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A FEEDBACK SYNTHESIS OF BOUNDARY CONTROL PROBLEM FOR A PLATE EQUATION WITH STRUCTURAL DAMPING†

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A boundary control problem for a Kirchhoff plate equation with a structural damping is considered. A distinctive feature of this problem is the lack of strong coercivity with respect to control variable. It is shown that the optimal control admits a pointwise state feedback synthesis via a solution of nonstandard Riccati equation. The novelty of the problem with respect to the literature is that both: the associated Riccati equation *and* the feedback control operator, are nonstandard.

1. Introduction

1.1. Model

Let Ω be an open, bounded domain in $\mathbb{R}^2$. It is assumed that the boundary of Ω, denoted by Γ, is smooth (say C^2). We consider the following model of a Kirchhoff plate (see Lagnese, 1989) in the variable w representing the displacement of the plate.

$$\begin{cases} \text{(i)} & \rho h w_{tt} - \rho \frac{h^3}{12}\Delta w_{tt} + \alpha \Delta^2 w_t + D\Delta^2 w = 0 & \text{in } Q \equiv \Omega \times (0,\infty) \\ \text{(ii)} & w = 0 & \text{on } \Sigma \equiv \Gamma \times (0,\infty) \\ \text{(iii)} & D\Delta w = u & \text{on } \Sigma \equiv \Gamma \times (0,\infty) \\ \text{(iv)} & w(\,\cdot\,,t=0) = w_0,\, w_t(\,\cdot\,,t=0) = w_1 & \text{in } \Omega \end{cases} \tag{1}$$

Here, the constant ρ is mass density per unit of volume, h represents the thickness of the plate (assumed to be small). The modulus of flexural rigidity D is given by

$$D \equiv Eh^3/12(1-\mu^2)$$

with μ Poisson's ratio ($0 < \mu < \frac{1}{2}$ in physical situations), and E Young's modulus. The parameter $\alpha \geq 0$ represents structural damping of the plate which in physical situations is usually small.

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The function $u \in L_2(\Sigma)$ appearing in the second boundary condition (1iii) represents boundary control which acts via a bending moment about the direction tangent to the edge of the plate.
The second term in the equation (1i) represents rotational inertia and may be neglected in some studies of the system.

1.2. Control Problem

With dynamics represented by (1) we associate functional cost given by

$$\begin{aligned} J(w,u) \equiv & \int_0^\infty \int_\Omega \left\{ \beta_1^2 w^2(t,x) + \beta_2^2 |\nabla w(t,x)|^2 \right\} \,dx dt \\ & + \int_0^\infty \int_\Gamma u^2(t,x) \,dx dt \end{aligned} \tag{2}$$

The control problem (P) to be studied is as follows: given w_0, w_1, in appropriate spaces (to be determined later), and $u(t=0) \in L_2(\Gamma)$, find an optimal control $u^0 \in L_2(0,\infty; L_2(\Gamma))$ such that cost function (2) is minimized for all $u \in L_2(0,\infty; L_2(\Gamma))$ subject to the dynamics in (1).

The main goal of the paper is to determine a feedback structure of the optimal control u^0. This is to say, we are seeking representation of the form

$$u^0(t) = CP(w^0(t), w_t^0(t)) \tag{3}$$

with a suitable (typically unbounded) operator $C : L_2(\Omega) \times L_2(\Omega) \to L_2(\Gamma)$ where P is a solution of an appropriate (nonstandard) Riccati equation.

The control problem formulated above is not a *standard* LQR control problem. The reasons are twofold:

(i) The presence of a boundary control coupled with structural damping gives rise, as we shall see in Section 3, to an abstract model of the type

$$z_t = Az + Bu + Bu_t \tag{4}$$

associated with the functional cost

$$J(z,u) = \int_0^\infty |Rz|_Z^2 + |u|_U^2 \,dt \tag{5}$$

(A, B, Z, U will be specified later). The above control problem is not *coercive* with respect to control variable (which also accounts for *velocity* u_t). As a result, the standard LQR methods cannot be applied. Moreover, as we shall see later, this problem leads to the so-called "non-standard" Riccati equations and "non-standard" synthesis problem which appears to be new even in the context of finite-dimensional theory.

(ii) Boundary controls appearing in (1iii) give rise to *unbounded* control operators B in the abstract model (4). Handling of this problem requires a careful mathematical/PDE analysis of the problem. One of the consequences is that the synthesizing operator C in (3) is also unbounded. Thus, it is necessary to develop *regularity* theory for Riccati operators to ensure that the composition operator CP is meaningful, and it is properly defined.

Boundary control problems for (structurally) damped wave equation were considered earlier in (Bucci, 1992). However, the problem treated in (Bucci, 1992) involves penalization of the velocity of the control. This is to say that instead of (5), Bucci (1992) takes

$$J(z,u) = \int_0^\infty |Cz|_Z^2 + |u|_U^2 + |u_t|_U^2 \, dt \tag{6}$$

This problem is, of course, coercive and allows for application of standard LQR theory with unbounded control operators (see also Balakrishnan, 1976; Bensoussan *et al.*, 1992; Lasiecka and Triggiani, 1983).

Thus the two new features of our problem (see (i), (ii) above) make the techniques developed in the literature nonapplicable, and solution of the problem requires a new approach.

The following *notation* will be used in the paper. $H^s(\Omega)$ denotes, as usual, Sobolev's spaces at order $s \geq 0$. $H_0^s(\Omega)$ is a completion of $C_0^\infty(\Omega)$ with respect to $H^s(\Omega)$ norm. $H^{-s}(\Omega) \equiv (H_0^s(\Omega))'$. $\mathcal{D}(A)$-denotes a domain of a closed, linear operator $A : H \to H$. $(\mathcal{D}(A^*))'$ denotes a dual (pivotal) space to $\mathcal{D}(A)$), i.e. $\mathcal{D}(A) \subset H \subset (\mathcal{D}(A^*))'$. If A is positive, $|x|_{(\mathcal{D}(A))'} = |A^{*-1}x|_H$. A^γ denotes fractional powers of a positive operator A (Pazy, 1983).

$(x,y)_\Omega \equiv \int_\Omega xy \, d\Omega$, $[x,y]_\Gamma \equiv \int_\Gamma xy \, d\Gamma$. ν denotes an exterior normal to the boundary.

1.3. Statement of Main Results

In order to present the main results, it is convenient to simplify the writing of the system by making a change of the time scale $t \to t\sqrt{\frac{D}{\rho h}}$. Then (1) is brought to the form

$$\begin{cases} w_{tt} - \gamma\Delta w_{tt} + \Delta^2 w + \alpha\Delta^2 w_t = 0 & \text{in } Q \\ w = 0 & \text{on } \Sigma \\ \Delta w = u & \text{on } \Sigma \\ w(t=0) = w_0, w_t(t=0) = w_1 & \text{in } \Omega \end{cases} \tag{7}$$

Here γ is proportional to the square of the thickness of the plate, i.e., $\gamma = \frac{h^2}{12}$. We shall consider separately two cases: case $\gamma = 0$ and case $\gamma > 0$. Case $\gamma > 0$ (resp. $\gamma = 0$) corresponds to the situation when rotational forces are accounted (resp. not accounted) for.

Remark 1. We note that a strict positivity of γ changes the character of undamped dynamics. Indeed, when $\alpha = 0$ and $\gamma > 0$, model (1) is of hyperbolic type with a finite speed at propagation while the case $\gamma = 0$ corresponds to "Petrovski" type of systems which is characterized by an infinite speed of propagation.

Theorem 1. *Assume that* $\gamma = 0$ *in (7) and* $\beta_2 = 0$ *in (2).*

I. *(Existence and Regularity) For any initial data* $w_0, w_1 \in L_2(\Omega) \times L_2(\Omega)$, $u(0) \in L_2(\Gamma)$, *there exists a unique solution to control problem* $(\mathbf{P})$: $u^0(t)$, $w^0(t)$ *such that*

$$u^0 \in C((0,\infty); L_2(\Gamma))$$
$$w^0 \in C([0,\infty); L_2(\Omega))$$

II. *(Riccati Equations) There exists a unique, positive, self-adjoint solution* $P \in \mathcal{L}(L_2(\Omega) \times L_2(\Omega))$ *satisfying the following Algebraic Riccati Equation.*

$$(x_1, \Delta^2 p_1 y)_\Omega + (\Delta^2 p_1 x, y_1)_\Omega - (x_2, p_1 y - \alpha\Delta^2 p_2 y)_\Omega$$
$$- (p_1 x - \alpha\Delta^2 p_2 x, y_2)_\Omega - \beta_1^2 (x_1, y_1)_\Omega$$
$$= \left[\frac{\partial}{\partial\nu}[-p_2 x + \alpha p_1 x - \alpha^2\Delta^2 p_2 x], \frac{\partial}{\partial\nu}[-p_2 y + \alpha p_1 y - \alpha^2\Delta^2 p_2 y]\right]_\Gamma$$

for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in L_2(\Omega) \times L_2(\Omega)$ *where*

$$P\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \equiv \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \equiv \begin{bmatrix} p_1 x \\ p_2 x \end{bmatrix}$$

Moreover, for all $x \in L_2(\Omega) \times L_2(\Omega)$,

$$p_2 x \in H^4(\Omega), \; p_1 x - \alpha\Delta^2 p_2 x \in H^{4-\epsilon}(\Omega); \;\; \forall\epsilon > 0 \tag{8}$$

III. *(Synthesis) Define the operator* $K : L_2(\Gamma) \to L_2(\Gamma)$ *by*

$$\frac{1}{\alpha} K u \equiv \frac{\partial}{\partial\nu}\left[(-p_{22} + \alpha p_{12} - \alpha^2\Delta^2 p_{22}) M u\right]_\Gamma \tag{9}$$

where $(Mu, \xi)_\Omega = -\left[u, \frac{\partial}{\partial\nu}\xi\right]_\Gamma$ *for all* $u \in L_2(\Gamma)$, $\xi \in H^2(\Omega) \cap H_0^1(\Omega)$.
Then, $K \in \mathcal{L}(L_2(\Gamma))$ *and, moreover,* $(I - K)^{-1} \in \mathcal{L}(L_2(\Gamma)$. *The optimal control* u^0 *admits the following feedback representation*

$$u^0(t) = [I - K]^{-1}\left[\frac{\partial}{\partial\nu}[p_2 z^0(t) - \alpha p_1 z^0(t) + \alpha^2\Delta^2 p_2 z^0(t)]\right]_\Gamma$$

with $z^0(t) = (w^0(t), w_t^0(t))$.

Theorem 2. *Assume* $\gamma > 0$.

I. *(Existence and regularity) For any initial data* w_0, $w_1 \in H_0^1(\Omega)$, $u(0) \in L_2(\Gamma)$, *there exists a unique solution to optimal control problem* $(\mathbf{P})$: $u^0(t)$, $w^0(t)$ *such that*

$$u^0 \in C((0,\infty); L_2(\Gamma))$$
$$w^0 \in C([0,\infty); H_0^1(\Omega))$$

II. *(Riccati Equations) There exists a unique, positive, self-adjoint solution* $P \in \mathcal{L}(H_0^1(\Omega))$ *satisfying the following Riccati equation*

$$\begin{aligned}(x_1, \overline{p_2 y})_\Omega &+ \gamma(\nabla x_1, \nabla \overline{p_2 y})_\Omega - (x_2, p_1 y)_\Omega - \gamma(\nabla x_2, \nabla p_1 y)_\Omega \\ &+ \alpha(x_2, \overline{p_2 y})_\Omega + \alpha\gamma(\nabla x_2, \nabla \overline{p_2 y})_\Omega + (y_1, \overline{p_2 x})_\Omega \\ &+ \gamma(\nabla \overline{y_1}, \nabla \overline{p_2 x})_\Omega - (y_2, p_1 x)_\Omega - \gamma(\nabla y_2, \nabla p_1 x)_\Omega \\ &+ \alpha(y_2, \overline{p_2 x})_\Omega + \alpha\gamma(\nabla y_2, \nabla \overline{p_2 x})_\Omega \\ &- \beta_1^2(x_1, y_1)_\Omega - \beta_2^2(\nabla x_1, \nabla y_1)_\Omega \\ &= \left[\frac{\partial}{\partial\nu}\left[-p_2 x + \alpha p_1 x - \alpha^2 \overline{p_2 x}\right], \frac{\partial}{\partial\nu}\left[-p_2 y + \alpha p_1 y - \alpha^2 \overline{p_2 y}\right]\right]_\Gamma\end{aligned}$$

for all $x = (x_1, x_2) \in H_0^1(\Omega) \times H_0^1(\Omega),\ y = (y_1, y_2) \in H_0^1(\Omega) \times H_0^1(\Omega)$.
Here $p_2 x$ *and* $\overline{p_2 x}$ *are related in 1-1 manner through the following system of equations*

$$\begin{cases} -\Delta p_2 x = \ell + \gamma \overline{p_2 x} & in \quad \Omega \\ p_2 x|_\Gamma = 0 \end{cases} \tag{10}$$

and ℓ *satisfies*

$$\begin{cases} -\Delta \ell = \overline{p}_2 x & in \quad \Omega \\ \ell = 0 & on \quad \Gamma \end{cases} \tag{10'}$$

Moreover, the following additional regularity holds

$$\begin{cases} p_2 x \in H^3(\Omega) \\ p_1 x - \alpha \overline{p_2 x} \in H^{3-\epsilon}(\Omega) \\ \overline{p_2 x} \in H_0^1(\Omega) \end{cases} \tag{11}$$

III. *(Synthesis) Define the boundary operator* $K : L_2(\Gamma) \to L_2(\Gamma)$

$$\frac{1}{\alpha} K u \equiv \frac{\partial}{\partial\nu}\left[(p_{22} - \alpha p_{12} + \alpha^2 \overline{p_{22}})k\right]_\Gamma \tag{12}$$

where the relation between k *and* u *is defined via duality*

$$\left[u, \frac{\partial}{\partial\nu}\xi\right]_\Gamma = (k, \xi)_\Omega + \gamma(\nabla k, \nabla \xi)_\Omega, \quad \xi \in H^2(\Omega)$$

Moreover, $\overline{p_{22}}x$ *is related to* $p_{22}x$ *via the same relations as in (10). Then,* $K \in \mathcal{L}(L_2(\Gamma))$ *and* $(I - K)^{-1} \in \mathcal{L}(L_2(\Gamma))$.

The optimal control u^0 can be written in the following feedback form

$$u^0(t) = [I - K]^{-1} \frac{\partial}{\partial \nu}[p_2 z^0(t) - \alpha p_1 z^0(t) + \alpha^2 \overline{p_2 z}^0(t)]_\Gamma \tag{13}$$

Remarks. (1) Notice that the structure of the feedback synthesis involves inversion of the boundary operator $I - K$ with K given by (9) or (12). This is a distinct feature of the problem reflecting the fact that the control problem is not a *standard* LQR problem. This phenomenon appears new even in the context of finite–dimensional theory.

(2) Regularity of optimal controls in parts I of both theorems, together with regularity of Riccati operators in (8) and (11) represent the additional regularity of the problem (i.e., this does not simply follow from minimization principle). These regularity results are critical to: (i) assert invertibility of the boundary operator K appearing in a feedback synthesis, and (ii) give the meaning to each term of Riccati equation.

(3) Notice that the optimal control may admit a discontinuity at the origin.

The remainder of this paper is devoted to the proofs of Theorems 1 and 2.

2. Reformulation of Control Problem (P) as an Abstract Control Problem

The goal of this section is to reformulate control problem **(P)** as an abstract control problem. To accomplish this we shall introduce the following spaces and operators

$$\begin{cases} \mathcal{A} : L_2(\Omega) \to L_2(\Omega) \quad \text{defined by} \\ \mathcal{A}u = -\Delta u, \ u \in \mathcal{D}(\mathcal{A}) = H_0^1(\Omega) \cap H^2(\Omega) \end{cases} \tag{14}$$

$$\begin{cases} D : L_2(\Gamma) \to L_2(\Omega) \quad \text{defined by} \\ Dg = v \text{ iff } \Delta v = 0 \text{ in } \Omega \text{ and } v|_\Gamma = g \end{cases} \tag{15}$$

With the above notation, equation (7) can be rewritten as (see Lasiecka and Triggiani, 1992),

$$\begin{cases} w_{tt} + \gamma \mathcal{A} w_{tt} + \mathcal{A}(\mathcal{A}w + Du) + \alpha \mathcal{A}(\mathcal{A}w_t + Du_t) = 0 \\ w(t=0) = w_0, \ w_t(t=0) = w_1 \end{cases} \tag{16}$$

Or, equivalently, using

$$(I + \gamma \mathcal{A})^{-1} \in \mathcal{L}(L_2(\Omega)) \tag{17}$$

as

$$\begin{cases} w_{tt} + (I+\gamma\mathcal{A})^{-1}\mathcal{A}^2 w + \alpha(I+\gamma\mathcal{A})^{-1}\mathcal{A}^2 w_t + (I+\gamma\mathcal{A})^{-1}\mathcal{A}Du \\ \qquad + \alpha(I+\gamma\mathcal{A})^{-1}\mathcal{A}Du_t = 0 \\ w(t=0) = w_0, \ w_t(t=0) = w_1 \end{cases} \tag{18}$$

Case $\gamma>0$. We define the following spaces and operators;

$$\begin{cases} H \equiv H_0^1(\Omega) \times H_0^1(\Omega) \\ U \equiv L_2(\Gamma), \quad Z = L_2(\Omega) \times L_2(\Omega) \times L_2(\Omega) \end{cases} \tag{19}$$

$A : H \to H$ is given by

$$A = \begin{bmatrix} 0 & -I \\ \mathcal{A}_\gamma & \alpha \mathcal{A}_\gamma \end{bmatrix} \tag{20}$$

where $\mathcal{A}_\gamma \equiv (I + \gamma \mathcal{A})^{-1} \mathcal{A}^2$;

$$D(A) = \{(w, z) \in H_0^1(\Omega) \times H_0^1(\Omega);\ \mathcal{A}_\gamma(w + \alpha z) \in H_0^1(\Omega)\} \tag{21}$$

$$B : L_2(\Gamma) \to L_2(\Omega) \times L_2(\Omega)$$

$$Bu = \begin{bmatrix} 0 \\ -(I + \gamma \mathcal{A})^{-1} \mathcal{A} D \end{bmatrix} \tag{22}$$

With the above notation dynamics in (18) can be written in the variable $z \equiv (w, w_t)$ as

$$\begin{cases} z_t + Az + Bu + \alpha B u_t = 0 \quad \text{in} \quad \mathcal{D}(A^*)' \\ z(0) = (w_0, w_1) \in H \end{cases} \tag{23}$$

The performance index (2) associated with (23) takes the form

$$J(z, u) = \int_0^\infty \left[|Rz|_Z^2 + |u|_U^2\right] dt \tag{24}$$

where

$$R = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} \tag{25}$$

and $C : H_0^1(\Omega) \to [L_2(\Omega)]^{n+1}$ is given by

$$Cw = [\beta_1 w, \beta_2 \nabla w]$$

Case $\gamma=0$, $\beta_2=0$. We define the following spaces and operators:

$$H = L_2(\Omega) \times L_2(\Omega), \quad U = L_2(\Gamma), \quad Z = H \tag{26}$$

$$A = \begin{bmatrix} 0 & -I \\ \mathcal{A}^2 & \alpha \mathcal{A}^2 \end{bmatrix} \tag{27}$$

$$D(A) \equiv \{(w, z) \in L_2(\Omega) \times L_2(\Omega);\ \mathcal{A}^2(w + \alpha z) \in L_2(\Omega)\}$$

$$B = \begin{bmatrix} 0 \\ -AD \end{bmatrix} \tag{28}$$

$$R = \begin{bmatrix} \beta_1 I & 0 \\ 0 & 0 \end{bmatrix} \tag{29}$$

Conclusion. The original control problem consisting of minimizing (2) subject to the dynamics (1) can be abstractly rewritten as minimization of (24) subject to (23) with the operator and spaces defined by (19), (20), (22), (25) in the case $\gamma > 0$, and by (26)–(28) in the case $\gamma = 0$.

3. Nonstandard Riccati Equations Associated with Abstract Control Problem

We consider an abstract differential equation given by

$$\begin{cases} z_t + Az + Bu + \alpha B u_t = 0 \quad \text{in} \quad \mathcal{D}(A^*)' \\ z(0) = z_0 \in H \end{cases} \tag{30}$$

where we are given: H, Z, and U Hilbert spaces, the operators (generally unbounded)

$$\begin{cases} A : H \to H \quad \text{with} \quad \mathcal{D}(A) \subset H \\ B : U \to \mathcal{D}(A^*)' \end{cases}$$

With (30) we associate functional cost

$$J(z,u) = \int_0^\infty [|Rz(t)|_Z^2 + |u(t)|_U^2]\, dt \tag{31}$$

Our *abstract control problem* is formulated as follows: given $z(0) \in H$ and $u(0) \in U$, find the optimal $u^0 \in L_2(0,\infty;U)$ such that (31) is minimized subject to dynamics (30).

Remark. In order that state variable $z(t)$ be uniquely defined with controls $u \in L_2(0,\infty;U)$, it is necessary to prescribe the value $u(0)$. Thus, the variables: $u(0) \in U$ and $u \in L_2(0,\infty;U)$ are two *independent* variables.

The following technical assumptions are imposed on the data of the problems.

(H-1) A is a generator of an analytic, stable semigroup e^{At} on H.

(H-2) There exists $1 > \gamma_0 \geq 0$ such that

$$A^{-\gamma_0} B \in \mathcal{L}(U,H)$$
$$A^{-\gamma} B : U \to H \quad \text{is compact for} \quad \gamma > \gamma_0$$

(H-3) $\quad RA^{\gamma_1} \in \mathcal{L}(H;Z) \quad$ for some $\quad \gamma_1 > \gamma_0$

Under the above assumptions, it was shown (Lasiecka *et al.*, 1994; Triggiani, 1993) that there exists a unique optimal solution to the control problem such that

$$\begin{cases} \text{(i)} \quad u^0 \in C[(0,\infty);U] \\ \text{(ii)} \quad z^0 \in C\big[(0,\infty);\, \mathcal{D}(A^{*\gamma})'\big] \end{cases} \tag{32}$$

Moreover, it was shown in (Lasiecka *et al.*, 1994) that the optimal solution u^0 can be synthesized "on line" via a state feedback operator. Precise formulation of this result is given below.

Theorem 3. (Lasiecka *et al.*, 1994) *Assume (H-1)–(H-3). Then*

(i) *(Existence of the solution to the Riccati Equation) There exists a positive, self-adjoint operator $P \in \mathcal{L}(H)$ which satisfies the following Riccati Equation:*

$$\begin{aligned} &-(Ax, Py)_H - (Px, Ay)_H + (R^*Rx, y)_H \\ &\quad = (\alpha B^*R^*Rx + (B^* + \alpha B^*A^*)Px, T_0^{-1}(\alpha B^*R^*Ry + (B^* + \alpha B^*A^*)Py))_U \\ &\qquad\qquad \forall x, y \in \mathcal{D}(A^{*\gamma_1})' \end{aligned} \tag{33}$$

where $T_0^{-1} = (I + \alpha^2 B^*R^*RB)^{-1} \in \mathcal{L}(U)$

(ii) *(Regularity of P) The operator $P \in \mathcal{L}(H)$ satisfies the following regularity property:*

$$A^{*1+\gamma}PA^{\gamma} \in \mathcal{L}(H), \quad \gamma < \gamma_1 \tag{34}$$

hence

$$B^*PA^{\gamma} \in \mathcal{L}(H;U) \tag{35}$$

(iii) *(Uniqueness of (33)) The solution P to (33) is unique within the class of self-adjoint positive operators in $\mathcal{L}(H)$ subject to the regularity property in (34).*

(iv) *(Synthesis of the optimal control) We have*

$$[I - \alpha(B^* + \alpha B^*A^*)PB]^{-1} \in \mathcal{L}(U) \tag{36}$$

Moreover, for each $z(0) \in H$, $t \geq 0$,

(v) $u^0(t) = [I - \alpha(B^* + \alpha B^*A^*)PB]^{-1}[\alpha B^*R^*R + (B^* + \alpha B^*A^*)P]z^0(t) \qquad (37)$

(vi) $$\min_{u \in L_2((0,\infty);U)} J(z,u) = (P(z_0 - \alpha Bu(0)), z_0 - \alpha Bu(0))_H \tag{38}$$

■

4. Proof of Theorem 1 and 2

We shall apply abstract result of Theorem 3 to the problem in hand. To this end, we need to verify assumptions (H-1)–(H-3).

4.1. Proof of Theorem 1 (Case γ=0)

Assumption (H-1). Since $\mathcal{A}^2$ is a positive, self–adjoint operator, analyticity and stability of the operator A defined by (26) follows from Chen and Triggiani (1990) (see also Chen and Russell (1982) for related results).

Assumption (H-2). We shall show that (H-2) is satisfied with $\gamma_0 > \frac{3}{8}$, i.e.,

$$A^{-\gamma_0}B : U \to H \quad \text{is compact for} \quad \gamma_0 > \frac{3}{8} \tag{39}$$

where A, B are defined by (27), (28) (without loss of generality we take $\alpha = 1$). Since

$$-A^{-1}B = \begin{bmatrix} -I & \mathcal{A}^{-2} \\ I & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \mathcal{A}D \end{bmatrix} = \begin{bmatrix} \mathcal{A}^{-1}D \\ 0 \end{bmatrix} \tag{40}$$

(39) is equivalent showing that

$$A^{1-\gamma_0} \begin{bmatrix} \mathcal{A}^{-1}D \\ 0 \end{bmatrix} : L_2(\Gamma) \to H = L_2(\Omega) \times L_2(\Omega) \quad \text{is compact} \tag{41}$$

By using compactness of $\mathcal{A}^{\rho}D : L_2(\Gamma) \to L_2(\Omega)$ for $\rho < \frac{1}{4}$ (see Lions and Magenes (1971) or Grisvard (1985)), (41) will follow from the boundedness of

$$A^{1-\gamma_0} \begin{bmatrix} \mathcal{A}^{-1-\rho} \\ 0 \end{bmatrix} : L_2(\Omega) \to H \quad \text{is bounded for} \quad \gamma_0 > \frac{3}{8}, \quad \rho < \frac{1}{4} \tag{42}$$

This boundedness is established by the claim that

$$\mathcal{D}_A(q,2) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in H;\ x + y \in \mathcal{D}(\mathcal{A}^{2q}) \right\} \tag{43}$$

where (see Triebel, 1978)

$$\mathcal{D}_A(q,2) \equiv \left\{ x \in H;\ \int_1^{\infty} \frac{|t^q AR(t,A)x|_H^2}{t} dt < \infty \right\} \tag{44}$$

The explicit computations for the resolvent yield

$$R(\lambda, A) = \frac{1}{\lambda+1} \begin{bmatrix} (\lambda I + \mathcal{A}^2) R\left(\frac{\lambda^2}{\lambda+1}, -\mathcal{A}^2\right) & R\left(\frac{\lambda^2}{\lambda+1}, -\mathcal{A}^2\right) \\ -\mathcal{A}^2 R\left(\frac{\lambda^2}{\lambda+1}, -\mathcal{A}^2\right) & \lambda R\left(\frac{\lambda^2}{\lambda+1}, -\mathcal{A}^2\right) \end{bmatrix}$$

hence

$$AR(\lambda, A) \begin{pmatrix} x \\ y \end{pmatrix} = \frac{-1}{\lambda+1} \begin{bmatrix} \mathcal{A}^2 R\left(\frac{\lambda^2}{\lambda+1}, -\mathcal{A}^2\right) x - \lambda R\left(\frac{\lambda^2}{\lambda+1}, -\mathcal{A}^2\right) y \\ \lambda \mathcal{A}^2 R\left(\frac{\lambda^2}{\lambda+1}, -\mathcal{A}^2\right)(x+y) + \mathcal{A}^2 R\left(\frac{\lambda^2}{\lambda+1}, -\mathcal{A}^2\right) y \end{bmatrix} \tag{45}$$

It is clear that the "dominant" term in (45) is

$$\frac{\lambda}{\lambda+1}\mathcal{A}^2 R\left(\frac{\lambda^2}{\lambda+1}, -\mathcal{A}^2\right)(x+y); \quad x, y \in L_2(\Omega)$$

Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{D}_A(q,2) \Leftrightarrow \int_1^\infty \frac{|t^q A^2 R\left(\frac{t^2}{t+1}, -\mathcal{A}^2\right)(x+y)|_{L_2}^2}{t}\,dt < \infty$$

By (44), this is equivalent to

$$x + y \in \mathcal{D}_{\mathcal{A}^2}(q,2) = \mathcal{D}(\mathcal{A}^{2q})$$

as desired for (43).

Applying (43) to (42) and recalling (see Triebel, 1978)

$$\mathcal{D}_A(q+\epsilon, 2) \subset \mathcal{D}(A^{\tilde{q}}) \subset \mathcal{D}_A(q,2) \quad \text{for} \quad \tilde{q} \in (q, q+\epsilon) \tag{46}$$

we infer that

$$\begin{pmatrix} \mathcal{A}^{-1-\rho} w \\ 0 \end{pmatrix} \in \mathcal{D}(A^{1-\gamma_0}) \text{ for } w \in L_2(\Omega),\ \gamma_0 > \frac{3}{8},\ \rho < \frac{1}{4} \Leftrightarrow w \in \mathcal{D}(\mathcal{A}^{1-2\gamma_0-\rho})$$

which is satisfied as long as $1 - 2\gamma_0 - \rho \le 0$ or $\gamma_0 \ge 1 - \rho > \frac{3}{4}$, which holds with $\gamma_0 > \frac{3}{8}$. This completes the proof of (42), hence of (39), as required for (H-2).

Assumption (H-3). From (27) and (29) we have

$$RA = \begin{pmatrix} 0 & -\beta_1 I \\ 0 & 0 \end{pmatrix} \tag{47}$$

hence clearly $RA \in \mathcal{L}(H)$. Assumption (H-3) holds with $\gamma_1 = 1$.

Since all the assumptions (H-1)–(H-3) are verified, we are in a position to apply conclusions of Theorem 3. To this end we note that an application of Green's formula gives

$$B^* v = \frac{\partial}{\partial \nu} v_2|_\Gamma, \quad v = (v_1, v_2) \tag{48a}$$

Moreover, it can be easily verified that

$$A^* = \begin{bmatrix} 0 & \mathcal{A}^2 \\ -I & \alpha\mathcal{A}^2 \end{bmatrix} \tag{48b}$$

$$B^* R^* R = 0 \tag{48c}$$

Performing now rather straightforward computations and taking into account the above relations (48), we obtain the result of Theorem 1. ■

4.2. Proof of Theorem 2 (Case $\gamma>0$)

It is convenient to use the following topology on $H_0^1(\Omega)$

$$(x,v)_{H_0^1(\Omega)} \equiv \int_\Omega xv \, d\Omega + \gamma \int_\Omega \nabla x \nabla v \, d\Omega = ((I+\gamma\mathcal{A})x, v)_\Omega \tag{49}$$

Assumption (H-1). With respect to the norm induced by (49), $\mathcal{A}_\gamma$ is self-adjoint on $H_0^1(\Omega)$. Therefore, the argument of (Chen and Triggiani, 1990) applies to assert the analyticity of the generator A defined by (20).

Assumption (H-2) is satisfied with the value of $\gamma_0 > \frac{1}{4}$. To prove this, it suffices to show that

$$A^{-\gamma_0}B : L_2(\Gamma) \to H_0^1(\Omega) \times H_0^1(\Omega) \quad \text{is compact for} \quad \gamma_0 > \frac{1}{4} \tag{50}$$

with A and B given by (20), (22), respectively.

As before, we compute

$$A^{-1}B = \begin{bmatrix} \alpha I & -\mathcal{A}_\gamma^{-1} \\ -I & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -(I+\gamma\mathcal{A})^{-1}\mathcal{A}D \end{bmatrix} = \begin{bmatrix} \mathcal{A}^{-1}D \\ 0 \end{bmatrix} \tag{51}$$

By virtue of (51), (50) is equivalent showing that the operator

$$A^{1-\gamma_0} \begin{bmatrix} \mathcal{A}^{-1}D \\ 0 \end{bmatrix} : L_2(\Gamma) \to H_0^1(\Omega) \times H_0^1(\Omega) \tag{52}$$

is compact for $\gamma_0 > \frac{1}{4}$.

Since $\mathcal{A}^\rho D : L_2(\Gamma) \to L_2(\Omega)$ for $\rho < \frac{1}{4}$ is compact (see Lions and Magenes, 1971), it suffices to show that

$$A^{1-\gamma_0} \begin{bmatrix} \mathcal{A}^{-1-\rho} \\ 0 \end{bmatrix} : L_2(\Omega) \to H_0^1(\Omega) \times H_0^1(\Omega) \tag{53}$$

is bounded for $\gamma_0 > \frac{1}{4}$ and $0 \le \rho < \frac{1}{4}$.

By the same arguments as those used to prove (43) (replacing $\mathcal{A}^2$ by $\mathcal{A}_\gamma$) we obtain the following characterization of $\mathcal{D}_A(q,2)$

$$\mathcal{D}_A(q,2) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in H_0^1(\Omega) \times H_0^1(\Omega);\ \mathcal{A}_\gamma^q(x+y) \in H_0^1(\Omega) \right\} \tag{54}$$

From (54) and (46) we infer that

$$\begin{pmatrix} \mathcal{A}^{-1-\rho}x \\ 0 \end{pmatrix} \in \mathcal{D}(A^{1-\gamma_0}) \iff \mathcal{A}_\gamma^{1-\gamma_0}\mathcal{A}^{-1-\rho}x \in H_0^1(\Omega) \tag{55}$$

Since $\mathcal{A}_\gamma \mathcal{A}^{-1}$ is an isomorphism on $H_0^1(\Omega)$, (55) is satisfied provided

$$\mathcal{A}^{-\gamma_0-\rho}x \in H_0^1(\Omega) \quad \text{for} \quad x \in L_2(\Omega) \tag{56}$$

But this is true for $\gamma_0 + \rho \geq \frac{1}{2}$ iff $\gamma_0 > \frac{1}{4}$, as desired.

Hypothesis (H-3). From (25) and (20)

$$RA = \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix}$$

Since $C \in \mathcal{L}(H_0^1(\Omega); L_2(\Omega))$, $RA \in \mathcal{L}(H, Z)$ and (H-3) is satisfied with $\gamma_1 = 1$.

We have verified hypotheses (H-1)–(H-3), hence the conclusion of Theorem 3 is applicable to our case. Calculating the adjoints of the operators A and B we obtain

$$A^* = \begin{bmatrix} 0 & \mathcal{A}_\gamma \\ -I & \alpha\mathcal{A}_\gamma \end{bmatrix}$$

$$B^* v = \frac{\partial}{\partial\nu} v_2, \quad v = (v_1, v_2)$$

$$B^* R^* R = 0$$

Introducing the change of variables

$$\overline{p_2}x = \mathcal{A}_\gamma p_2 x = (1 + \gamma\mathcal{A})^{-1}\mathcal{A}^2 p_2 x$$

and specializing the result of Theorem 3 to our problem yields the assertion of Theorem 2. ■

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