easy to verify that

$$\int_0^{t_f} \bar{u}(\tau)^T Q \bar{u}(\tau) d\tau = \int_0^{t_f} \hat{u}(\tau)^T Q \hat{u}(\tau) d\tau + \int_0^{t_f} [\bar{u}(\tau) - \hat{u}(\tau)]^T Q [\bar{u}(\tau) - \hat{u}(\tau)] d\tau \quad (19)$$

From (19) it follows that  $I(\hat{u}) < I(\bar{u})$  since the second term on the right-hand side of the inequality is nonnegative. To find the minimal value of the performance index (11), we substitute (13) into (11) and obtain

$$I(\hat{u}) = \int_0^{t_f} \hat{u}^T(\tau) Q u(\hat{u}) d\tau = x_f^T W^{-1} \int_0^{t_f} \Phi(t_f - \tau) B Q^{-1} B^T \Phi^T(t_f - \tau) d\tau W^{-1} x_f = x_f^T W^{-1} x_f \quad (20)$$

since (12) holds. ■

### 4. Procedure and a numerical example

To find  $t \in [0, t_f]$  for which  $\hat{u}(t) \in \mathbb{R}_+^n$  reaches its minimal value using (13), we compute the derivative

$$\frac{d\hat{u}(t)}{dt} = Q^{-1}B^T\Psi(t)W^{-1}(t_f)x_f, \quad t \in [0, t_f], \quad (21)$$

where

$$\Psi(t) = \frac{d}{dt}[\Phi^T(t_f - t)]. \quad (22)$$

Given  $\Psi(t)$  and using the equality

$$\Psi(t)W^{-1}(t_f)x_f = 0, \quad (23)$$

we can find  $t \in [0, t_f]$  for which  $\hat{u}(t)$  reaches its maximal value.

Note that if the system is asymptotically stable,  $\lim_{t \rightarrow \infty} \Phi(t) = 0$ , then  $\hat{u}(t)$  reaches its maximal value for  $t = t_f$  and if it is unstable then for  $t = 0$ .

From the above we have the following procedure for computation of optimal inputs satisfying the condition (10) that steer the state of the system from  $x_0 = 0$  to  $x_f \in \mathbb{R}_+^n$  and minimize the performance index (11).

**Procedure 1.**

*Step 1.* Given  $A \in M_n$  and using (5), compute  $\Phi(t)$ .

*Step 2.* Using (12), compute the matrix  $W(t_f)$  for given  $A, B, Q, \alpha$  and some  $t_f$ .

*Step 3.* Using (13) and (23), find  $t_f$  for which  $\hat{u}(t)$  satisfying (10) reaches its maximal value and the desired  $\hat{u}(t)$  for given  $U \in \mathbb{R}_+^n$  and  $x_f \in \mathbb{R}_+^n$ .

*Step 4.* Using (15), compute the maximal value of the performance index.

**Example 1.** Consider the fractional positive system (2) for  $\alpha = 0.5$  with matrices

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad (24)$$

and the performance index (11) with

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}. \quad (25)$$

Compute the bounded input  $\hat{u}(t) \in \mathbb{R}_+^2$  satisfying the condition

$$\hat{u}(t) = \begin{bmatrix} \hat{u}_1(t) \\ \hat{u}_2(t) \end{bmatrix} < \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{for } t \in [0, t_f] \quad (26)$$

that steers the state of the system from zero state to  $x_f = [1 \ 1]^T \in \mathbb{R}_+^2$  ( $T$  denotes the transpose) and minimizes the performance index (11) with (25).

Using Procedure 1, we obtain what follows.

*Step 1.* From (5) and (24), we obtain

$$\begin{aligned} \Phi(t) &= \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \\ &= \sum_{k=0}^{\infty} \begin{bmatrix} 2^k & 0 \\ 0 & 1 \end{bmatrix} \frac{t^{(k-1)0.5}}{\Gamma[(k+1)0.5]}. \end{aligned} \quad (27)$$

*Step 2.* Using (12), (24) and (25), we get

$$\begin{aligned} W(t_f) &= \int_0^{t_f} \Phi(t_f - \tau)BQ^{-1}B^T\Phi^T(t_f - \tau) d\tau \\ &= \int_0^{t_f} \Phi(\tau)BQ^{-1}B^T\Phi^T(\tau) d\tau \\ &= \frac{1}{2} \int_0^{t_f} \Phi^2(\tau) d\tau \\ &= \sum_{k=0}^{\infty} \begin{bmatrix} 2^k & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{\Gamma[(k+1)0.5]} \int_0^{t_f} \tau^{k-1} d\tau \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{\ln t_f}{\Gamma(0.5)} \\ &\quad + \sum_{k=1}^{\infty} \begin{bmatrix} 2^{2k} & 0 \\ 0 & 1 \end{bmatrix} \frac{t_f^k}{k(\Gamma[(k+1)0.5])^2}. \end{aligned} \quad (28)$$

*Step 3.* The system with (24) is unstable. Therefore,  $\hat{u}(t)$  reaches its maximal value for  $t = 0$ ,

$$\hat{u}(0) = Q^{-1}B^T\Phi^T(t_f)W^{-1}(t_f)x_f < \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (29)$$

Using (29), we can find  $t_f$  satisfying the condition (26),

$$\begin{aligned} \hat{u}(0) &= Q^{-1}B^T\Phi(t_f)W^{-1}(t_f)x_f \\ &= \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^T \\ &\quad \times \sum_{k=0}^{\infty} \begin{bmatrix} 2^k & 0 \\ 0 & 1 \end{bmatrix} \frac{t_f^{(k-1)0.5}}{\Gamma[(k+1)0.5]} \\ &\quad \times \left[ \frac{\ln t_f}{\Gamma(0.5)} + \sum_{k=1}^{\infty} \frac{2^k t_f^k}{k(\Gamma[(k+1)0.5])^2} \right. \\ &\quad \left. \frac{0}{\Gamma(0.5)} + \sum_{k=1}^{\infty} \frac{t_f^k}{k(\Gamma[(k+1)0.5])^2} \right]^{-1} \\ &\quad \times \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned} \quad (30)$$



## 5. Concluding remarks

Necessary and sufficient conditions for the reachability of fractional positive continuous-time linear systems have been established (Theorem 2). The minimum energy control problem for fractional positive continuous-time linear systems with bounded inputs was formulated and solved. Sufficient conditions for the existence of a solution to the problem were given (Theorem 4) and a procedure for computation of an optimal input satisfying the condition (10) and the minimal value of the performance index was proposed. The effectiveness of the procedure was demonstrated on a numerical example. The presented method can be extended to positive discrete-time linear systems and to fractional positive discrete-time linear systems with bounded inputs.

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