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GENERAL RESPONSE FORMULA FOR 2–D BILINEAR SYSTEMS

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Two new models of 2–D bilinear systems are introduced. The general response formulae for the models are derived.

1. Introduction

Recently the classical 2–D state–space Roesser model (Roesser, 1975), the Fornasini–Marchesini models (Fornasini and Marchesini, 1976; 1978) and the Kurek model (Kurek, 1985) have been extended to the singular case (Kaczorek, 1988; 1992; Lewis, 1992). In this paper two models of 2–D bilinear systems will be introduced. They can be considered as an extension for bilinear systems of the models introduced by Fornasini and Marchesini for 2–D linear case. The general response formula for the models will be derived.

2. Models of 2–D Bilinear Systems and Problem Formulation

Consider the following two models of 2–D bilinear systems

$$x_{i+1,j+1} = A_1 x_{i+1,j} + A_2 x_{i,j+1} + \sum_{k=1}^{m} u_{ij}^k B_k x_{ij} + C u_{ij} \tag{1}$$

$$x_{i+1,j+1} = A_1 x_{i+1,j} + A_2 x_{i,j+1} + \sum_{k=1}^{n} x_{ij}^k B'_k u_{ij} + C u_{ij} \tag{2}$$

where $i, j \in Z$; Z is the set of nonnegative integers; $x_{ij} = [x_{ij}^1, x_{ij}^2, \ldots, x_{ij}^n]^T$ – the local state vector at the point (i,j); T denotes the transposition; $u_{ij} = [u_{ij}^1, u_{ij}^2, \ldots, u_{ij}^m]^T$ – the input vector; A_1, A_2, B_k are $n \times n$ real matrices and B'_k is an $n \times m$ real matrix.

The output equation of the models is of the form

$$y_{ij} = D x_{ij} + E u_{ij} \tag{3}$$

where y_{ij} is a p–dimensional output vector and D, E are real matrices of appropriate dimensions.
Boundary conditions for (1) and (2) are given by

$$x_{i0} \quad \text{for} \quad i \in Z \qquad \text{and} \qquad x_{0j} \quad \text{for} \quad j \in Z \tag{4}$$

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Define

$$\overline{u}_{ij} := u_{ij} \otimes I_n = \begin{bmatrix} u_{ij}^1 I_n \\ u_{ij}^2 I_n \\ \vdots \\ u_{ij}^m I_n \end{bmatrix}, \quad \overline{x}_{ij} := x_{ij}^T \otimes I_n = [x_{ij}^1 I_n, x_{ij}^2 I_n, \dots, x_{ij}^n I_n]$$

and

$$\overline{B} = [B_1, B_2, \dots, B_m], \qquad \overline{B}' = \begin{bmatrix} B_1' \\ B_2' \\ \vdots \\ B_n' \end{bmatrix}$$

where I_n is the $n \times n$ identity matrix and $\otimes$ denotes the Kronecker product. Using the above notation we may write (1) and (2) in the form

$$x_{i+1,j+1} = A_1 x_{i+1,j} + A_2 x_{i,j+1} + \overline{B}\overline{u}_{ij} x_{ij} + C u_{ij} \tag{5}$$

and

$$x_{i+1,j+1} = A_1 x_{i+1,j} + A_2 x_{i,j+1} + \overline{x}_{ij} \overline{B}' u_{ij} + C u_{ij} \tag{6}$$

The problem can be stated as follows. Given the matrices of (1) ((2)) and (3), input sequence u_{ij} and the boundary conditions (4), find solutions to (1) ((2)) and the general response formula.

3. Solutions to the Models and General Response Formula

In this section solutions to models (1) and (2) with boundary conditions (4) will be derived.

Theorem 1. *The solution x_{ij} to (1) with boundary conditions (4) is given by*

$$x_{ij} = \sum_{k=1}^{i} T_{i-k,j-1} A_1 x_{k0} + \sum_{l=1}^{j} T_{i-1,j-l} A_2 x_{0l} + \sum_{k_1=0}^{i-1} \sum_{l_1=0}^{j-1} T_{i-k_1-1,j-l_1-1} \overline{B}\overline{u}_{k_1 l_1} M_{k_1 l_1}$$

$$+ \Big(\sum_{k_2=1}^{i-1} \sum_{l_2=1}^{j-1} T_{i-k_2-1,j-l_2-1} \overline{B}\overline{u}_{k_2 l_2} \Big) \Big(\sum_{k_1=0}^{k_2-1} \sum_{l_1=0}^{l_2-1} T_{k_2-k_1-1,l_2-l_1-1} \overline{B}\overline{u}_{k_1 l_1} M_{k_1 l_1} \Big)$$

$$+ \dots + \Big(\sum_{k_r=r-1}^{i-1} \sum_{l_r=r-1}^{j-1} T_{i-k_r-1,j-l_r-1} \overline{B}\overline{u}_{k_r l_r} \Big)$$

$$\times \Big(\sum_{k_{r-1}=r-2}^{k_r-1} \sum_{l_{r-1}=r-2}^{l_r-1} T_{k_r-k_{r-1}-1,l_r-l_{r-1}-1} \overline{B}\overline{u}_{k_{r-1} l_{r-1}} \Big)$$

$$\times \dots \times \Big(\sum_{k_1=0}^{k_2-1}\sum_{l_1=0}^{l_2-1} T_{k_2-k_1-1,l_2-l_1-1}\overline{B}\overline{u}_{k_1l_1}M_{k_1l_1}\Big)$$

$$+\sum_{k_1=0}^{i-1}\sum_{l_1=0}^{j-1} T_{i-k_1-1,j-l_1-1}Cu_{k_1l_1} + \Big(\sum_{k_2=1}^{i-1}\sum_{l_2=1}^{j-1} T_{i-k_2-1,j-l_2-1}\overline{B}\overline{u}_{k_2l_2}\Big)$$

$$\times\Big(\sum_{k_1=0}^{k_2-1}\sum_{l_1=0}^{l_2-1} T_{k_2-k_1-1,l_2-l_1-1}Cu_{k_1l_1}\Big) + \dots + \Big(\sum_{k_r=r-1}^{i-1}\sum_{l_r=r-1}^{j-1} T_{i-k_r-1,j-l_r-1}\overline{B}\overline{u}_{k_rl_r}\Big)$$

$$\times\Big(\sum_{k_{r-1}=r-2}^{k_r-1}\sum_{l_{r-1}=r-2}^{l_r-1} T_{k_r-k_{r-1}-1,l_r-l_{r-1}-1}\overline{B}\overline{u}_{k_{r-1}l_{r-1}}\Big)$$

$$\times \dots \times \Big(\sum_{k_1=0}^{k_2-1}\sum_{l_1=0}^{l_2-1} T_{k_2-k_1-1,l_2-l_1-1}Cu_{k_1l_1}\Big) \qquad \Big(r \le \max(i,j)\Big) \tag{7}$$

where T_{pq} *is the transition matrix defined as follows*

$$\begin{aligned} &T_{00} := I_n \\ &T_{pq} := T_{p,q-1}A_1 + T_{p-1,q}A_2 = A_1T_{p,q-1} + A_2T_{p-1,q} \quad \textit{for } i,j \in Z \\ &T_{pq} := 0 \qquad \textit{(the zero matrix)} \quad \textit{for } i < 0 \quad \textit{and/or } j < 0 \end{aligned} \tag{8}$$

and

$$M_{ij} := \begin{cases} \sum_{k=1}^{i} T_{i-k,j-1}A_1x_{k0} + \sum_{l=1}^{j} T_{i-1,j-l}A_2x_{0l} & \textit{for } i>0 \textit{ and } j>0 \\ x_{ij} & \textit{for } i=0 \textit{ and/or } j=0 \end{cases} \tag{9}$$

Proof. The proof will be accomplished by induction on the pair (i,j). From (7) we have

$$x_{11} = A_1x_{10} + A_2x_{01} + \overline{B}\overline{u}_{00}x_{00} + Cu_{00} \qquad \text{for } i=j=1$$

$$\begin{aligned} x_{21} &= T_{10}A_1x_{10} + A_1x_{20} + T_{10}A_2x_{01} \\ &\quad + T_{10}\overline{B}\overline{u}_{00}x_{00} + \overline{B}\overline{u}_{10}x_{10} + T_{10}Cu_{00} + Cu_{10} \qquad \text{for } i=2,\ j=1 \end{aligned}$$

and

$$\begin{aligned} x_{12} &= T_{01}A_1x_{10} + T_{01}A_2x_{01} + A_2x_{02} \\ &\quad + T_{01}\overline{B}\overline{u}_{00}x_{00} + \overline{B}\overline{u}_{01}x_{01} + T_{01}Cu_{00} + Cu_{01} \qquad \text{for } i=1,\ j=2 \end{aligned}$$

The same results follow from (5) for $i=j=0$; $i=1,\ j=0$; and $i=0,\ j=1$. Therefore, the hypothesis is true for $i=j=1$; $i=2,\ j=1$ and $i=1,\ j=2$.

Assuming that the hypothesis holds for the pairs $(i,j),(i+1,j)$ and $(i,j+1)$, $i>0,\ j>0$, we shall show that it is also valid for the pair $(i+1,j+1)$.
Using (5), (7) and (8) we may write

$$
\begin{aligned}
x_{i+1,j+1} &= A_1 x_{i+1,j} + A_2 x_{i,j+1} + \overline{B}\overline{u}_{ij} x_{ij} + C u_{ij} \\
&= A_1 \Bigg[\sum_{k=1}^{i+1} T_{i-k+1,j-1} A_1 x_{k0} + \sum_{l=1}^{j} T_{i,j-l} A_2 x_{0l} + \sum_{k_1=0}^{i} \sum_{l_1=0}^{j-1} T_{i-k_1,j-l_1-1} \overline{B}\overline{u}_{k_1 l_1} M_{k_1 l_1} \\
&+ \Big(\sum_{k_2=1}^{i} \sum_{l_2=1}^{j-1} T_{i-k_2,j-l_2-1} \overline{B}\overline{u}_{k_2 l_2} \Big) \Big(\sum_{k_1=0}^{k_2-1} \sum_{l_1=0}^{l_2-1} T_{k_2-k_1-1,l_2-l_1-1} \overline{B}\overline{u}_{k_1 l_1} M_{k_1 l_1} \Big) \\
&+ \ldots + \Big(\sum_{k_r=r-1}^{i} \sum_{l_r=r-1}^{j-1} T_{i-k_r,j-l_r-1} \overline{B}\overline{u}_{k_r l_r} \Big) \\
&\times \Big(\sum_{k_{r-1}=r-2}^{k_r-1} \sum_{l_{r-1}=r-2}^{l_r-1} T_{k_r-k_{r-1}-1,l_r-l_{r-1}-1} \overline{B}\overline{u}_{k_{r-1} l_{r-1}} \Big) \\
&\times \ldots \times \Big(\sum_{k_1=0}^{k_2-1} \sum_{l_1=0}^{l_2-1} T_{k_2-k_1-1,l_2-l_1-1} \overline{B}\overline{u}_{k_1 l_1} M_{k_1 l_1} \Big) + \sum_{k_1=0}^{i} \sum_{l_1=0}^{j-1} T_{i-k_1,j-l_1-1} C u_{k_1 l_1} \\
&+ \Big(\sum_{k_2=1}^{i} \sum_{l_2=1}^{j-1} T_{i-k_2,j-l_2-1} \overline{B}\overline{u}_{k_2 l_2} \Big) \Big(\sum_{k_1=0}^{k_2-1} \sum_{l_1=0}^{l_2-1} T_{k_2-k_1-1,l_2-l_1-1} C u_{k_1 l_1} \Big) \\
&+ \ldots + \Big(\sum_{k_r=r-1}^{i} \sum_{l_r=r-1}^{j-1} T_{i-k_r,j-l_r-1} \overline{B}\overline{u}_{k_r l_r} \Big) \\
&\times \Big(\sum_{k_{r-1}=r-2}^{k_r-1} \sum_{l_{r-1}=r-2}^{l_r-1} T_{k_r-k_{r-1}-1,l_r-l_{r-1}-1} \overline{B}\overline{u}_{k_{r-1} l_{r-1}} \Big) \\
&\times \ldots \times \Big(\sum_{k_1=0}^{k_2-1} \sum_{l_1=0}^{l_2-1} T_{k_2-k_1-1,l_2-l_1-1} C u_{k_1 l_1} \Big) \Bigg] \\
&+ A_2 \Bigg[\sum_{k=1}^{i} T_{i-k,j} A_1 x_{k0} + \sum_{l=1}^{j+1} T_{i-1,j-l+1} A_2 x_{0l} \sum_{k_1=0}^{i-1} \sum_{l_1=0}^{j} T_{i-k_1-1,j-l_1} \overline{B}\overline{u}_{k_1 l_1} M_{k_1 l_1} \\
&+ \Big(\sum_{k_2=1}^{i-1} \sum_{l_2=1}^{j} T_{i-k_2-1,j-l_2} \overline{B}\overline{u}_{k_2 l_2} \Big) \Big(\sum_{k_1=0}^{k_2-1} \sum_{l_1=0}^{l_2-1} T_{k_2-k_1-1,l_2-l_1-1} \overline{B}\overline{u}_{k_1 l_1} M_{k_1 l_1} \Big)
\end{aligned}
$$

$$+\dots+\Big(\sum_{k_r=r-1}^{i-1}\sum_{l_r=r-1}^{j}T_{i-k_r-1,j-l_r}\overline{B}\overline{u}_{k_rl_r}\Big)$$

$$\times\Big(\sum_{k_{r-1}=r-2}^{k_r-1}\sum_{l_{r-1}=r-2}^{l_r-1}T_{k_r-k_{r-1}-1,l_r-l_{r-1}-1}\overline{B}\overline{u}_{k_{r-1}l_{r-1}}\Big)$$

$$\times\dots\times\Big(\sum_{k_1=0}^{k_2-1}\sum_{l_1=0}^{l_2-1}T_{k_2-k_1-1,l_2-l_1-1}\overline{B}\overline{u}_{k_1l_1}M_{k_1l_1}\Big)+\sum_{k_1=0}^{i-1}\sum_{l_1=0}^{j}T_{i-k_1-1,j-l_1}Cu_{k_1l_1}$$

$$+\Big(\sum_{k_2=1}^{i-1}\sum_{l_2=1}^{j}T_{i-k_2-1,j-l_2}\overline{B}\overline{u}_{k_2l_2}\Big)\Big(\sum_{k_1=0}^{k_2-1}\sum_{l_1=0}^{l_2-1}T_{k_2-k_1-1,l_2-l_1-1}Cu_{k_1l_1}\Big)$$

$$+\dots+\Big(\sum_{k_r=r-1}^{i-1}\sum_{l_r=r-1}^{j}T_{i-k_r-1,j-l_r}\overline{B}\overline{u}_{k_rl_r}\Big)$$

$$\times\Big(\sum_{k_{r-1}=r-2}^{k_r-1}\sum_{l_{r-1}=r-2}^{l_r-1}T_{k_r-k_{r-1}-1,l_r-l_{r-1}-1}\overline{B}\overline{u}_{k_{r-1}l_{r-1}}\Big)$$

$$\times\dots\times\Big(\sum_{k_1=0}^{k_2-1}\sum_{l_1=0}^{l_2-1}T_{k_2-k_1-1,l_2-l_1-1}Cu_{k_1l_1}\Big)\Bigg]+\overline{B}\overline{u}_{ij}\Bigg[\sum_{k=1}^{i}T_{i-k,j-1}A_1x_{k0}$$

$$+\sum_{l=1}^{j}T_{i-1,j-l}A_2x_{0l}+\sum_{k_1=0}^{i-1}\sum_{l_1=0}^{j-1}T_{i-k_1-1,j-l_1-1}\overline{B}\overline{u}_{k_1l_1}M_{k_1l_1}$$

$$+\Big(\sum_{k_2=1}^{i-1}\sum_{l_2=1}^{j-1}T_{i-k_2-1,j-l_2-1}\overline{B}\overline{u}_{k_2l_2}\Big)\Big(\sum_{k_1=0}^{k_2-1}\sum_{l_1=0}^{l_2-1}T_{k_2-k_1-1,l_2-l_1-1}\overline{B}\overline{u}_{k_1l_1}M_{k_1l_1}\Big)$$

$$+\dots+\Big(\sum_{k_r=r-1}^{i-1}\sum_{l_r=r-1}^{j-1}T_{i-k_r-1,j-l_r-1}\overline{B}\overline{u}_{k_rl_r}\Big)$$

$$\times\Big(\sum_{k_{r-1}=r-2}^{k_r-1}\sum_{l_{r-1}=r-2}^{l_r-1}T_{k_r-k_{r-1}-1,l_r-l_{r-1}-1}\overline{B}\overline{u}_{k_{r-1}l_{r-1}}\Big)$$

$$\times\dots\times\Big(\sum_{k_1=0}^{k_2-1}\sum_{l_1=0}^{l_2-1}T_{k_2-k_1-1,l_2-l_1-1}\overline{B}\overline{u}_{k_1l_1}M_{k_1l_1}\Big)+\sum_{k_1=0}^{i-1}\sum_{l_1=0}^{j-1}T_{i-k_1-1,j-l_1-1}Cu_{k_1l_1}$$

$$+\Big(\sum_{k_2=1}^{i-1}\sum_{l_2=1}^{j-1}T_{i-k_2-1,j-l_2-1}\overline{B}\overline{u}_{k_2l_2}\Big)\Big(\sum_{k_1=0}^{k_2-1}\sum_{l_1=0}^{l_2-1}T_{k_2-k_1-1,l_2-l_1-1}Cu_{k_1l_1}\Big)$$

$$+\ldots+\Big(\sum_{k_r=r-1}^{i-1}\sum_{l_r=r-1}^{j-1}T_{i-k_r-1,j-l_r-1}\overline{B}\overline{u}_{k_r l_r}\Big)$$

$$\times\Big(\sum_{k_{r-1}=r-2}^{k_r-1}\sum_{l_{r-1}=r-2}^{l_r-1}T_{k_r-k_{r-1}-1,l_r-l_{r-1}-1}\overline{B}\overline{u}_{k_{r-1}l_{r-1}}\Big)$$

$$\times\ldots\times\Big(\sum_{k_1=0}^{k_2-1}\sum_{l_1=0}^{l_2-1}T_{k_2-k_1-1,l_2-l_1-1}Cu_{k_1l_1}\Big)\Big]+Cu_{ij}$$

$$=\sum_{k=1}^{i+1}T_{i-k+1,j}A_1x_{k0}+\sum_{l=1}^{j+1}T_{i,j-l+1}A_2x_{0l}+\sum_{k_1=0}^{i}\sum_{l_1=0}^{j}T_{i-k_1,j-l_1}\overline{B}\overline{u}_{k_1l_1}M_{k_1l_1}$$

$$+\Big(\sum_{k_2=1}^{i}\sum_{l_2=1}^{j}T_{i-k_2,j-l_2}\overline{B}\overline{u}_{k_2l_2}\Big)\Big(\sum_{k_1=0}^{k_2-1}\sum_{l_1=0}^{l_2-1}T_{k_2-k_1-1,l_2-l_1-1}\overline{B}\overline{u}_{k_1l_1}M_{k_1l_1}\Big)$$

$$+\ldots+\Big(\sum_{k_{r+1}=r}^{i}\sum_{l_{r+1}=r}^{j}T_{i-k_{r+1},j-l_{r+1}}\overline{B}\overline{u}_{k_{r+1}l_{r+1}}\Big)$$

$$\times\Big(\sum_{k_r=r-1}^{k_{r+1}-1}\sum_{l_r=r-1}^{l_{r+1}-1}T_{k_{r+1}-k_r-1,l_{r+1}-l_r-1}\overline{B}\overline{u}_{k_rl_r}\Big)$$

$$\times\ldots\times\Big(\sum_{k_1=0}^{k_2-1}\sum_{l_1=0}^{l_2-1}T_{k_2-k_1-1,l_2-l_1-1}\overline{B}\overline{u}_{k_1l_1}M_{k_1l_1}\Big)+\sum_{k_1=0}^{i}\sum_{l_1=0}^{j}T_{i-k_1,j-l_1}Cu_{k_1l_1}$$

$$+\Big(\sum_{k_2=1}^{i}\sum_{l_2=1}^{j}T_{i-k_2,j-l_2}\overline{B}\overline{u}_{k_2l_2}\Big)\Big(\sum_{k_1=0}^{k_2-1}\sum_{l_1=o}^{l_2-1}T_{k_2-k_1-1,l_2-l_1-1}Cu_{k_1l_1}\Big)$$

$$+\ldots+\Big(\sum_{k_{r+1}=r}^{i}\sum_{l_{r+1}=r}^{j}T_{i-k_{r+1},j-l_{r+1}}\overline{B}\overline{u}_{k_{r+1}l_{r+1}}\Big)$$

$$\times\ldots\times\Big(\sum_{k_r=r-1}^{k_{r+1}-1}\sum_{l_r=r-1}^{l_{r+1}-1}T_{k_{r+1}-k_r-1,l_{r+1}-l_r-1}\overline{B}\overline{u}_{k_rl_r}\Big)$$

$$\times\ldots\times\Big(\sum_{k_1=0}^{k_2-1}\sum_{l_1=0}^{l_2-1}T_{k_2-k_1-1,l_2-l_1-1}Cu_{k_1l_1}\Big)$$

Therefore, the hypothesis is valid for the pair $(i+1,j+1)$. ■

In a similar way the following dual theorem can be proved.

Theorem 2. *The solution x_{ij} to (2) with boundary conditions (4) is given by*

$$x_{ij} = \sum_{k=1}^{i} T_{i-k,j-1} A_1 x_{k0} + \sum_{l=1}^{j} T_{i-1,j-l} A_2 x_{0l}$$

$$+ \sum_{k_1=0}^{i-1} \sum_{l_1=0}^{j-1} T_{i-k_1-1,j-l_1-1} \left(M_{k_1 l_1}^T \otimes I_n \right) \overline{B}' u_{k_1 l_1} + \sum_{k_2=1}^{i-1} \sum_{l_2=1}^{j-1} T_{i-k_2-1,j-l_2-1}$$

$$\times \left\{ \left(\sum_{k_1=0}^{k_2-1} \sum_{l_1=0}^{l_2-1} T_{k_2-k_1-1,l_2-l_1-1} \left[M_{k_1 l_1}^T \otimes I_n \right] \overline{B}' u_{k_1 l_1} \right)^T \otimes I_n \right\} \overline{B}' u_{k_2 l_2}$$

$$+ \ldots + \sum_{k_r=r-1}^{i-1} \sum_{l_r=r-1}^{j-1} T_{i-k_r-1,j-l_r-1} \left\{ \left[\sum_{k_{r-1}=r-2}^{k_r-1} \sum_{l_{r-1}=r-2}^{l_r-1} T_{k_r-k_{r-1}-1,l_r-l_{r-1}-1} \right. \right.$$

$$\left\{ \left[\ldots \left(\sum_{k_1=0}^{k_2-1} \sum_{l_1=0}^{l_2-1} T_{k_2-k_1-1,l_2-l_1-1} \left[M_{k_1 l_1}^T \otimes I_n \right] \overline{B}' u_{k_1 l_1} \right)^T \otimes I_n \ldots \right]^T \otimes I_n \right\}$$

$$\times \left. \left. \overline{B}' u_{k_{r-1} l_{r-1}} \right]^T \otimes I_n \right\} \overline{B}' u_{k_r l_r} + \sum_{k_1=0}^{i-1} \sum_{l_1=0}^{j-1} T_{i-k_1-1,j-l_1-1} C u_{k_1 l_1}$$

$$+ \sum_{k_2=1}^{i-1} \sum_{l_2=1}^{j-1} T_{i-k_2-1,j-l_2-1} \left\{ \left(\sum_{k_1=0}^{k_2-1} \sum_{l_1=0}^{l_2-1} T_{k_2-k_1-1,l_2-l_1-1} C u_{k_1 l_1} \right)^T \otimes I_n \right\} \overline{B}' u_{k_2 l_2}$$

$$+ \ldots + \sum_{k_r=r-1}^{i-1} \sum_{l_r=r-1}^{j-1} T_{i-k_r-1,j-l_r-1} \left\{ \left[\sum_{k_{r-1}=r-2}^{k_r-1} \sum_{l_{r-1}=r-2}^{l_r-1} T_{k_r-k_{r-1}-1,l_r-l_{r-1}-1} \right. \right.$$

$$\left\{ \left[\ldots \left(\sum_{k=0}^{k_2-1} \sum_{l_1=0}^{l_2-1} T_{k_2-k_1-1,l_2-l_1-1} C u_{k_1 l_1} \right)^T \otimes I_n \ldots \right]^T \otimes I_n \right\}$$

$$\times \left. \left. \overline{B}' u_{k_{r-1} l_{r-1}} \right]^T \otimes I_n \right\} \overline{B}' u_{k_r l_r} \qquad \left(r = \min(i,j) \right) \tag{10}$$

where T_{pq} and M_{ij} are defined by (8) and (9), respectively

Substitution of (7) and (10) into (3) yields the desired general response formulae.

4. Concluding Remarks

The new models of 2–D bilinear systems have been introduced. The general response formulae for the models have been derived. Employing solutions (7) and (10) to the models, conditions for the local reachability and controllability can be established by an extension for 2–D case of the approach presented in (Klamka, 1991) for 1–D case. The above considerations can be extended for other models of 2–D bilinear systems, for example for the model.

$$x_{i+1,j+1} = A_1 x_{i+1,j} + A_2 x_{i,j+1} + \sum_{k=1}^{n} x_{i+1,j}^k B_{1k} u_{i+1,j} + \sum_{k=1}^{n} x_{i,j+1}^k B_{2k} u_{i,j+1} + C_1 u_{i+1,j} + C_2 u_{i,j+1}$$

with the boundary conditions:

x_{ij} are given for all (i,j) such that $i+j=0$

The same approach can be applied to the 2–D bilinear model of the Roesser type

$$\begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + \sum_{k=1}^{m} u_{ij}^k \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \right) \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} u_{ij}$$

where $i,j \in Z$; $x_{ij}^h \in \mathbb{R}^{n_1}$ is the horizontial state vector, $x_{ij}^v \in \mathbb{R}^{n_1}$ – the vectical state vector, $u_{ij} \in \mathbb{R}^m$ – the input vector, u_{ij}^k – the k–th component of u_{ij} and A_{pq}, B_{pq}, C_p $(p,q=1,2)$ are real matrices of appropriate dimensions.

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