Appl. Math. and Comp. Sci., 1995, vol.5, No.1, 33–48

DETERMINATION OF THE STABILITY OF A NON-LINEAR ORDINARY DIFFERENTIAL EQUATION BY LEAST SQUARE APPROXIMATION. COMPUTATIONAL PROCEDURE

TAYEB BENOUAZ*, OVIDE ARINO*

The purpose of this paper is to present a computational procedure, based on the minimization in the least square sense, which is associated with a given non-linear ordinary differential equation, when considered in the neighborhood of one of its equilibrium points P, a linear ordinary differential equation, having the same stability property as the non-linear equation at P.

1. Introduction

In the study of non-linear ordinary differential equations, linearization methods play an important role. The principal method, when studying the stability of an equilibrium point, is to consider the linear equation, which one gets by differentiating (in the Frechet sense) the nonlinearity at this point. The linearized equation has the same behaviour as the non-linear equation in the hyperbolic case i.e. when the eigenvalues are not purely imaginary. However, there are three setbacks to this method:

1. Impossibility to locate the eigenvalues with respect to the imaginary axis; in particular if one or more eigenvalues are close to this axis.
2. If the equilibrium is a center, i.e. in case there are purely imaginary eigenvalues and/or if 0 is an eigenvalue. The behaviour of the solution of the non-linear equation in the neighborhood of such a point can be anything.
3. If the nonlinearity is not smooth enough in the neighborhood of a stationary point, then, in general, one cannot calculate the Frechet derivative. This can happen, if for instance, the function is only locally Lipschitzian.

These three problems justify the introduction of other techniques, methods or computations to analyze the stability. Among them is the method of optimal linearization, which was introduced by Vujanovic (1973). This method was used recently by Jordan *et al.* (1987a; 1987b).

To the best of our knowledge, however, no theoretical evidence of the validity of the method has been given up to now. It is our intention to make some progress in that direction, in order to, later on, apply the method to some specific problems. In

* Universite de Tlemcen, Institut de sciences exactes, B.P. 119, Tlemcen 13000, Algerie
* Laboratoire de mathématiques appliquées, URA 1204 CNRS, Avenue de l'Université 64000 Pau, France

the present work, we propose a new method, introducing an iterative computation of the optimal linearization. In fact, our method yields many optimal equations. We conjecture, however, that all the optimal equations have the same type of stability, which should be the same as the stability of the non-linear equation. This fact holds in the scalar case. The non-linear vector case is more difficult and will be dealt with in a subsequent work.

2. Presentation of the Method

2.1. Position of the Problem

Consider the following system of non-linear ordinary differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x(t)), \qquad x(0) = x_0 \tag{1}$$

where $x = (x_1, \ldots, x_n)$ is the unknown function, and $F = (f_1, \ldots, f_n)$ is a given function on an open subset Ω of $\mathbb{R}^n$, which is locally Lipschitzian.

Our purpose is to elaborate a method of linearization, which will associate to system (1) a system of the form:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A^* x(t), \qquad x(0) = x_0 \tag{2}$$

where $A^* \in \mathcal{M}_n(\mathbb{R})$ is to be determined. For this, we shall assume:

i) $F(0) = 0$

ii) The spectrum $\sigma(DF(x))$ is contained in the set $\{z : Re z < 0\}$ for every $x \neq 0$, in the neighborhood of 0, for which $DF(x)$ exists.

System (2), corresponding to system (1), will give an optimal approximation with respect to curves, starting at the initial point x_0 and tending to 0 as t goes to infinity.

2.2. Formalism

Consider the functional defined by

$$G(A) = \int_0^{+\infty} \|F(x(t)) - Ax(t)\|^2 \, \mathrm{d}t \tag{3}$$

where F is defined on an open subset Ω of $\mathbb{R}^n$, $A \in \mathcal{M}_n(\mathbb{R})$ is to be determined successively, $x(t)$ represents a function such that 0 suitably fast as $t \to +\infty$. Later on, we shall introduce functions $x(t)$ that are solutions of linear equations.

The minimization of the functional $G(A)$ with respect to A will allow us to get the optimal system (2). We have

$$DG(A)\alpha = 2\int_0^{+\infty} \Big\langle Ax(t) - F(x(t)), \alpha x(t) \Big\rangle \mathrm{d}t = 0 \tag{4}$$

for every matrix α. In particular for matrices α such that $\alpha_{l,m} = 1$, $\alpha_{i,j} = 0$, if $(i,j) \neq (l,m)$, we have

$$\left\langle Ax(t) - F(x(t)), \alpha x(t) \right\rangle = \left[Ax(t) - F(x(t))\right]_l x_m(t) \tag{5}$$

and

$$\int_0^{+\infty} \left[Ax(t) - F(x(t))\right]_l x_m(t)\, \mathrm{d}t = 0,\ \forall 1 \leq l, m \leq n \tag{6}$$

Let $(a_{i,j})$ denote the elements of matrix A. Then (6) yields

$$\sum_{j=1}^{n} a_{l,j} \left(\int_0^{+\infty} x_j(t)\, x_m(t)\, \mathrm{d}t \right)_{1 \leq j,m \leq n} = \left(\int_0^{+\infty} f_l(x(t))\, x_m(t)\, \mathrm{d}t \right)_{1 \leq l,m \leq n} \tag{7}$$

Let

$$\Gamma(x) = \left(\int_0^{+\infty} x_j(t)\, x_m(t)\, \mathrm{d}t \right)_{1 \leq j,m \leq n} = \int_0^{+\infty} \left[x(t)\right] \left[x(t)\right]^T \mathrm{d}t \tag{8}$$

Then,

$$A = \left[\int_0^{+\infty} \left[F(x(t))\right] \left[x(t)\right]^T \mathrm{d}t \right] \left[\Gamma(x)\right]^{-1} \tag{9}$$

We have assumed the matrix Γ to be non-singular. A necessary and sufficient condition for this assumption to be satisfied is that the trajectory corresponding to x does not lie in any strict subspace of $\mathbb{R}^n$. If it is the case that x takes values in a strict subspace of $\mathbb{R}^n$, we can perturb x by a vector of the form

$$\varepsilon \begin{pmatrix} e^{-at} \\ \vdots \\ e^{-nat} \end{pmatrix}, \qquad a > 0,\ \varepsilon > 0 \tag{10}$$

Hence, if Γ is an invertible matrix, the matrix A is uniquely determined.

2.3. Principle of the Method

The above computation will be used inductively. We shall assume that the successive matrices are stable, that is, their spectrum is in $\{z : Re z < 0\}$. Verifying this fact is a delicate problem. We have given conditions that ensure that this property is satisfied. These results will be developed subsequently. The initial matrix is the Jacobian matrix of F at x_0, assuming that x_0 is a point at which $DF(x_0)$ exists.

Consider system (1)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x(t)), \qquad x(0) = x_0$$

Step 1. Compute $A_0 = DF(x_0)$.

Step 2. Compute A_1 from the solution of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = A_0 y(t), \qquad y(0) = x_0 \tag{11}$$

by minimizing the functional

$$G(A) = \int_0^{+\infty} \|F(y(t)) - Ay(t)\|^2 \,\mathrm{d}t \tag{12}$$

A_1 is uniquely determined by formula (9), where x is replaced by y, solution of (11).

Step 3. To compute A_j from A_{j-1}, we first solve

$$\frac{\mathrm{d}y}{\mathrm{d}t} = [A_{j-1}]\, y(t), \qquad y(0) = x_0 \tag{13}$$

Let y_j be the solution of equation (13).
The minimization of the functional

$$G_j(A) = \int_0^{+\infty} \|F(y_j(t)) - Ay_j(t)\|^2 \,\mathrm{d}t \tag{14}$$

yields A_j.
In fact, we have the following relationship between A_j and A_{j-1}.

$$A_j \Gamma(y_j) = \int_0^{+\infty} \Big[F(y_j(t))\Big]\Big[y_j(t)\Big]^T \,\mathrm{d}t \tag{15}$$

Assuming that $\Gamma(y_j)$ invertible, A_j can be written as

$$A_j = \left[\int_0^{+\infty} \Big[F(y_j(t))\Big]\Big[y_j(t)\Big]^T \,\mathrm{d}t\right] [\Gamma(y_j)]^{-1} \tag{16}$$

If the sequence A_j converges, then the limit A^* is by definition the optimal linearization of F at x_0. The optimal matrix depends on x_0. We conjecture that the stability property is independent of the initial point. Note, however, that if F is linear, then the procedure gives F at the first iteration.
Indeed, in this case, formula (9) can be written

$$\left[\int_0^{+\infty} \Big[Fx(t)\Big]\Big[x(t)\Big]^T \mathrm{d}t\right] \left[\int_0^{+\infty} \Big[x(t)\Big]\Big[x(t)\Big]^T \mathrm{d}t\right]^{-1} \tag{17}$$

hence,

$$F\left[\int_0^{+\infty} \Big[x(t)\Big]\Big[x(t)\Big]^T \mathrm{d}t\right] \left[\int_0^{+\infty} \Big[x(t)\Big]\Big[x(t)\Big]^T \mathrm{d}t\right]^{-1} = F \tag{18}$$

Of course this is possible only when: $\Gamma(x) = \left[\int_0^{+\infty} [x(t)]\,[x(t)]^T \,\mathrm{d}t\right]$ is an invertible matrix. This condition is satisfied if x_0 is chosen so that it has a non-zero component on all the eigenspaces of F. Then $x(t)$ should not be contained in any proper subspace of $\mathbb{R}^n$. In this case, Γ is non degenerate. Therefore, the optimal approximation of a linear system is the system itself. In this case, the result is independent of x_0.

3. Applications

3.1. Examples of Non-linear Systems

3.1.1. Case of a System of Two Equations, One Linear, One Non-linear

Consider the following system

$$\left.\begin{aligned} \frac{dx}{dt} &= -7.10^2 x - 2.10^3 x^2 - 2.10^5 y \\ \frac{dy}{dt} &= 2.10^3 x - 2.10^5 y \end{aligned}\right\}, \qquad (x_0, y_0) = (5, 0) \tag{19}$$

with the value of $DF(x, y)$ computed at (x_0, y_0)

$$DF(x_0, y_0) = \begin{bmatrix} -207.10^2 & -2.10^5 \\ 2.10^3 & -2.10^5 \end{bmatrix} \tag{20}$$

After the 4-th iteration, the computational procedure gives

$$A^* = \begin{bmatrix} -13.27\ 10^3 & 4.31\ 10^5 \\ 2.10^3 & -2.10^5 \end{bmatrix} \tag{21}$$

The curves in Figs. 1 and 2 represent the variation of solutions $(x(t), y(t))$ of systems (19), (21) as a function of time. In both figures, curve 1 represents the exact result obtained by solving numerically the non-linear system (19), curve 2 corresponds to the solution given by the optimal linear equation.

Remark. Note that the second equation in (19) is linear, and so is unchanged by the transformation into the optimal system (21). Consider a more general system of non-linear equations with a nonlinearity of the form

$$F(x) = M\, x + \tilde{F}(x) \tag{22}$$

where M is linear.

The computation of matrix A_1 gives

$$A_1 = \left[\int_0^{+\infty} \left[F(x(t))\right]\left[x(t)\right]^T dt\right]\left[\Gamma(x)\right]^{-1} \tag{23}$$

which can be written as

$$A_1 = \left[M\,\Gamma(x) + \left(\int_0^{+\infty} \left[\tilde{F}(x(t))\right]\left[x(t)\right]^T dt\right)\right]\left[\Gamma(x)\right]^{-1} \tag{24}$$

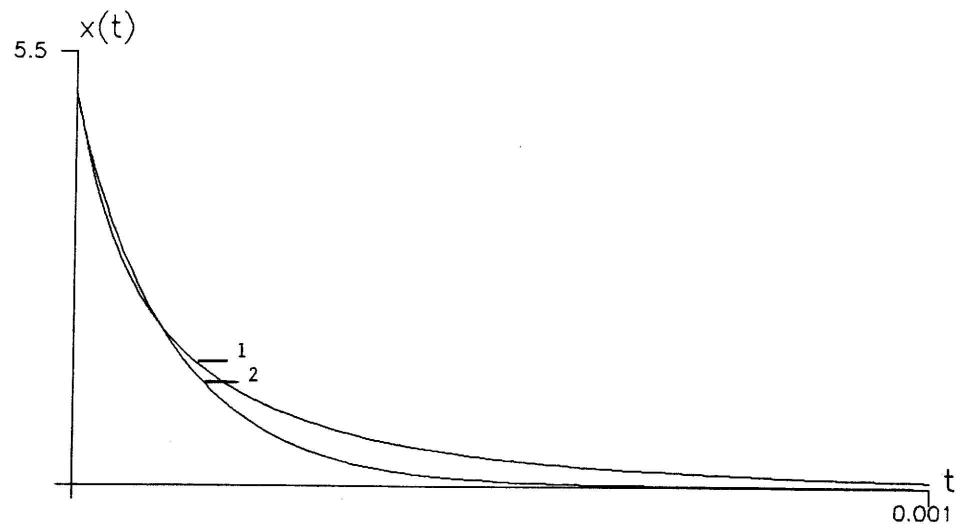

Fig. 1. The variation of the solution $x(t)$ as a function of time for the initial conditions $(x_0, y_0) = (5, 0)$.
Curve 1 corresponds to the solution of system (19).
Curve 2 corresponds to the solution of system (21).

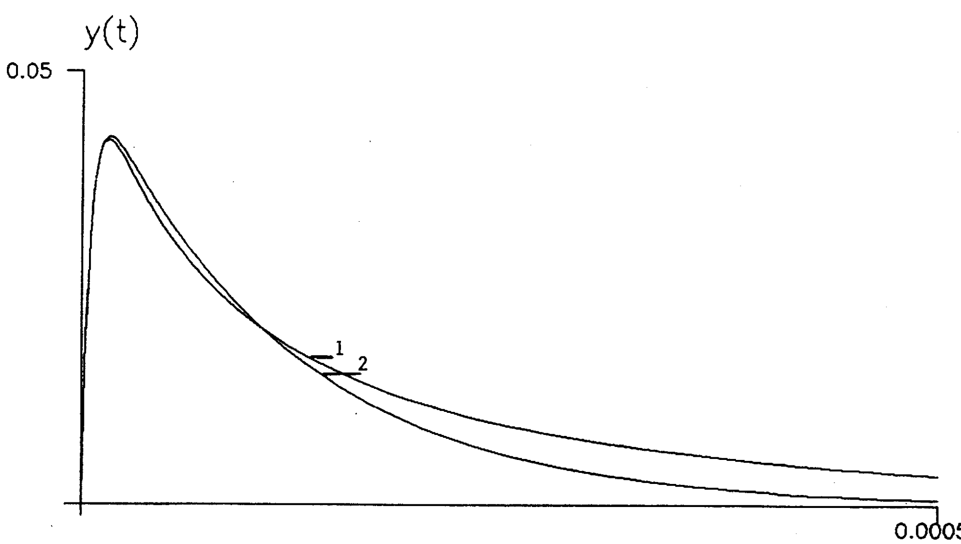

Fig. 2. The variation of the solution $y(t)$ as a function of time for the initial conditions $(x_0; y_0) = (5, 0)$.
Curve 1 corresponds to the solution of system (19).
Curve 2 corresponds to the solution of system (21).

and finally

$$A_1 = M + \left[\int_0^{+\infty} \left[\widetilde{F}(x(t)) \right] \left[x(t) \right]^T \mathrm{d}t \right] \left[\Gamma(x) \right]^{-1} \tag{25}$$

Hence

$$A_1 = M + \tilde{A}_1 \text{ with } \tilde{A}_1 = \left[\int_0^{+\infty} \left[\widetilde{F}(x(t)) \right] \left[x(t) \right]^T \mathrm{d}t \right] \left[\Gamma(x) \right]^{-1}$$

Then, for all j we have

$$A_j = M + \tilde{A}_j \tag{26}$$

with

$$\tilde{A}_j = \left[\int_0^{+\infty} \left[\widetilde{F}(x_j(t)) \right] \left[x_j(t) \right]^T \mathrm{d}t \right] \left[\Gamma(x_j) \right]^{-1} \tag{27}$$

If, in particular, some components of F are linear, then the corresponding components of $\widetilde{F}$ are zero, and the corresponding components of A_j are those of F. If f_k is linear, then the k-th row of matrix A_j is equal to f_k.

3.1.2. Case of a System Which Cannot be Linearized at 0 Using the Frechet Derivative

Consider a system with a function of the absolute value type that is non differentiable at 0

$$\left. \begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= -x + \alpha \sin(|y|) \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -y + \alpha \sin(|x|) \end{aligned} \right\}, \qquad (x_0, y_0) = (1, 0.5), \ |\alpha| < 1 \tag{28}$$

Then we have for $\alpha = 0.5$

$$DF(x_0, y_0) = \begin{bmatrix} -1 & 0.4387 \\ 0.2701 & -1 \end{bmatrix} \tag{29}$$

After the 5-th iteration, the computational procedure gives

$$A^* = \begin{bmatrix} -1.0207 & 0.5172 \\ 0.3502 & -0.8336 \end{bmatrix} \tag{30}$$

The curves in Figs. 3 and 4 represent the variation of solutions $(x(t), y(t))$ of systems (28) and (30) as a function of time. In both figures, the exact result obtained by solving numerically the non-linear system (28) and the approximate solution given by the optimal linear equation (30) are the same. Note that the method enables us to associate a linear system (optimal approximation) to a non-linear system in the neighborhood of 0.

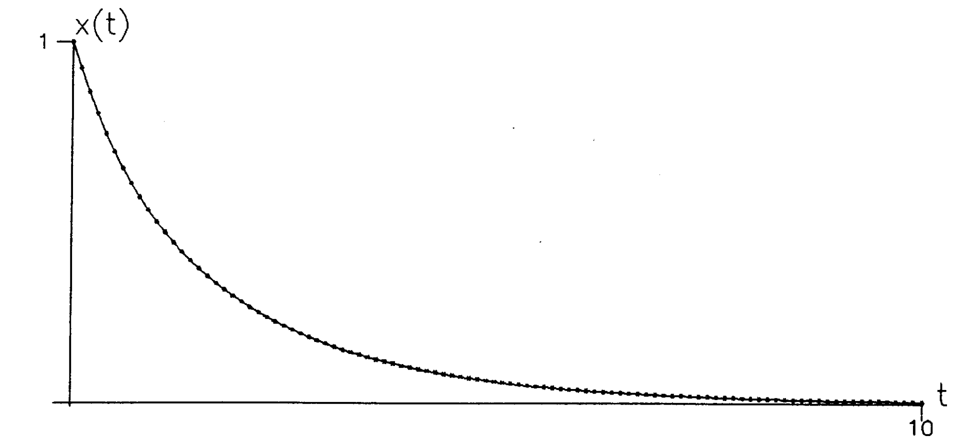

Fig. 3. The variation of the solution $x(t)$ as a function of time for the initial conditions $(x_0, y_0) = (1, 0.5)$. In this case, the solution of the non-linear system and optimal approximation are the same.

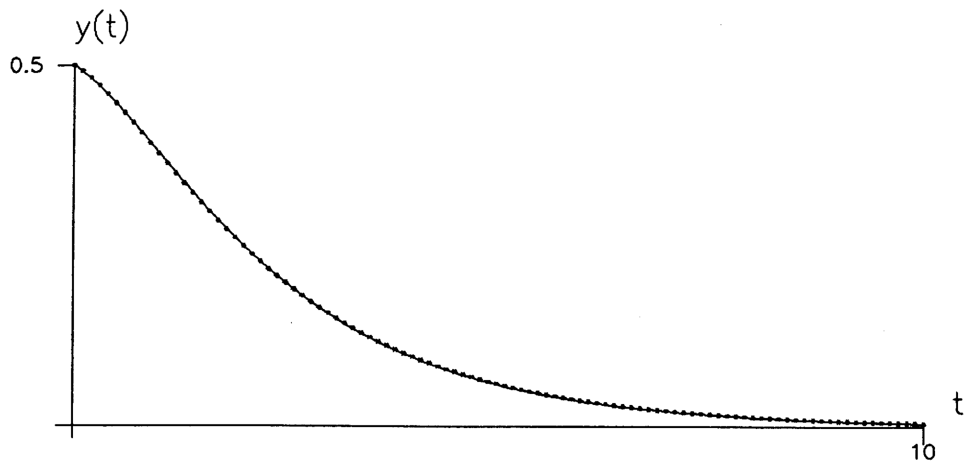

Fig. 4. The variation of the solution $y(t)$ as a function of time for the initial conditions $(x_0, y_0) = (1, 0.5)$. In this case, the solution of the non-linear system and optimal approximation are the same.

3.2. Analytic Expression in the Scalar Case

Consider the following non-linear scalar equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x(t)), \qquad x(0) = x_0 \tag{31}$$

where $f : \mathbb{R} \to \mathbb{R}$, is locally Lipschitzian and satisfies the following conditions:

i) $f(0) = 0$,

ii) $f'(x) < 0$ at every point where $f'(x)$ exists, in an interval $]-\alpha, +\alpha[, \alpha > 0$.

Choose $x_0 \in]-\alpha, +\alpha[$ such that $f'(x_0)$ exists. Set $a_0 = f'(x_0)$ and use the method presented in 2.3.

We solve the linear equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a_0 x(t), \qquad x(0) = x_0 \tag{32}$$

to obtain

$$x(t) = \exp(a_0 t)\, x_0 \tag{33}$$

Substituting f for F in expression (9), we get:

$$a_1 = \frac{\left(\int_0^{+\infty} f(x(t))\, e^{a_0 t}\, \mathrm{d}t\right)}{\left(\int_0^{+\infty} e^{2a_0 t}\, \mathrm{d}t\right)} \frac{1}{x_0} \tag{34}$$

For $x_0 \neq 0$, $f(x(t))$ is almost everywhere differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[f\left(e^{a_0 t} x_0\right)\right] = f'\left(e^{a_0 t} x_0\right) e^{a_0 t}\, x_0\, a_0 \tag{35}$$

This gives

$$\int_0^{+\infty} f(x(t))\, e^{a_0 t}\, \mathrm{d}t$$

$$= \frac{1}{a_0}\left[f(x(t))\, e^{a_0 t}\right]_0^{+\infty} - \frac{1}{a_0}\int_0^{+\infty} \left(f'(x(t))\, e^{2a_0 t} \mathrm{d}t\right) x_0\, a_0 \tag{36}$$

from which we obtain a_1

$$a_1 = 2\left(\frac{f(x_0)}{x_0} + a_0 \int_0^{+\infty} f'(x(t))\, e^{2a_0 t}\, \mathrm{d}t\right) \tag{37}$$

Note that $a_1 < 0$.

Assume we have constructed the sequence a_j for $j = 1$ to n. We are going to determine a_{n+1}. For this, consider the function given by

$$x_{n+1} = \exp(a_n t)\, x_0 \tag{38}$$

which stays in $]-\alpha, +\alpha[$, $\forall\ t \geq 0$. Substituting f for F and x_n for x in (9), we get

$$a_{n+1} = \frac{\left(\int_0^{+\infty} f\left(x_{n+1}(t)\right) e^{a_n t} \mathrm{d}t\right)}{\left(\int_0^{+\infty} e^{2a_n t} \mathrm{d}t\right)} \frac{x_0}{x_0^2} \tag{39}$$

which we can write as follows

$$a_{n+1} = 2\left(\frac{f(x_0)}{x_0} + a_n \int_0^{+\infty} f'\left(e^{a_n t} x_0\right) e^{2a_n t} \mathrm{d}t\right) \tag{40}$$

Let $z = e^{a_n t} x_0$ in the integral $\int_0^{+\infty} f'\left(e^{a_n t} x_0\right) e^{2a_n t} \mathrm{d}t$, to get

$$\int_0^{+\infty} f'\left(e^{a_n t} x_0\right) e^{2a_n t} \mathrm{d}t = \frac{-1}{a_n x_0^2} \int_0^{x_0} f'(z)\, z\, \mathrm{d}z \tag{41}$$

Finally, a^* is exactly a_{n+1}, so we have

$$a^*(x_0) = 2\left(\frac{f(x_0)}{x_0} - \frac{1}{x_0^2} \int_0^{x_0} f'(z)\, z\, \mathrm{d}z\right) \tag{42}$$

Remarks

1) a^* is obtained at the first iteration.

2) if $f'(0)$ exists, then $a^*(x_0) \rightarrow f'(0)$ as $x_0 \rightarrow 0$.

3.3. Relationship Between the Stability of the Non-linear Ordinary Differential Equation and the Stability of the Optimal Linear Equation in the Scalar Case (in $\mathbb{R}$)

Consider the optimal linear equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a^* x(t), \qquad x(0) = x_0 \tag{43}$$

where a^* is given by relation (42).

To study the sign of a^*, we consider the following quantity:

$$h(x_0) = x_0 f(x_0) - \int_0^{x_0} f'(z)\, z\, \mathrm{d}z \tag{44}$$

Note that

$$a^*(x_0) = \frac{2}{x_0^2} h(x_0) \tag{45}$$

Let us calculate the derivative of $h(x_0)$. We have

$$h'(x_0) = x_0 f'(x_0) + f(x_0) - f'(x_0)\, x_0 = f(x_0) \tag{46}$$

Therefore, we have

$$h(x_0) = \int_0^{x_0} f(s)\, \mathrm{d}s \tag{47}$$

If x is a solution of

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x(t)) \tag{48}$$

then,

$$\frac{\mathrm{d}}{\mathrm{d}t} h(x(t)) = \frac{\mathrm{d}}{\mathrm{d}x} h(x(t)) \frac{\mathrm{d}x}{\mathrm{d}t} = \left(f(x(t))\right)^2 > 0 \tag{49}$$

If $h(x)$ has a constant sign for $x \neq 0$, then $h(x)$ is a Lyapunov function for equation (31)

Conclusion. If $h(x) < 0$ for $0 < |x| < \alpha$, then: $x(t) \to 0$ as $t \to +\infty$.
If $h(x) > 0$ for $0 < |x| < \alpha$, then: $x(t) \to 0$ as $t \to -\infty$.

In the scalar case, $a^*(x_0)$ is the value (up to a multiplicative constant) of a Lyapunov function. Consequently, if $a^*(x_0)$ has a constant sign on $\{x_0; 0 < |x_0| < \alpha\}$, then the solution is

- *Asymptotically stable,* if $a^*(x_0) < 0$
- *Unstable,* if $a^*(x_0) > 0$

In this case, the method of optimal approximation is equivalent to the Lyapunov method.

4. Examples of Application of the Optimal Method for Analyzing the Stability in the Vector Case

Note that the study of the vector case was motivated by the scalar case.

Example 1. Consider the following non-linear system (Kalman and Bertran, 1960)

$$\left.\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= y + ax\left(x^2 + y^2\right) \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -x + ay\left(x^2 + y^2\right) \end{aligned}\right\}, \quad (x_0, y_0) = (0, 1) \tag{50}$$

For this system, the stability of the stationary point sets a problem. In fact, the linearized equation has a center, i.e. the eigenvalues are purely imaginary.

Case 1. $a = -1$, system (50) reduces to

$$\left.\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= y - x\left(x^2 + y^2\right) \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -x - y\left(x^2 + y^2\right) \end{aligned}\right\}, \quad (x_0, y_0) = (0, 1) \tag{51}$$

The linearized equation $DF(x,y)$ at $(x_0, y_0) = (0,0)$ gives

$$DF(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{52}$$

whose eigenvalues are $\lambda_{1,2} = \pm i$. Hence, we cannot conclude from the linear equation. In Fig. 5, curve 1 represents the solution $(x(t), y(t))$ of system (52) in the phase space.

Now, compute the optimal matrix corresponding to system (51), starting from the value of $DF(x,y)$ computed at $(x_0, y_0) = (0,1)$

$$DF(x_0, y_0) = \begin{bmatrix} -1 & 1 \\ -1 & -3 \end{bmatrix} \tag{53}$$

after the 10-th iteration, the computational procedure gives

$$A^* = \begin{bmatrix} -0.2669 & 0.9053 \\ -1.0566 & -0.5465 \end{bmatrix} \tag{54}$$

whose eigenvalues are

$$\begin{aligned} \lambda_1^* &= -0.4067 + i0.9679 \\ \lambda_1^* &= -0.4067 - i0.9679 \end{aligned} \tag{55}$$

$\mathrm{Re}(\lambda_1^*) < 0$ and $\mathrm{Re}(\lambda_2^*) < 0$. Hence, the matrix is exponentially stable. Therefore, the origin is asymptotically stable for the non-linear equation, which is in agreement with the Lyapunov method (Kalman and Bertran, 1960). In Fig. 5, we have plotted the solutions $(x(t), y(t))$ in the phase space. Curve 2 represents the solution of system (54), and curve 3 represents the exact solution of system (51).

Remark. Note that the type of stability of non-linear system (51) is a focus and the eigenvalues of optimal approximation (54) are complex.

Case 2. $a = +1$. System (50) has the form

$$\left. \begin{aligned} \frac{dx}{dt} &= y + x\left(x^2 + y^2\right) \\ \frac{dy}{dt} &= -x + y\left(x^2 + y^2\right) \end{aligned} \right\}, \quad (x_0, y_0) = (0,1) \tag{56}$$

$DF(0,0)$ is the same as in case 1, $DF(x_0, y_0)$ reads

$$DF(x_0, y_0) = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \tag{57}$$

After the 10-th iteration, the procedure gives

$$A^* = \begin{bmatrix} 0.2669 & 0.9053 \\ -1.0566 & 0.5465 \end{bmatrix} \tag{58}$$

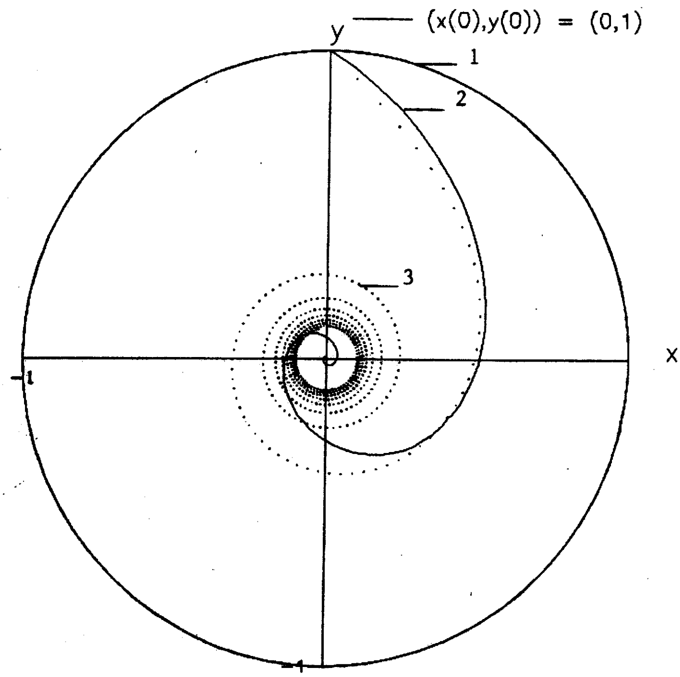

Fig. 5. The solutions $(x(t), y(t))$ in the phase space.
Curve 1 corresponds to the solution of system (52).
Curve 2 corresponds to the solution of system (54).
Curve 3 corresponds to the solution of system (51).

whose eigenvalues are

$$\begin{aligned} \lambda_1^* &= 0.4067 + i0.9679 \\ \lambda_1^* &= 0.4067 - i0.9679 \end{aligned} \tag{59}$$

$\mathrm{Re}(\lambda_1^*) > 0$ and $\mathrm{Re}(\lambda_2^*) > 0$. In this case, A^* is unstable. The origin is asymptotically unstable for the non-linear equation, as shown by the Lyapunov method (Kalman and Bertran, 1960).

Remark. We know, from the beginning, that system (56) is unstable. So we have to deal with the problem of the convergence of the integral defined by (3). In the two dimensional case, we integrate from 0 to $-\infty$ i.e. we write

$$G(A) = \int_0^{-\infty} \|F(x(t)) - Ax(t)\|^2 \, \mathrm{d}t \tag{60}$$

Example 2. Here, we consider a non-linear system (Demailly, 1991) for which the first order linearization at $(0,0)$ has a node, while the analysis of the non-linear system shows that the system has a focus.

Consider the following system

$$\left.\begin{aligned}\frac{dx}{dt} &= -x - \frac{2y}{\ln(x^2+y^2)}\\ \frac{dy}{dt} &= -y + \frac{2x}{\ln(x^2+y^2)}\end{aligned}\right\}, \qquad (x_0, y_0) = (0, 0.5) \tag{61}$$

in the open unit disk $\{(x,y) \in \mathbb{R}^2;\ x^2+y^2 < 1\}$.

Note that $\dfrac{2y}{\ln(x^2+y^2)}$ has a C^1 extension in the neighborhood of $(0,0)$. It has a limit equal to zero at the origin. We can make a similar remark for $\dfrac{2x}{\ln(x^2+y^2)}$. Hence, the origin is a singular point.

The linearization gives

$$DF(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{62}$$

The eigenvalues are :$\lambda_1 = \lambda_2 = -1$. Hence system (62) has a stable proper node. In Fig. 6, curve 1 represents the solution $(x(t), y(t))$ of the linearized equation in the phase space.

The Jacobian matrix $DF(x,y)$ computed at $(x_0, y_0) = (0, 0.5)$ gives

$$DF(x_0, y_0) = \begin{bmatrix} -1 & 3.524 \\ -1.4426 & -1 \end{bmatrix} \tag{63}$$

After the 8-th iteration, the computational procedure gives

$$A^* = \begin{bmatrix} -1.4934 & 1.2489 \\ -0.5213 & -1.1254 \end{bmatrix} \tag{64}$$

whose eigenvalues are

$$\begin{aligned}\lambda_1^* &= -1.3094 - i0.7856\\ \lambda_1^* &= -1.3094 + i0.7856\end{aligned} \tag{65}$$

Therefore, $\mathrm{Re}(\lambda_1^*) < 0$ and $\mathrm{Re}(\lambda_2^*) < 0$, and consequently the origin is a stable focus. In Fig. 6, curve 2 corresponds to the solution of system (64) in the phase space.

Now, we are going to solve the original non-linear system (61) using polar coordinates (r, θ). This yields

$$\begin{aligned}\frac{dr}{dt} &= -\frac{x^2+y^2}{r} = -r\\ \frac{d\theta}{dt} &= \frac{1}{x^2+y^2}\,\frac{2x^2+2y^2}{\ln(x^2+y^2)} = \frac{1}{\ln r}\end{aligned} \tag{66}$$

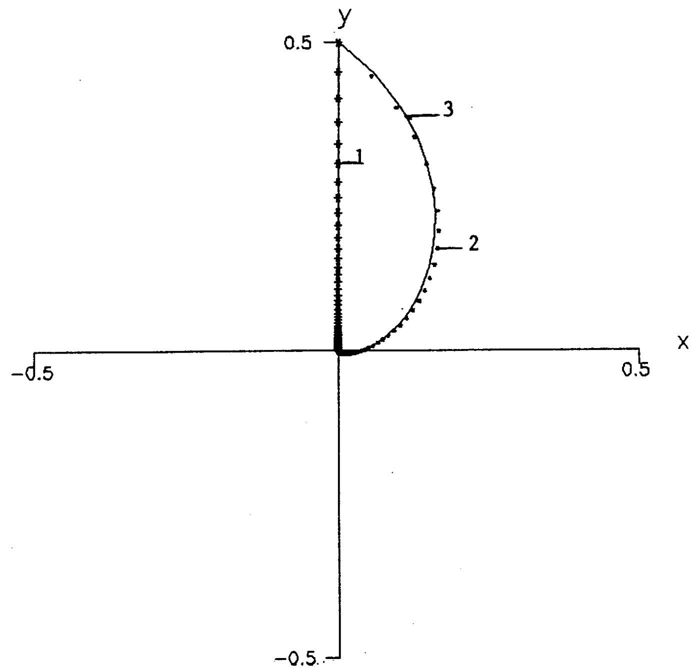

Fig. 6. The solutions $(x(t), y(t))$ in the phase space.
Curve 1 corresponds to the solution of the first order linearized (63).
Curve 2 corresponds to the solution of system (64).
Curve 3 corresponds to the solution of system (61).

The solution of the Cauchy problem is given by

$$r(t) = r_0 e^{-t}, \qquad r_0 < 1$$
$$\theta(t) = \theta_0 - \ln\left(1 - \frac{t}{\ln r_0}\right) \tag{67}$$

for an initial data (r_0, θ_0), given at $t = 0$. The solution is defined on $[\ln r_0, +\infty]$. Also, we have $r(t) \to 0$ and $\theta(t) \to -\infty$, as $t \to +\infty$. This shows that we have a spiral converging to 0. The curve spirals inside the unit circle $r = 1$ when $t \to \ln(r_0)$ from above, which shows that the system has a stable focus. In Fig. 6, curve 3 corresponds to the solution of the non-linear system (61) in the phase space. Observe that the procedure gives an answer which agrees with the analysis of the non-linear system.

5. Comments

We have presented a computational procedure which we have applied to several examples. The differential equations have been solved using the fourth order Runge-Kutta method. The computational approximation procedure is based on the algorithm presented in 2.3, and written in Fortran language. Details can be provided on request.

We should point out that

1) The optimal approximation is obtained after the first iteration in the scalar case. It was possible to give a relationship between the stability of the non-linear equation and that of the optimal linear equation. In this case, the method of optimal approximation is equivalent to the method of Lyapunov.
2) The method enables us to associate a linear equation (optimal approximation) to a non-linear equation in the neighborhood of 0 in the vector case, even though the latter equation cannot be linearized around the origin. This is the case notably when the functions involved are not smooth near the origin.

For the stability issue, the study of the vector case was motivated by the scalar case. The examples we have seen illustrate some problems set in analyzing stability.

In the first example, the linearization based on the Frechet derivative does not allow us to conclude the stability problem (eigenvalues are imaginary).

As shown in the second example the behaviour of the solutions of the linearized equation (in the Frechet sense) is not the same as the behaviour of the solutions of the non-linear equation, since the linear equation has a stable node, while the non-linear equation has a stable focus. In both cases, the proposed procedure gives results in agreement with the analysis of the original non-linear equation (same behaviour).

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Received: June 7, 1994