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OPTIMAL FEEDBACK CONTROL PROPORTIONAL TO THE SYSTEM STATE CAN BE FOUND FOR NON-CAUSAL DESCRIPTOR SYSTEMS (A REMARK ON A PAPER BY P.C. MÜLLER)

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Optimal feedback control depending only on the system state is constructed for a control problem by the non-causal descriptor system for which optimal feedback control depending on state derivatives was considered in the paper (Müller, 1998). To this end, a non-symmetric solution of the algebraic operator Riccati equation is used.

Keywords: optimal feedback control, non-causal descriptor systems

1. Problem Statement

There are many works devoted to the study of optimal control problems for systems with the state equation which is not solvable with respect to the derivative (Lewis, 1986; Mehrmann, 1991; Kurina, 1992). In the scientific literature, such systems are called descriptor, singular, implicit or differential-algebraic systems. Causality plays an important role in studying optimal control problems for descriptor systems. Causality and non-causality distinguish between the cases where the descriptor system is exclusively governed by the control input or additionally by its time-derivatives (Müller, 1998; Kostjukova, 2000).

In (Müller, 1998), the following simple academic problem of minimizing the functional

$$J = \frac{1}{2}\int_0^{+\infty} \left(q_1 x_1^2 + q_2 x_2^2 + q_3 x_3^2 + r u^2\right) \mathrm{d}t \qquad (1)$$

on the trajectories of the system

$$\begin{aligned} \dot{x}_2 &= x_1 + b_1 u, \\ \dot{x}_3 &= x_2 + b_2 u, \\ 0 &= x_3 + b_3 u, \end{aligned} \qquad (2)$$

where $b_i,\ q_i \geq 0,\ i = \overline{1,3}$, and $r > 0$ are some parameters, is considered in order to illustrate the difficulties and the surprising results of the optimal control design of non-causal descriptor systems. The index of the last system is equal to three and this system is causal for $b_2 = 0,\ b_3 = 0$ and non-causal in other cases.

In (Müller, 1998), some optimal feedback control is given. It depends on the state variables and their derivatives for the non-causal case, i.e. it has the form

$$u = \frac{1}{r + q_2 b_1 b_3}\big[(b_1 q_1 - b_3 q_2)x_1 + b_2 q_2 x_2 + b_3 q_3 x_3 - b_2 q_1 \dot{x}_1 + b_3 q_1 \ddot{x}_1\big].$$

The related closed-loop control system is a standard system of the fourth order $(b_3 \neq 0,\ q_1 > 0)$. In order to find a unique solution for such a system, there must be given, e.g., the initial conditions $x_1(0),\ \dot{x_1}(0),\ x_2(0),\ x_3(0)$. The characteristic polynomial of the closed-loop system is bi-quadratic and hence the trajectory is asymptotically unstable. In (Müller, 1998), it is said that this is a consequence of an irregularly formulated optimization problem.

In the second approach by Müller, the state variables $\xi_1 = u,\ \xi_2 = \dot{u}$ are introduced, $\ddot{u}$ being considered as the control. The functional (1) is also modified using the relations for the state variables $x_1\ x_2,\ x_3$, as the functions of the control u and its derivatives, obtained from the state equation (2). Then the classical linear-quadratic optimal control problem is obtained to which the standard method, using the symmetric solution of the algebraic operator Riccati equation, can be applied. Thus $\ddot{u}$ is represented in the form of a linear combination of u and $\dot{u}$.

In the third approach by Müller, a possibility of applying a generalized Riccati equation is indicated but it is said that this equation cannot always be used even in the case of causal systems, and for non-causal systems its application is not possible because the typical time-derivative feedback part is not represented in this approach directly.

In the present paper, optimal feedback control which depends only on the system state is constructed for the control problem of the form (1), (2). For that purpose, the solution of the algebraic operator Riccati equation is used. In contrast to the classical case, this solution is non-symmetric and it satisfies the special symmetry condition (Kurina, 1993).

2. Problem Solution

The problem (1), (2) is a particular case of the problem of minimizing the functional

$$J(u)=\frac{1}{2}\int_0^{+\infty}\big(\langle x(t),Wx(t)\rangle+\langle u(t),Ru(t)\rangle\big)\,\mathrm{d}t \quad (3)$$

on the trajectories of the system

$$\frac{\mathrm{d}(Ax(t))}{\mathrm{d}t}=Cx(t)+Bu(t), \quad (4)$$

$$Ax(0)=x^0, \quad (5)$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in the appropriate space, $W=W'\geq 0$, $R=R'>0$ (the prime denotes the adjoint operator).

The problem (3)–(5) was considered in (Kurina, 1993). It was proved that if $K(t)$ is a solution to the algebraic operator Riccati equation of the form

$$K'C+C'K-K'BR^{-1}B'K+W=0, \quad (6)$$

satisfying the symmetry condition

$$A'K=K'A, \quad (7)$$

and the trajectory of (4), (5) for the control

$$u^*(t)=-R^{-1}B'Kx(t) \quad (8)$$

satisfies the condition

$$Ax(+\infty)=0, \quad (9)$$

then the feedback control (8) is optimal for the problem (3)–(5) and the minimal value of the functional (3) is equal to

$$J(u^*)=\frac{1}{2}\langle x(0),A'Kx(0)\rangle.$$

Some sufficient conditions for the existence of a solution to the problem (6), (7) are also given in (Kurina, 1993). It should be noted that the causality of the system and also the regularity of the pencil of the operators from (4) was not assumed in (Kurina, 1993).

For simplicity and clarity of calculations, with no loss of generality, we will assume that the parameters in the problem (1), (2) have the following values:

$$q_1=r=b_1=b_3=1,\quad q_2=q_3=b_2=0. \quad (10)$$

Taking into account the notation of the problem (3), (4), we obtain

$$W=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad R=1,\quad A=\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix},$$

$$C=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad B=\begin{pmatrix}1\\0\\1\end{pmatrix}.$$

It is not difficult to verify that the solution to the problem (6), (7), satisfying the condition (9), is the operator defined by the matrix

$$K=\begin{pmatrix}0&k_1&k_2\\0&k_2&k_3\\k_4&k_5&k_6\end{pmatrix},$$

where $k_1=\sqrt{2(\sqrt{2}-1)}$, $k_2=\sqrt{2}-1$, $k_3=2\sqrt{\sqrt{2}-1}$, $k_4=\pm 1$, $k_5=k_1(1/k_4-1)$, $k_6=k_2(1/k_4-1)+1$. It should be noted that, in view of the condition (7), the problem of finding nine elements of the matrix K is reduced to finding six elements only.

The optimal feedback control (8) for the problem (1), (2), (10) has the form

$$u^*(t)=-k_4x_1-\frac{k_1}{k_4}x_2-\left(\frac{k_2}{k_4}+1\right)x_3, \quad (11)$$

and the minimal value of the functional is equal to

$$J(u^*)=\frac{1}{2}\big(k_1x_2^2(0)+2k_2x_2(0)x_3(0)+k_3x_3^2(0)\big). \quad (12)$$

Remarks: 1. The optimal feedback control (11) is not unique. In particular, we obtain a different form taking into account the last equality in (2).

2. It is easy to see that the problem (1), (2), (10) can be reduced to the problem of minimizing the functional

$$\frac{1}{2}\int_0^{+\infty}\big((u+\ddot{u})^2+u^2\big)\,\mathrm{d}t.$$

3. The classical linear-quadratic optimal control problem of minimizing the functional

$$\frac{1}{2}\int_0^{+\infty}(x_1^2+x_3^2)\,\mathrm{d}t$$

on the trajectories of the system

$$\dot{x_2}=x_1-x_3,\quad \dot{x_3}=x_2,$$

$$x_2(0)=x_{20},\quad x_3(0)=x_{30}$$

corresponds to the problem (1), (2), (10). Here the control is $x_1 = x_1(t)$. The solution to this problem can be found with the help of the standard operator algebraic Riccati equation. The obtained solution corresponds to the relations (11), (12).

4. The equation of the form (6) under the condition (7) is also studied in (Kawamoto *et al*., 1998).

5. The special operator algebraic Riccati equation of the form

$$A'ZC + C'ZA - A'ZBR^{-1}B'ZA + W = 0 \qquad (13)$$

is used in (Mehrmann, 1991) for the introduction of some optimal feedback control. It is not difficult to show that this equation is not solvable for the problem (1), (2), (10). For that purpose it is sufficient to compare the elements on the upper left of all the additional matrix terms in (13). This element is equal to one for W and it is equal to zero for other terms.

6. The formula for optimal feedback control, using a non-symmetric (in the general case) solution to the differential operator Riccati equation non-solvable with respect to the derivative, was obtained in (Kurina, 1982; 1984) for the regulation problem by descriptor systems on a finite interval.

3. Conclusion

We have obtained that optimal feedback control, depending on the system state only, can be found for non-causal descriptor systems despite the statement of (Müller, 1998). For that purpose, the solution of the algebraic operator Riccati equation (6) satisfying the symmetry condition (7) must be used.

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References

Kawamoto A., Takaba K. and Katayama T. (1998): *On the generalized algebraic Riccati equation for continuous-time descriptor systems.* — Proc. 13-th Int. Symp. *Mathematical Theory of Networks and Systems*, Padova, Italy, pp. 649–652.

Kostjukova O.I. (2000): *Optimality criterion for a linear-quadratic optimal control problem by a descriptor system.* — Differencial'nye Uravnenija, Vol. 36, No. 11, pp. 1475–1481 (in Russian).

Kurina G.A. (1982): *Design of feedback control for linear control systems non-solvable with respect to the derivative.* — Unpublished paper, VINITI, No. 3619–82, Voronezh Forestry Institute (in Russian)).

Kurina G.A. (1984): *Feedback control for linear systems non-solvable with respect to the derivative.* — Avtomatika i Telemekhanika, No. 6, pp. 37–41 (in Russian).

Kurina G.A. (1992): *Singular perturbations of control problems with state equation non-solvable with respect to the derivative. A survey.* — Izvestija RAN. Tekhnicheskaya Kibernetika, No. 4, pp. 20–48 (in Russian).

Kurina G.A. (1993): *On regulating by a descriptor system in an infinite interval.* — Izvestija RAN, Tekhnicheskaya Kibernetika, No. 6, pp. 33–38 (in Russian).

Lewis F.L. (1986): *A survey of linear singular systems.* — Circ. Syst. Signal Process., Vol. 5, No. 1, pp. 3–36.

Mehrmann V. (1991): *The Autonomous Linear Quadratic Control Problem.* Berlin: Springer.

Müller P.C. (1998): *Stability and optimal control of nonlinear descriptor systems: A survey.* — Appl. Math. Comput. Sci., Vol. 8, No. 2, pp. 269–286.

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