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AN INVERSE PROBLEM OF A DIFFERENTIAL EQUATION IN A BANACH SPACE

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A problem of dynamical reconstruction of the unknown right hand side for a linear differential equation in a Banach space is considered. A numerical solution method based on certain constructions of the theory of positional (closed–loop) control and modelling is suggested.

1. Introduction

In recent years, the problem of stable real–time numerical reconstruction of unknown parameters of dynamical systems subject to incomplete or averaged information appears in various scientific and technical researches. The goal of the present paper is to give a method to solve the above problem for a concrete dynamical system.

2. Problem Formulation

An abstract differential wave equation of the form

$$\ddot{x}(t) + Ax(t) = Bu(t) + f(t), \quad t \in T = [0, v], \tag{1}$$

$$x(0) = x_0 \in V, \qquad \dot{x}(0) = x_{10} \in H$$

is considered. Here $A : D(A) \to V^*$ is a self adjoint positively definite linear operator whose domain $D(A)$ is dense in a Hilbert space $(H, |\cdot|), f(\cdot) \in L_2(T; H)$ is a given function, $u(t) \in U$ is a control, $(U, |\cdot|_U)$ is a real Banach space, $B : U \to H$ is a linear continuous operator, $V = D(A^{1/2})$ is a separable Hilbert space with the scalar product

$$(v_1, v_2)_V = \sum_{j=1}^{\infty} \lambda_j (v_1, \varphi_j)_H (v_2, \varphi_j)_H \quad \forall v_1, v_2 \in V,$$

which is continuously and densely imbedded in $H (D(A) \subset V \subset H)$. We assume that A has a complete sequence of orthonormal eigenelements $(\varphi_i)_{j=1}^{\infty}$, $(\lambda_j)_{j=1}^{\infty}$ is a corresponding sequence of real eigenvalues λ_j

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$$0 < \lambda_1 \le \lambda_2 \le \ldots \quad \text{and} \quad \lim_{j\to\infty} \lambda_j = \infty$$

For every $f(\cdot) \in L_2(T;H)$, $u(\cdot) \in L_2(T;U)$, $x_0 \in V$, $x_{10} \in H$, the weak solution of (1) is a function $x(\cdot) : T \to V$ with the following properties:

i) $x(\cdot) \in C(T;V)$, for all $t \in T$ there exists the strong derivative $\dot{x}(t)$, and $\dot{x}(t) \in C(T;H)$,

ii) $\lim_{t\to 0+} |x(t) - x_0|_V = 0$, $\lim_{t\to 0+} |\dot{x}(t) - x_{10}|_H = 0$,

iii) $\ddot{x}(\cdot)$, understood in the sense of distributions, can be identified with a function in $L_2(T;V^*)$, as follows

$$< \ddot{x}(t) + \overline{A}x(t), v >_{V^* \times V} = (Bu(t) + f(t), v)_H$$

$\forall v \in V$ and $t \in T$

The linear mapping $\overline{A} : V \to V^*$ is defined by

$$< \overline{A}v, w >_{V^* \times V} = (A^{1/2}v, A^{1/2}w)_H \; \forall v, \; w \in V$$

It is known (Krabs, 1985) that weak solution $x(\cdot) = x(\cdot; x_0, x_{10}, u(\cdot))$ of (1) is given by

$$x(t) = S(t)x_0 + T(t)x_{10} + \int_0^t T(t-\tau)(Bu(\tau) + f(\tau))d\tau$$

for all $t \in T$, and the strong derivative $\dot{x}(\cdot)$ is given by

$$\dot{x}(t) = -AT(t)x_0 + S(t)x_{10} + \int_0^t S(t-\tau)(Bu(\tau) + f(\tau))d\tau$$

for all $t \in T$. Here, $S(t)$ is the cosine operator (Krabs, 1985), generated by A, $T(t)$ is the operator associated with the sine operator

$$T(t)x = \int_0^t S(\tau)x d\tau$$

For the properties of the operators $S(t)$ and $T(t)$ and the weak solutions see (Fattorini, 1969) and (Lasiecka and Triggiani, 1981; 1983).

Let Y be a Banach space, $C : H \to Y$ be a linear continuous operator. The problem can be explained as follows. An unknown control $u_*(\cdot) \in L_2(T;U)$ generating an unknown realization, i.e. the weak solution $x_*(\cdot) = x(\cdot; x_0, x_{10}, u_*(\cdot))$ of (1) acts on the system (1). The time interval T is put into parts by intervals $[\tau_i, \tau_{i+1})$, $\tau_{i+1} = \tau_i + \delta$, $\delta > 0$, $i \in [0 : m_\delta]$. At time instants τ_i the elements $C\dot{x}_*(\tau_i)$ are measured approximately, i.e. elements $\xi_i = \xi(\tau_i) \in Y$ being approximations to $C\dot{x}_*(\tau_i)$ are measured

$$|C\dot{x}_*(\tau_i) - \xi_i|_Y \leq \varepsilon \tag{2}$$

An approximation algorithm of the $u_*(\cdot)$ is to be found.

3. Problem Solution

Below, a solution algorithm for the problem based on the approach to the inverse problems of dynamics developed by Kryazhimski and Osipov (1983, 1987), Osipov (1987), Osipov *et. al.* (1991) is proposed. For the case where $Y = H$, $C = I$ (the identity operator), i.e. the elements $x_*(\tau_i)$ are measured, the various algorithms approximating $u_*(\cdot)$ for systems of the form (1) (based on the above approach) were considered by Osipov and Maksimov (1992), Korotki and Osipov (1991) and Maksimov (1991).

Let the following conditions be fulfilled:

i) $Y = U$, U is a real Hilbert space,

ii) x_0 and x_{10} are known, $f(\cdot) \in C(T; H)$,

iii) the function $t \to \xi(t) = Cx_*(t) \in C^2(T; U)$, $Cx_0 = \xi(0)$, $Cx_{10} = \dot{\xi}(0)$,

iv) the operator CB is invertible in U, and $(CB)^{-1} : U \to U$ is a linear continuous operator,

v) $\overline{CA} : H \to U$ is a linear continuous operator ($\overline{CA}$ is the closure of the operator CA),

vi) the sine operator $T(t)$ is a function of type ω

$$|T(t)x| \leq K \exp(\omega t)|x| \qquad \forall x \in H,$$

vii) $\int_0^\vartheta \int_0^t |K(t,\tau)|_U^2 d\tau dt < 1$

Here $K(t,\tau) = -(CB)^{-1}\overline{CA}T(t-\tau)B$, if $t \geq \tau$, $K(t,\tau) = 0$, in the opposite case.

Condition (5) is fulfilled if $U = H = L_2(\Omega)$, $B = I$, $\Omega \subset R^n$ is a bounded open set with the smooth boundary, $A : L_2(\Omega) \to L_2(\Omega)$, $Ax = -\Delta x$ (Δ is the Laplace operator), $D(A) = H_0^1(\Omega) \bigcap H^2(\Omega)$

$$(Cx)(\eta) = \int_\Omega M(\eta,\nu)x(\nu)d\nu \quad \text{for} \quad \eta \in \Omega \quad \text{and} \quad \forall x \in H$$

functions $M(\eta,\nu)$ and $\Delta_\nu M(\eta,\nu)$ are measured in the sense of Lebesque on $\Omega \times \Omega$,

$$\nu \to M(\eta,\nu) \in H_0^1(\Omega) \bigcap H^2(\Omega) \quad \text{for} \quad \eta \in \Omega$$

$$\int_{\Omega\times\Omega} |M(\eta,\nu)|_U^2 d\nu d\eta < +\infty, \quad \int_{\Omega\times\Omega} |\Delta_\nu M(\eta,\nu)|_U^2 d\nu d\eta < +\infty$$

Besides,

$$(\overline{CA}x)(\eta) = \int_\Omega \Delta_\nu M(\eta,\nu)x(\nu)d\nu \quad \text{for} \quad \eta \in \Omega \quad \text{and} \quad \forall x \in H$$

To calculate approximately $u_*(\cdot)$, we apply the method of closed-loop control with a model (Krasovski, 1985; Kryazhimski and Osipov, 1983; 1987). Let us describe the algorithm, i.e. the sequence of actions forming an approximation to $u_*(\cdot)$. First, a family $\Delta_\varepsilon = \{\tau_{\varepsilon,i}\}_{i=0}^{m_\delta}$, $\tau_{\varepsilon,0} = 0$, $\tau_{\varepsilon,m_\delta} = \vartheta$, $\tau_{\varepsilon,i+1} = \tau_{\varepsilon,i} + \delta(\varepsilon)$, of partition of the interval T with diameters $\delta(\varepsilon) = \delta(\Delta_\varepsilon)$, $\delta(\varepsilon) \to 0$, $\varepsilon\delta^{-1}(\varepsilon) \to 0$ as $\varepsilon \to 0$, and an auxiliary system M (*a model*) functioning synchronically with the real system (1) are chosen. The model M is given by the system

$$\begin{aligned}
&\dot w(t) = v^\varepsilon(t) && w(0) = 0\\
&\ddot w_1(t) + Aw_1(t) = f(t) && w_1(0) = x_0, \dot w_1(0) = x_{10}\\
&\dot w_2(t) - \overline{CA}w_1(t) = Cf(t) && w_2(0) = 0\\
&\ddot w_3(t) + Aw_3(t) = Bv^\varepsilon(t) && w_3(0) = \dot w_3(0) = 0\\
&\dot w_3^{(1)}(t) = (CB)^{-1}\overline{CA}w_3(t) && w_3^{(1)}(0) = 0
\end{aligned}$$

for $t \in T$, and

$$\begin{aligned}
&\ddot w_4(t) + Aw_4(t) = 0 \qquad w_4(\tau_{\varepsilon,i}) = w_3(\tau_{\varepsilon,i}) \qquad \dot w_4(\tau_{\varepsilon,i}) = 0\\
&\ddot w_5(t) + Aw_5(t) = 0 \qquad w_5(\tau_{\varepsilon,i}) = 0 \qquad \dot w_5(\tau_{\varepsilon,i}) = \dot w_3(\tau_{\varepsilon,i})
\end{aligned}$$

for $t \in \delta_{\varepsilon,i} = [\tau_{\varepsilon,i}, \tau_{\varepsilon,i+1})$

with the control $v^\varepsilon(\cdot)$. Before the initial time of the process, a value ε and the partition $\Delta = \Delta_\varepsilon$ are fixed. The work of the algorithm starting at time t_0 is decomposed into $m_\delta - 1$ steps. At the i-th step carried out during the time interval $\delta_{\varepsilon,i}$, the control $v^\varepsilon(t) = v_i^\varepsilon$, $t \in \delta_{\varepsilon,i}$

$$v_i^\varepsilon = \begin{cases} |(CB)^{-1}\delta^{-1}(\varepsilon)[\{(\xi_i - \xi_{i-1}) - (w_2(\tau_{\varepsilon,i}) - w_2(\tau_{\varepsilon,i-1}))\} - \\ \overline{CA}\int_{\tau_{\varepsilon,i-1}}^{\tau_{\varepsilon,i}}\{w_4(\tau) + w_5(\tau)\}d\tau]|_U s_i/|s_i|_U, & \text{if } |s_i|_U \neq 0 \\ 0, & \text{in the opposite case}\end{cases}$$

$$s_i = (CB)^{-1}\{\xi_{i-1} - \xi_0 - w_2(\tau_{\varepsilon,i-1})\} - w(\tau_{\varepsilon,i-1}) - w_3^{(1)}(\tau_{\varepsilon,i-1})$$

$$\xi_i = \xi(\tau_{\varepsilon,i})$$

is calculated. After that, we transform the state $p_{\varepsilon,i} = \{w(\tau_{\varepsilon,i}),\ w_1(\tau_{\varepsilon,i}),\ \dot{w}_1(\tau_{\varepsilon,i}),\ w_2(\tau_{\varepsilon,i}),\ w_3(\tau_{\varepsilon,i}),\ \dot{w}_3(\tau_{\varepsilon,i}),\ w_3^{(1)}(\tau_{\varepsilon,i})\}$ of the model into $p_{\varepsilon,i+1}$. The procedure stops at time ϑ. Due to the properties of a weak solution of system (1), the phase trajectory of the model is determined correctly.

The following Lemma is true.

Lemma. *The bounds*

$$\|v^{\varepsilon}(\cdot)\|_{L_2(T;U)} \leq k_0$$

$$\varepsilon(t) \leq k\{\delta(\Delta_{\varepsilon}) + \varepsilon\delta^{-1}(\Delta_{\varepsilon})\} \quad \forall t \in T$$

hold uniformly with respect to all $\varepsilon \in (0,1]$, *partitions* $\{\Delta_{\varepsilon}\}$ *with diameter* $\delta(\Delta_{\varepsilon})$, *and measurement results* $\xi_i = \xi(\tau_{\varepsilon,i})$, *satisfying (2).*

Here, k, k_0 are constants written out explicitly

$$\varepsilon(t) = |\int_0^t \{g(\tau) - v^{\varepsilon}(\tau) + \int_0^T K(\tau,\eta)v^{\varepsilon}(\eta)d\eta\}d\tau|_U^2,$$

$$g(t) = (CB)^{-1}\{\xi(t) - \overline{CA}x(t;x_0,x_{10},f(\cdot)) - Cf(t)\}$$

Let the weak norm $|\cdot|_w$ be introduced on the set

$$S(k_0) = \{v(\cdot) \in L_2(T;U) : \|v(\cdot)\|_{L_2(T;U)} \leq k_0\}$$

(it can be done by the Theorem of Bishop (Warga, 1972)).

Then the following Theorem is formulated.

Theorem. *Let* $\varepsilon \to 0$, *then* $\|u_*(\cdot)\|_{L_2(T;U)} \leq k_0$ *and*

$$\|v^{\varepsilon}(\cdot) - u_*(\cdot)\|_{L_2(T;U)} \to 0$$

The proof of the Theorem is based on the results given by Orlovski (1991) and on the Lemma as well as on the fact that $u_*(\cdot)$ is the unique in $L_2(T;U)$ solution of the Volterra equation

$$u(t) = g(t) + \int_0^t K(t,\tau)u(\tau)d\tau, \qquad t \in T$$

4. Conclusion

The paper presents an algorithm for reconstruction of unknown controls action on a dynamical system described by a differential equation in a Banach space. The algorithm is stable with respect to informational noises and computational errors, and allows to calculate an input with an arbitrary accuracy, basing on averaged measurements of variable states.

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