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# GENERALIZED 2-D CONTINUOUS-DISCRETE LINEAR SYSTEMS WITH DELAYS

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A general Kurek (K) type model and a general Roesser (R) type model of 2-D continuous-discrete linear systems with deviating arguments are introduced. A solution and the general response formula for the regular general type model of 2-D continuous-discrete linear systems with delays are derived. The necessary and sufficient conditions for the local reachability and the local controllability of the regular general K type model are established. The minimum energy control of the regular model is solved.

## 1. Introduction

The most popular models of two-dimensional (2-D) linear systems are the discrete state space models introduced by Roesser (1975), Fornasini and Marchesini (1976; 1978) and Kurek (1985). The models have been extended for singular (implicit) linear discrete systems in (Gregor, 1992; Kaczorek, 1988b; 1990). A review of singular 2-D discrete linear systems has been given in (Kaczorek, 1993c; Lewis, 1992). Generalized multidimensional linear systems with deviating arguments (specially with delays) have been investigated in (Kaczorek, 1992; 1993a; 1993b). Continuous 2-D models of linear and non-linear systems have been considered in (Bergman *et al.*, 1989; Idczak and Walczak, 1992; Walczak, 1988). Recently in (Kaczorek, 1994a; 1994b) singular 2-D continuous-discrete models of linear systems have been introduced. In 2-D continuous-discrete systems one independent variable is continuous and the other independent variable is discrete. Such continuous-discrete models appear for example in the iterative learning control synthesis (Kurek and Zaremba, 1993) and the repetitive processes analysis (Rogers and Owens, 1992).

In this paper a general Kurek (K) type model of 2-D continuous-discrete systems with deviating arguments and a general Roesser (R) type model of 2-D continuous-discrete systems with deviating arguments will be introduced.

The general response formula for the regular general K type model of 2-D continuous-discrete linear systems with delays will be derived and the necessary and sufficient conditions for the local reachability and local controllability will be established. The minimum energy control problem for the general K type model will be solved.

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## 2. Models of 2-D Continuous-Discrete Linear Systems

Consider a general Kurek (K) type model of 2-D continuous-discrete model linear system with deviating arguments described by the equations

$$\begin{aligned} E\dot{x}(t,k+1) &= A_0x(t,k) + A_1\dot{x}(t,k) + A_2x(t,k+1) + B_0x(t-d_1,k-d_2) \\ &+B_1\dot{x}(t-d_1,k-d_2) + B_2x(t-d_1,k-d_2+1) \\ &+C_0u(t,k) + C_1\dot{u}(t,k) + C_2u(t,k+1) \\ &\qquad \text{for } t,d_1 \in \mathbb{R}_+,\ \ k,d_2 \in \mathbb{Z}_+ \end{aligned} \tag{1a}$$

$$y(t,k) = D_0x(t,k) + D_1u(t,k) \tag{1b}$$

where $\dot{x}(t,k) = \dfrac{\partial x(t,k)}{\partial t}$, $x(t,k) \in \mathbb{R}^n$ is the semistate vector, $u(t,k) \in \mathbb{R}^m$ is the input vector, $y(t,k) \in \mathbb{R}^p$ is the output vector, $E \in \mathbb{R}^{q\times n}$, $A_i \in \mathbb{R}^{q\times n}$, $B_i \in \mathbb{R}^{q\times n}$, $C_i \in \mathbb{R}^{q\times m}$, $i = 0,1,2$, $D_0 \in \mathbb{R}^{p\times n}$, $D_1 \in \mathbb{R}^{p\times m}$, and $\mathbb{R}^{q\times n}$ is the set of $q\times n$ real matrices, $\mathbb{R}_+$ and $\mathbb{Z}_+$ is the set of non-negative real numbers and integers, respectively.

If $q \neq n$ or $\det E = 0$ when $q = n$, model (1) is called singular (implicit).

If $q = n$ and $\det E \neq 0$ the model will be called standard. In this case premultiplying (1a) by $E^{-1}$ we obtain a model with $E = I$ (the identity matrix).

If $q = n$ and $\det E = 0$ but

$$\det\left[Esz - A_0 - A_1s - A_2z - B_0e^{-sd_1}z^{-d_2} - B_1se^{-d_1}z^{-d_2} - B_2e^{-d_1s}z^{1-d_2}\right] \neq 0$$

for some $s \in \mathbb{C}$ (the field of complex numbers) model (1) will be called regular. If $d_1 > 0$ and $d_2 > 0$ then (1) will be called a model with delays (retarded arguments) and $d_1 < 0$ and $d_2 < 0$ then it will be called a model with advanced arguments.
When all or some entries of the matrices $E, A_i, B_i, C_i$, $i = 0,1,2$, $D_0$ and $D_1$ depend on $t$ and $k$ then (1) will be called a model with variable coefficients.
From (1) for $C_1 = 0$ and $C_2 = 0$ we obtain the first generalized Fornasini-Marchesini model of 2-D continuous-discrete linear systems with deviating arguments. Similarily, from (1) for $A_0 = 0$, $B_0 = 0$ and $C_0 = 0$ we obtain the second generalized Fornasini-Marchesini model of 2-D continuous-discrete linear systems with deviating arguments.

The general Roesser (R) type model of 2-D continuous-discrete linear systems with deviating arguments has the form

$$\begin{aligned} E\begin{bmatrix} \dot{x}^h(t,k) \\ x^v(t,k+1) \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x^h(t,k) \\ x^v(t,k) \end{bmatrix} \\ &+ \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}\begin{bmatrix} x^h(t-d_1,k-d_2) \\ x^v(t-d_1,k-d_2) \end{bmatrix} + \begin{bmatrix} C_{10} \\ C_{20} \end{bmatrix}u(t,k) \end{aligned} \tag{2a}$$

$$y(t,k) = \begin{bmatrix} D_{01} & D_{02} \end{bmatrix}\begin{bmatrix} x^h(t,k) \\ x^v(t,k) \end{bmatrix} + D_1u(t,k) \tag{2b}$$

where $\dot{x}^h(t,k) = \dfrac{\partial x^h(t,k)}{\partial t}$, $x^h(t,k) \in \mathbb{R}^{\overline{n}_1}$ is the horizontal semistate vector, $x^\nu(t,k) \in \mathbb{R}^{\overline{n}_2}$ is the vertical semistate vector, $u(t,k) \in \mathbb{R}^m$ is the input vector, $y(t,k) \in \mathbb{R}^p$ is the output vector, and

$$A_{11}, B_{11} \in \mathbb{R}^{q_1 \times \overline{n}_1}, \quad A_{12}, B_{12} \in \mathbb{R}^{q_1 \times \overline{n}_2}, \quad A_{21}, B_{21} \in \mathbb{R}^{q_2 \times \overline{n}_1}, \quad A_{22}, B_{22} \in \mathbb{R}^{q_2 \times \overline{n}_2}$$

$$C_{10} \in \mathbb{R}^{q_1 \times m}, \quad C_{20} \in \mathbb{R}^{q_2 \times m}, \quad D_{01} \in \mathbb{R}^{p \times \overline{n}_1}, \quad D_{02} \in \mathbb{R}^{p \times \overline{n}_2}, \quad D_1 \in \mathbb{R}^{p \times m}$$

Similarly as for (1) the generalized Roesser model (2) will be called singular if $q_1 \neq \overline{n}_1$, $q_2 \neq \overline{n}_2$ or $\det E = 0$ when $q_1 = \overline{n}_1$, $q_2 = \overline{n}_2$. If $E$ is square and $\det E \neq 0$ then model (2) will be called standard.

Model (2) will be called regular if $q_1 = \overline{n}_1$, $q_2 = \overline{n}_2$ and $\det E = 0$ but

$$\det \begin{bmatrix} E_{11}s - A_{11} - B_{11}e^{-sd_1}z^{-d_2}, & E_{12}z - A_{12} - B_{12}e^{-sd_1}z^{-d_2} \\ E_{21}s - A_{21} - B_{21}e^{-sd_1}z^{-d_2}, & E_{22}z - A_{22} - B_{22}e^{-sd_1}z^{-d_2} \end{bmatrix} \neq 0$$

for some $s, z \in \mathbb{C}$, where

$$E = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}, \qquad E_{11} \in \mathbb{R}^{q_1 \times \overline{n}_1}, \qquad E_{22} \in \mathbb{R}^{q_2 \times \overline{n}_2}$$

It can be shown that the generalized Roesser model (2) is a special case of the model (1).

## 3. Solution to the Regular General Model

To simplify the notion we write (1a) in the form

$$E\dot{x}(t,k+1) = A_0x(t,k) + A_1\dot{x}(t,k) + A_2x(t,k+1) + B_0x(t-d_1,k-d_2)$$
$$+B_1\dot{x}(t-d_1,k-d_2) + B_2x(t-d_1,k-d_2+1) + f(t,k) \tag{3a}$$

where

$$f(t,k) := C_0u(t,k) + C_1\dot{u}(t,k) + C_2u(t,k+1) \tag{3b}$$

It is assumed that $q = n$, $\det E = 0$ and

$$\det[Es - A_2] \neq 0 \text{ for some } s \in \mathbb{C} \text{ (the field of complex numbers)} \tag{4}$$

It is well-known that if (4) holds then there exist non-singular matrices $P, Q \in \mathbb{R}^{n \times n}$ such that

$$P[Es - A_2]Q = \begin{bmatrix} I_{n_1}s - A_{21} & 0 \\ 0 & Ns - I_{n_2} \end{bmatrix} \tag{5}$$

where $n_1$ is the degree of $\det[Es - A_2]$, $n_2 := n - n_1$, $A_{21} \in \mathbb{R}^{n_1 \times n_1}$, $N \in \mathbb{R}^{n_2 \times n_2}$ is a nilpotent matrix with index $\nu(N^{\nu-1} \neq 0$ and $N^\nu = 0)$.

In a similar way as in (Kaczorek, 1993c) it can be shown that if (4) holds then the model (3a) is regular. Premultiplying (3a) by $P$, introducing the new vector

$$Q^{-1}x(t,k) := \begin{bmatrix} x_1(t,k) \\ x_2(t,k) \end{bmatrix}, \quad x_1(t,k) \in \mathbb{R}^{n_1}, \quad x_2(t,k) \in \mathbb{R}^{n_2}$$

and using (5) we obtain

$$\begin{aligned} \dot{x}(t,k+1) = {} & A_{01}x_1(t,k) + A_{02}x_2(t,k) + A_{11}\dot{x}_1(t,k) + A_{12}\dot{x}_2(t,k) \\ & +A_{21}x_1(t,k+1) + B_{01}x_1(t-d_1,k-d_2) + B_{02}x_2(t-d_1,k-d_2) \\ & +B_{11}\dot{x}_1(t-d_1,k-d_2) + B_{12}\dot{x}_2(t-d_1,k-d_2) \\ & +B_{21}x_1(t-d_1,k-d_2+1) + B_{22}x_2(t-d_1,k-d_2+1) + f_1(t,k) \end{aligned} \tag{6a}$$

$$\begin{aligned} N\dot{x}_2(t,k+1) = {} & A_{03}x_1(t,k) + A_{04}x_2(t,k) + A_{13}\dot{x}_1(t,k) + A_{14}\dot{x}_2(t,k) \\ & +x_2(t,k+1) + B_{03}x_1(t-d_1,k-d_2) + B_{04}x_2(t-d_1,k-d_2) \\ & +B_{13}\dot{x}_1(t-d_1,k-d_2) + B_{14}\dot{x}_2(t-d_1,k-d_2) \\ & +B_{23}x_1(t-d_1,k-d_2+1) + B_{24}x_2(t-d_1,k-d_2+1) + f_2(t,k) \\ & \text{for } t \in \mathbb{R}_+, \quad k \in \mathbb{Z}_+ \end{aligned} \tag{6b}$$

where

$$PA_kQ = \begin{bmatrix} A_{k1} & A_{k2} \\ A_{k3} & A_{k4} \end{bmatrix}, \qquad k = 0,1,$$

$$PB_jQ = \begin{bmatrix} B_{j1} & B_{j2} \\ B_{j3} & B_{j4} \end{bmatrix}, \qquad j = 0,1,2,$$

$$Pf(t,k) = \begin{bmatrix} f_1(t,k) \\ f_2(t,k) \end{bmatrix}$$

The submatrices $A_{ki}$, $B_{ij}$, $k = 0,1$; $i = 1,2,3,4$; $j = 0,1,2$ and the vectors $f_1(t,k)$, $f_2(t,k)$ have dimensions compatible with the dimensions of $x_1$ and $x_2$, respectively.

Let the boundary conditions for (3a) be given by

$$x(t,k) \text{ and } \dot{x}(t,k) \text{ for } t \geq -d_1, \ -d_2 \leq k \leq 0 \text{ and } -d_1 \leq t \leq 0, \ k > 0 \tag{7}$$

It is assumed that the boundary conditions and $u(t,k)$ are $(\nu+1)$ – times differentable with respect to $t$. The boundary conditions (7) are called admissible if for a given $u(t,k)$ there exists a solution $x(t,k)$ to (3a).

Knowing (7) we may find the boundary conditions for (6)

$$\begin{bmatrix} x_1(t,k) \\ x_2(t,k) \end{bmatrix} = Q^{-1}x(t,k) \quad \text{and} \quad \begin{bmatrix} \dot{x}_1(t,k) \\ \dot{x}_2(t,k) \end{bmatrix} = Q^{-1}\dot{x}(t,k) \tag{8}$$

for $t \geq -d_1$, $-d_2 \leq k \leq 0$ and $-d_1 \leq t \leq 0$, $k > 0$.

Note that equations (6) are coupled by the matrices $A_{i2}, A_{i3}$, $i = 0, 1$, $B_{j2}, B_{j3}$, $j = 0, 1, 2$ and if at least one of the matrices is non-zero the equations cannot be solved independently.

To find the solutions $x_1(t, k)$ and $x_2(t, k)$ to (6) with boundary conditions (8) let us consider the equations for $k = 0$

$$\dot{x}_1(t, 1) = A_{21}x_1(t, 1) + F_{10}(t) \tag{9a}$$

$$N\dot{x}_2(t, 1) = x_2(t, 1) + F_{20}(t) \tag{9b}$$

where

$$\begin{aligned} F_{10}(t) := {} & A_{01}x_1(t, 0) + A_{02}x_2(t, 0) + A_{11}\dot{x}_1(t, 0) + A_{12}\dot{x}_2(t, 0) \\ & + B_{01}x_1(t - d_1, -d_2) + B_{02}x_2(t - d_1, -d_2) + B_{11}\dot{x}_1(t - d_1, -d_2) \\ & + B_{12}\dot{x}_2(t - d_1, -d_2) + B_{21}x_1(t - d_1, 1 - d_2) \\ & + B_{22}x_2(t - d_1, 1 - d_2) + f_1(t, 0) \end{aligned} \tag{10a}$$

$$\begin{aligned} F_{20}(t) := {} & A_{03}x_1(t, 0) + A_{04}x_2(t, 0) + A_{13}\dot{x}_1(t, 0) + A_{14}\dot{x}_2(t, 0) \\ & + B_{03}x_1(t - d_1, -d_2) + B_{04}x_2(t - d_1, -d_2) + B_{13}\dot{x}_1(t - d_1, -d_2) \\ & + B_{14}\dot{x}_2(t - d_1, -d_2) + B_{23}x_1(t - d_1, 1 - d_2) \\ & + B_{24}x_2(t - d_1, 1 - d_2) + f_2(t, 0) \end{aligned} \tag{10b}$$

are known for given (8), $f_1(t, k)$ and $f_2(t, k)$.

The solution $x_1(t, 1)$ to (9a) has the form (Gantmacher, 1959; Kaczorek, 1993c)

$$x_1(t, 1) = e^{A_{21}t}x_1(0, 1) + \int_0^t e^{A_{21}(t-\tau)}F_{10}(\tau)\, d\tau, \qquad t \in \mathbb{R}_+ \tag{11}$$

Premultiplying (9b) successively by $N, N^2, \dots, N^{\nu-1}$ and differentiating with respect to $t$ we obtain

$$\begin{aligned} & N\dot{x}_2(t, 1) - x_2(t, 1) = F_{20}(t) \\ & N^2x_2^{(2)}(t, 1) - N\dot{x}_2(t, 1) = N\dot{F}_{20}(t) \\ & \dots\dots\dots\dots\dots\dots \\ & N^\nu x_2^{(\nu)}(t, 1) - N^{\nu-1}x_2^{(\nu-1)}(t, 1) = N^{\nu-1}F_{20}^{(\nu-1)}(t) \end{aligned} \tag{12}$$

where $x_2^{(i)}(t, 1)$ and $F_{20}^{(i)}(t)$ denotes the $i$-th order derevative of $x_2(t, 1)$ and $F_{20}(t)$, respectively.

Adding equations (12) and taking into account that $N^\nu = 0$ we obtain the solution $x_2(t, 1)$ to (9b) in the form

$$x_2(t, 1) = -\sum_{i=0}^{\nu-1} N^i F_{20}^{(i)}(t) \tag{13}$$

If $d_2 \geq 2$ then substituting $k = 1$ into (6) and using (11) and (13) we obtain

$$\begin{aligned}
\dot{x}_1(t,2) &= A_{01}x_1(t,1) + A_{02}x_2(t,1) + A_{11}\dot{x}_1(t,1) + A_{12}\dot{x}_2(t,1) + A_{21}x_1(t,2) \\
&\quad + B_{01}x_1(t-d_1, 1-d_2) + B_{02}x_2(t-d_1, 1-d_2) \\
&\quad + B_{11}\dot{x}_1(t-d_1, 1-d_2) + B_{12}\dot{x}_2(t-d_1, 1-d_2) \\
&\quad + B_{21}x_1(t-d_1, 2-d_2) + B_{22}x_2(t-d_1, 2-d_2) + f_1(t,1) \\
&= A_{01}\Big[e^{A_{21}t}x_1(0,1) + \int_0^t e^{A_{21}(t-\tau)}F_{10}(\tau)\,\mathrm{d}\tau\Big] + A_{02}\Big[-\sum_{i=0}^{\nu-1} N^i F_{20}^{(i)}(t)\Big] \\
&\quad + A_{11}\Big[A_{21}e^{A_{21}t}x_1(0,1) + F_{10}(t) + A_{21}\int_0^t e^{A_{21}(t-\tau)}F_{10}(\tau)\,\mathrm{d}\tau\Big] \\
&\quad + A_{12}\Big[-\sum_{i=0}^{\nu-1} N^i F_{20}^{(i+1)}(t)\Big] + A_{21}x_1(t,2) \\
&\quad + B_{01}x_1(t-d_1, 1-d_2) + B_{02}x_2(t-d_1, 1-d_2) \\
&\quad + B_{11}\dot{x}_1(t-d_1, 1-d_2) + B_{12}\dot{x}_2(t-d_1, 1-d_2) \\
&\quad + B_{21}x_1(t-d_1, 2-d_2) + B_{22}x_2(t-d_1, 2-d_2) + f_1(t,1) \\
&= A_{21}x_1(t,2) + F_{11}(t)
\end{aligned} \tag{14a}$$

and

$$\begin{aligned}
N\dot{x}_2(t,2) &= A_{03}x_1(t,1) + A_{04}x_2(t,1) + A_{13}\dot{x}_1(t,1) + A_{14}\dot{x}_2(t,1) + x_2(t,2) \\
&\quad + B_{03}x_1(t-d_1, 1-d_2) + B_{04}x_2(t-d_1, 1-d_2) \\
&\quad + B_{13}\dot{x}_1(t-d_1, 1-d_2) + B_{14}\dot{x}_2(t-d_1, 1-d_2) \\
&\quad + B_{23}x_1(t-d_1, 2-d_2) + B_{24}x_2(t-d_1, 2-d_2) + f_2(t,1) \\
&= A_{03}\Big[e^{A_{21}t}x_1(0,1) + \int_0^t e^{A_{21}(t-\tau)}F_{10}(\tau)\,\mathrm{d}\tau\Big] + A_{04}\Big[-\sum_{i=0}^{\nu-1} N^i F_{20}^{(i)}(t)\Big] \\
&\quad + A_{13}\Big[A_{21}e^{A_{21}t}x_1(0,1) + F_{10}(t) + A_{21}\int_0^t e^{A_{21}(t-\tau)}F_{10}(\tau)\,\mathrm{d}\tau\Big] \\
&\quad + A_{14}\Big[-\sum_{i=0}^{\nu-1} N^i F_{20}^{(i+1)}(t)\Big] + x_2(t,2) \\
&\quad + B_{03}x_1(t-d_1, 1-d_2) + B_{04}x_2(t-d_1, 1-d_2) \\
&\quad + B_{13}\dot{x}_1(t-d_1, 1-d_2) + B_{14}\dot{x}_2(t-d_1, 1-d_2) \\
&\quad + B_{23}x_1(t-d_1, 2-d_2) + B_{24}x_2(t-d_1, 2-d_2) + f_2(t,1) \\
&= x_2(t,2) + F_{21}(t)
\end{aligned} \tag{14b}$$

where

$$F_{11}(t) := \overline{A}_1\Big[e^{A_{21}t}x_1(0,1) + \int_0^t e^{A_{21}(t-\tau)}F_{10}(\tau)\,\mathrm{d}\tau\Big] + A_{11}F_{10}(t)$$
$$-\sum_{i=0}^{\nu-1}\Big[A_{02}N^iF_{20}^{(i)}(t) + A_{12}N^iF_{20}^{(i+1)}(t)\Big] + B_{01}x_1(t-d_1,1-d_2)$$
$$+B_{02}x_2(t-d_1,1-d_2) + B_{11}\dot{x}_1(t-d_1,1-d_2) + B_{12}\dot{x}_2(t-d_1,1-d_2)$$
$$+B_{21}x_1(t-d_1,2-d_2) + B_{22}x_2(t-d_1,2-d_2) + f_1(t,1)$$

$$F_{21}(t) := \overline{A}_2\Big[e^{A_{21}t}x_1(0,1) + \int_0^t e^{A_{21}(t-\tau)}F_{10}(\tau)\,\mathrm{d}\tau\Big] + A_{13}F_{10}(t)$$
$$-\sum_{i=0}^{\nu-1}\Big[A_{04}N^iF_{20}^{(i)}(t) + A_{14}N^iF_{20}^{(i+1)}(t)\Big] + B_{03}x_1(t-d_1,1-d_2)$$
$$+B_{04}x_2(t-d_1,1-d_2) + B_{13}\dot{x}_1(t-d_1,1-d_2) + B_{14}\dot{x}_2(t-d_1,1-d_2)$$
$$+B_{23}x_1(t-d_1,2-d_2) + B_{24}x_2(t-d_1,2-d_2) + f_2(t,1)$$

$$\overline{A}_1 := A_{01} + A_{11}A_{21}, \qquad \overline{A}_2 := A_{03} + A_{13}A_{21}$$

The solutions $x_1(t,2)$, $x_2(t,2)$ to (14) have the form

$$x_1(t,2) = e^{A_{21}t}x_1(0,2) + \int_0^t e^{A_{21}(t-\tau)}F_{11}(\tau)\,\mathrm{d}\tau, \qquad t \in \mathbb{R}_+$$

$$x_2(t,2) = -\sum_{i=0}^{\nu-1} N^iF_{21}(t)$$

If $d_2 < 2$ we also have to substitute (11) and (13) instead of $x_1(t-d_1, 2-d_2)$ and $x_2(t-d_1, 2-d_2)$ into (14). For $d_2 = 1$ we have

$$F_{11}(t) := \overline{A}_1\Big[e^{A_{21}t}x_1(0,1) + \int_0^t e^{A_{21}(t-\tau)}F_{10}(\tau)\,\mathrm{d}\tau\Big] + A_{11}F_{10}(t)$$
$$-\sum_{i=0}^{\nu-1}\Big[A_{02}N^iF_{20}^{(i)}(t) + A_{12}N^iF_{20}^{(i+1)}(t) + B_{22}N^iF_{20}^{(i)}(t-d_1)\Big]$$
$$+B_{21}\Big[e^{A_{21}(t-d_1)}x_1(-d_1,1) + \int_0^t e^{A_{21}(t-d_1-\tau)}F_{10}(\tau)\,\mathrm{d}\tau\Big] + B_{01}x_1(t-d_1,0)$$
$$+B_{02}x_2(t-d_1,0) + B_{11}\dot{x}_1(t-d_1,0) + B_{12}\dot{x}_2(t-d_1,0) + f_1(t,1)$$

$$F_{21}(t) := \overline{A}_2\Big[e^{A_{21}t}x_1(0,1) + \int_0^t e^{A_{21}(t-\tau)}F_{10}(\tau)\,\mathrm{d}\tau\Big] + A_{13}F_{10}(t)$$
$$-\sum_{i=0}^{\nu-1}\Big[A_{04}N^iF_{20}^{(i)}(t) + A_{14}N^iF_{20}^{(i+1)}(t) + B_{24}N^iF_{20}^{(i)}(t-d_1)\Big]$$
$$B_{23}\Big[e^{A_{21}(t-d_1)}x_1(-d_1,1) + \int_0^t e^{A_{21}(t-d_1-\tau)}F_{10}(\tau)\,\mathrm{d}\tau\Big] + B_{03}x_1(t-d_1,0)$$
$$+B_{04}x_2(t-d_1,0) + B_{13}\dot{x}_1(t-d_1,0) + B_{14}\dot{x}_2(t-d_1,0) + f_2(t,1)$$

Continuing this procedure after $k$ steps we obtain

$$x_1(t,k) = e^{A_{21}t}x_1(0,k) + \int_0^t e^{A_{21}(t-\tau)}F_{1,k-1}(\tau)\,\mathrm{d}\tau \tag{15a}$$

$$x_2(t,k) = -\sum_{i=0}^{\nu-1} N^i F_{2,k-1}^{(i)}(t) \tag{15b}$$

where $F_{1,k-1}(t)$, $F_{2,k-1}(t)$ for $k \geq d_2$ satisfy the equations

$$\begin{aligned} F_{1,k}(t) = \overline{A}_1 \Big[e^{A_{21}t}x_1(0,k) + \int_0^t e^{A_{21}(t-\tau)}F_{1,k-1}(\tau)\,\mathrm{d}\tau\Big] + A_{11}F_{1,k-1}(t) \\ -\sum_{i=0}^{\nu-1}\Big[A_{02}N^iF_{2,k-1}^{(i)}(t) + A_{12}N^iF_{2,k-1}^{(i+1)}(t)\Big] + B_{01}x_1(t-d_1,k-d_2) \\ +B_{02}x_2(t-d_1,k-d_2) + B_{11}\dot{x}_1(t-d_1,k-d_2) + B_{12}\dot{x}_2(t-d_1,k-d_2) \\ +B_{21}x_1(t-d_1,k-d_2+1) + B_{22}x_2(t-d_1,k-d_2+1) + f_1(t,k) \end{aligned} \tag{16a}$$

$$\begin{aligned} F_{2,k}(t) := \overline{A}_2 \Big[e^{A_{21}t}x_1(0,k) + \int_0^t e^{A_{21}(t-\tau)}F_{1,k-1}(\tau)\,\mathrm{d}\tau\Big] + A_{13}F_{1,k-1}(t) \\ -\sum_{i=0}^{\nu-1}\Big[A_{04}N^iF_{2,k-1}^{(i)}(t) + A_{14}N^iF_{2,k-1}^{(i+1)}(t)\Big] + B_{03}x_1(t-d_1,k-d_2) \\ +B_{04}x_2(t-d_1,k-d_2) + B_{13}\dot{x}_1(t-d_1,k-d_2) + B_{14}\dot{x}_2(t-d_1,k-d_2) \\ +B_{23}x_1(t-d_1,k-d_2+1) + B_{24}x_2(t-d_1,k-d_2+1) + f_2(t,k) \end{aligned} \tag{16b}$$

Let $P_t := \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$ be a linear operator (map) defined as follows

$$\begin{aligned} P_{11}F_1(t) &:= \overline{A}_1 \int_0^t e^{A_{21}(t-\tau)}F_1(\tau)\,\mathrm{d}\tau + A_{11}F_1(t) \\ P_{12}F_2(t) &:= -\sum_{i=0}^{\nu-1}\Big[A_{02}N^iF_2^{(i)}(t) + A_{12}N^iF_2^{(i+1)}(t)\Big] \\ P_{21}F_1(t) &:= \overline{A}_2 \int_0^t e^{A_{21}(t-\tau)}F_1(\tau)\,\mathrm{d}\tau + A_{13}F_1(t) \\ P_{22}F_2(t) &:= -\sum_{i=0}^{\nu-1}\Big[A_{04}N^iF_2^{(i)}(t) + A_{14}N^iF_2^{(i+1)}(t)\Big] \end{aligned} \tag{17}$$

and $P^k := P \circ P \circ \ldots \circ P$ is the $k$ multiple composition of $P$ (by definition $P^0 := I$ (the identity operator)).

Using (17) and

$$\begin{aligned} h_1(t,k) &:= \overline{A}_1 e^{A_{21}t} x_1(0,k) + B_{01}x_1(t-d_1,k-d_2) \\ &\quad + B_{02}x_2(t-d_1,k-d_2) + B_{11}\dot{x}_1(t-d_1,k-d_2) + B_{12}\dot{x}_2(t-d_1,k-d_2) \\ &\quad + B_{21}x_1(t-d_1,k-d_2+1) + B_{22}x_2(t-d_1,k-d_2+1) + f_1(t,k) \\ h_2(t,k) &:= \overline{A}_2 e^{A_{21}t} x_1(0,k) + B_{03}x_1(t-d_1,k-d_2) \\ &\quad + B_{04}x_2(t-d_1,k-d_2) + B_{13}\dot{x}_1(t-d_1,k-d_2) + B_{14}\dot{x}_2(t-d_1,k-d_2) \\ &\quad + B_{23}x_1(t-d_1,k-d_2+1) + B_{24}x_2(t-d_1,k-d_2+1) + f_2(t,k) \end{aligned} \tag{18}$$

we may write equations (16) in the form

$$\begin{bmatrix} F_{1,k}(t) \\ F_{2,k}(t) \end{bmatrix} = P_t \begin{bmatrix} F_{1,k-1}(t) \\ F_{2,k-1}(t) \end{bmatrix} + \begin{bmatrix} h_1(t,k) \\ h_2(t,k) \end{bmatrix}, \qquad t \in \mathbb{R}_+, \quad k \in \mathbb{Z}_+ \tag{19}$$

It is easy to show that the solution to (19) has the form

$$\begin{bmatrix} F_{1,k}(t) \\ F_{2,k}(t) \end{bmatrix} = P_t^k \begin{bmatrix} F_{10}(t) \\ F_{20}(t) \end{bmatrix} + \sum_{i=0}^{k-1} P_t^{k-i-1} \begin{bmatrix} h_1(t,i+1) \\ h_2(t,i+1) \end{bmatrix}, \qquad k > 0 \tag{20}$$

where $F_{10}(t)$, $F_{20}(t)$ are defined by (10).

In a similar way we may define the operator $P$ and the vectors $h_1(t,k)$, $h_2(t,k)$ for $k < d_2$. Therefore, the following theorem has been proved.

**Theorem 1.** *If the assumption (4) holds then the solution $x(t,k)$ to (3a) with (7) has the form*

$$x(t,k) = Q \begin{bmatrix} e^{A_{21}t}x_1(0,k) + \int_0^t e^{A_{21}(t-\tau)} F_{1,k-1}(\tau)\,\mathrm{d}\tau \\ -\sum_{i=0}^{\nu-1} N^i F_{2,k-1}^{(i)}(t) \end{bmatrix}, \qquad t \in \mathbb{R}_+, \quad k \in \mathbb{Z}_+ \tag{21}$$

*where $F_{1,k}(t)$, $F_{2,k}(t)$ are given by (20).*

Substitution of (21) into (1b) yields the desired general response formula, which able us to find $y(t,k)$ for given $u(t,k)$ and admissible boundary conditions (7).

## 4. Local Reachability and Local Controllability

The local controllability (reachability) of 2-D discrete and continuous linear systems has been considered in many papers (Bergman *et al.*, 1989; Fornasini and Marchesini, 1976; 1978; Idczak and Walczak, 1992; Kaczorek, 1990; 1993b; 1988a; 1994b; Kaczorek and Klamka, 1987; 1986; Klamka, 1994; 1993; Roesser, 1975) and books (Kaczorek, 1993c; Klamka, 1991). In this section the necessary and sufficient conditions for the local reachability and local controllability of the regular system (3a) with $f(t,k) = Cu(t,k)$ will be established.

**Definition 1.** System (3a) with $f(t,k) = Cu(t,k)$ is called locally reachable in the rectangle

$$[h,r] := \{(t,k) \in \mathbb{R}_+ \times \mathbb{Z}_+ : 0 \le t \le h,\ 0 < k \le r\} \tag{22}$$

if for any admissible boundary conditions (7) and every vector $x_f \in \mathbb{R}^n$ there exists $u(t,k)$ for $0 \le t \le h,\ 0 < k \le r-1$ such that $x(h,r) = x_f$.

Using

$$P_t^i := \begin{bmatrix} P_{1,t}^i \\ P_{2,t}^i \end{bmatrix}, \quad i \in \mathbb{Z}_+; \qquad F_0(t) := \begin{bmatrix} F_{10}(t) \\ F_{20}(t) \end{bmatrix}$$

$$h(t,k) := \begin{bmatrix} h_1(t,k) \\ h_2(t,k) \end{bmatrix}, \quad t \in \mathbb{R}_+,\ k \in \mathbb{Z}_+$$

we may write (20) as

$$\begin{aligned} F_{1,k}(t) &= P_{1,t}^k F_0(t) + \sum_{i=0}^{k-1} P_{1,t}^{k-i-1} h(t,i+1) \\ F_{2,k}(t) &= P_{2,t}^k F_0(t) + \sum_{i=0}^{k-1} P_{2,t}^{k-i-1} h(t,i+1) \end{aligned} \qquad k > 0 \tag{23}$$

From (21) we have

$$x(t,k) = Q_1 \left[ e^{A_{21}t} x_1(0,k) + \int_0^t e^{A_{21}(t-\tau)} F_{1,k-1}(\tau)\, d\tau \right] - \sum_{i=0}^{\nu-1} Q_2 N^i F_{2,k-1}^{(i)}(t) \tag{24}$$

where $Q = [Q_1, Q_2]$, $Q_1 \in \mathbb{R}^{n \times n_1}$ and $Q_2 \in \mathbb{R}^{n \times n_2}$.

Substitution of (23) into (24) yields

$$\begin{aligned} x(t,k) &= Q_1 e^{A_{21}t} x_1(0,k) + \int_0^t Q_1 e^{A_{21}(t-\tau)} P_{1,\tau}^{k-1} F_0(\tau)\, d\tau \\ &+ \sum_{i=0}^{k-2} \int_0^t Q_1 e^{A_{21}(t-\tau)} P_{1,\tau}^{k-i-2} h(\tau, i+1)\, d\tau - \sum_{i=0}^{\nu-1} Q_2 N^i \left[ P_{2,t}^{k-1} F_0(t) \right]^{(i)} \\ &- \sum_{i=0}^{\nu-1} \sum_{j=0}^{k-2} Q_2 N^i \left[ P_{2,t}^{k-j-2} h(t, j+1) \right]^{(i)} \qquad k > 1 \end{aligned} \tag{25}$$

From (10) and (18) for $f(t,k) = Cu(t,k)$ we have

$$\begin{aligned} F_0(t) &= PA_0 x(t,0) + PA_1 \dot{x}(t,0) + PB_0 x(t-d_1, -d_2) \\ &+ PB_1 \dot{x}(t-d_1, -d_2) + PB_2 x(t-d_1, 1-d_2) \\ &= F_{bc}(t) + PCu(t,0) \end{aligned} \tag{26a}$$

and

$$h(t,k) = \begin{bmatrix} \overline{A}_1 \\ \overline{A}_2 \end{bmatrix} e^{A_{21}t} x_1(0,k) + PB_0 x(t-d_1, k-d_2) + PB_1 \dot{x}(t-d_1, k-d_2) + PB_2 x(t-d_1, k-d_2+1) = h_{bc}(t,k) + PCu(t,k) \tag{26b}$$

where

$$F_{bc}(t) := PA_0 x(t,0) + PA_1 \dot{x}(t,0) + PB_0 x(t-d_1,-d_2) + PB_1 \dot{x}(t-d_1,-d_2) + PB_2 x(t-d_1, 1-d_2)$$

$$h_{bc}(t,k) := \begin{bmatrix} \overline{A}_1 \\ \overline{A}_2 \end{bmatrix} e^{A_{21}t} x_1(0,k) + PB_0 x(t-d_1, k-d_2) + PB_1 \dot{x}(t-d_1, k-d_2) + PB_2 x(t-d_1, k-d_2+1)$$

Assume that

$$u(t,k) := u_k \quad \text{for} \quad 0 \le t \le h, \ \ 0 \le k \le r \tag{27}$$

where $u_k$ is independent of $t$.

Taking into account (26) and (27) we may write (25) in the form

$$x(t,k) = x_{bc}(t,k) + \int_0^t Q_1 e^{A_{21}(t-\tau)} P_{1,\tau}^{k-1} PC \, d\tau \, u_0 + \sum_{i=0}^{k-2} \int_0^t Q_1 e^{A_{21}(t-\tau)} P_{1,\tau}^{k-i-2} PC \, d\tau \, u_{i+1} - \sum_{i=0}^{\nu-1} Q_2 N^i \left[P_{2,t}^{k-1}\right]^{(i)} PCu_0 - \sum_{i=0}^{\nu-1}\sum_{j=0}^{k-2} Q_2 N^i \left[P_{2,t}^{k-j-2}\right]^{(i)} PCu_{j+1} \tag{28}$$

where

$$x_{bc}(t,k) := Q_1 e^{A_{21}t} x_1(0,k) + \int_0^t Q_1 e^{A_{21}(t-\tau)} P_{1,\tau}^{k-1} F_{bc}(\tau)\, d\tau + \sum_{i=1}^{k-2} \int_0^t Q_1 e^{A_{21}(t-\tau)} P_{1,\tau}^{k-i-2} h_{bc}(\tau, i+1) - \sum_{i=0}^{\nu-1} Q_2 N^i \left[P_{2,t}^{k-1} F_{bc}(t)\right]^{(i)} - \sum_{i=0}^{\nu-1}\sum_{j=0}^{k-2} Q_2 N^i \left[P_{2,t}^{k-j-2} h_{bc}(t, j+1)\right]^{(i)} \tag{29}$$

**Theorem 2.** *System (3a) with $f(t,k) = Cu(t,k)$ is locally reachable in the rectangle (22) if and only if*

$$rank[R_0, R_1, \ldots, R_{r-1}] = n \tag{30a}$$

*or*

$$rank[V_0, V_1, \dots, V_{r-1}] = n \tag{30b}$$

*where*

$$R_{r-i-1} = R(h) := \int_0^h Q_1 e^{A_{21}(h-\tau)} P_{1,\tau}^i PC \, \mathrm{d}\tau - \sum_{j=0}^{\nu-1} Q_2 N^j \left[P_{2,h}^i\right]^{(j)} PC \tag{30c}$$

$$i = 0, 1, \dots, r-1$$

$V_i = V_i(h) := R_i R_i^T$ *and* $T$ *denotes the transposition.*

*Proof.* Using (28) for $t = h$, $k = r$, (29) and $x(h,r) = x_f$ we obtain

$$x_f - x_{bc}(h,r) = \begin{bmatrix} R_0, & R_1, & \dots, & R_{r-1} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{r-1} \end{bmatrix} \tag{31}$$

From (31) it follows that for any admissible boundary conditions (7) and every vector $x_f$ there exists a sequence $u_0, u_1, \dots, u_{r-1}$ if and only if (28) holds. The equivalence of conditions (30a) and (30b) can be shown in a similar way as for 1-D case (Klamka, 1991). ■

**Definition 2.** System (3a) with $f(t,k) = Cu(t,k)$ is called locally controllable in the rectangle (22) if for any admissible boundary conditions (7) there exists $u(t,k)$ for $0 \le t \le h$, $0 \le k \le r-1$ such that $x(h,r) = 0$.

**Theorem 3.** *System (3a) with* $f(t,k) = Cu(t,k)$ *is locally controllable in the rectangle (22) if and only if condition (30a) or (30b) holds.*

*Proof.* Using (28) for $t = h$, $k = r$, (29) and $x(h,r) = 0$ we obtain (31) for $x_f = 0$. From (29) for $t = h$, $k = r$ it follows that for any $x_1(0,k)$ the term $Q_1 e^{A_{21}h} x_1(0,r)$ is an arbitrary $n$-dimensional vector. Therefore, there exists a sequance $u_0, u_1, \dots, u_{r-1}$ satisfying (31) for $x_f = 0$ if and only if (30a) holds. ■

From theorem 2 and theorem 3 we have the following important.

**Corollary.** *For the 2-D continuous-discrete regular system (3a) the local controllability is equivalent to its local reachability.*

In (1994) Klamka has considered the local controllability od 2-D continuous-discrete linear systems under the assumption that $u(t,k) \in \mathbb{L}_2([0,\infty], \mathbb{R}^m)$. The Klamka's approach can also be applied to the 2-D continuous-discrete linear systems with delays.

## 5. Minimum Energy Control

The minimum energy control problem for 2-D discrete linear systems has been considered in many papers (Kaczorek, 1988a; 1990; Kaczorek and Klamka, 1986; 1987;

Klamka, 1983; 1991; 1993). In this section the problem will be extended for regular 2-D continuous-discrete linear systems with delays.

Consider system (3a) with $f(t,k) = Cu(t,k)$ and the performance index

$$I(u) := \sum_{i=0}^{r-1} u_i^T Q u_i \tag{32}$$

where $Q$ is the $m \times m$ symmetric and positive definite weighting matrix.
The minimum energy control problem for system (3a) can be stated as follows. Given the matrices $A_i, B_i,\ i = 0,1,2, C$ of (3a), the weighting matrix $Q$, the numbers $h, r$ and the boundary conditions (7), find a sequence $u_0, u_1, \ldots, u_{r-1}$ defined by (27) which transfers the system to the desired final state $x_f$, $x(h,r) = x_f$ and and minimizes the performance index (32).

To solve the problem we define the matrix

$$W_Q := \sum_{i=0}^{r-1} R_i Q^{-1} R_i^T \tag{33}$$

where $R_i$ is defined by (30c).
It is easy to show that matrix (33) is non-singular if and only if system (3a) is reachable in (22).

We may define the input sequence

$$\widehat{u}_k := Q^{-1} R_k^T W_Q^{-1}(x_f - x_{bc}), \qquad k = 0,1,\ldots,r-1 \tag{34}$$

where $x_{bc}$ is given by (29) for $t = h,\ k = r$.

**Theorem 4.** *Let us assume that*
i) *system (3a) with* $f(t,k) = Cu(t,k)$ *is reachable in rectangle (22),*
ii) $\overline{u}_k$, $k \in [0, r-1]$ *is any input sequence which transfer the system to* $x_f$.
*Then the input sequence (34) accomplishes the same task and*

$$I(\widehat{u}) < I(\overline{u}) \tag{35}$$

*Moreover, the minimum value of (32) is given by*

$$I(\widehat{u}) = (x_f - x_{bc})^T W_Q^{-1}(x_f - x_{bc}) \tag{36}$$

*Proof.* First we shall show that the input sequence (34) provides $x(h,r) = x_f$. Using (28) for $t = h,\ k = r$, (30c), (34) and (33) we obtain

$$x(h,r) = x_{bc} + \sum_{i=0}^{r-1} R_i \widehat{u}_i = x_{bc} + \sum_{i=0}^{r-1} R_i Q^{-1} R_i^T W_Q^{-1}(x_f - x_{bc}) = x_f$$

Since $\overline{u}_i$ and $\widehat{u}_i$, $i \in [0, r-1]$ transfer the system to the same $x_f$, then

$$\sum_{i=0}^{r-1} R_i \overline{u}_i = \sum_{i=0}^{r-1} R_i \widehat{u}_i$$

and

$$\sum_{i=0}^{r-1} R_i[\overline{u}_i - \widehat{u}_i] = 0 \tag{37}$$

From (37) and (34) it follows that

$$\sum_{i=0}^{r-1}[\overline{u}_i - \widehat{u}_i]^T R_i^T W_Q^{-1}(x_f - x_{bx}) = \sum_{i=0}^{r-1}[\overline{u}_i - \widehat{u}_i]^T Q\widehat{u}_i = 0 \tag{38}$$

Using (38) it is easy to show that

$$\sum_{i=0}^{r-1} \overline{u}_i^T Q\overline{u}_i^T = \sum_{i=0}^{r-1} \widehat{u}_i^T Q\widehat{u}_i + \sum_{i=0}^{r-1}[\overline{u}_i - \widehat{u}_i]^T Q[\overline{u}_i - \widehat{u}_i] \tag{39}$$

Inequality (35) holds since the last term in (39) is always non-negative. To obtain the minimum value of (32) we substitute (34) into (32).

$$I(\widehat{u}) = \sum_{i=0}^{r-1} \widehat{u}_i^T Q\widehat{u}_i = \sum_{i=0}^{r-1} \Big[Q^{-1}R_i^T W_Q^{-1}(x_f - x_{bc})\Big]^T Q\Big[Q^{-1}R_i^T W_Q^{-1}(x_f - x_{bc}\Big]$$

$$= \sum_{i=0}^{r-1}(x_f - x_{bc})^T W_Q^{-1} R_i Q^{-1} R_i^T W_Q^{-1}(x_f - x_{bc}) = (x_f - x_{bc})^T W_Q^{-1}(x_f - x_{bc})$$

■

## 6. Concluding Remarks

The general Kurek type and the general Roesser type model of 2-D continuous-discrete linear systems with deviating arguments have been introduced. The solution (21) to the regular model (3a) satisfying condition (4) with admissible boundary conditions (7) has been derived. The necesary and sufficient conditions for the local reachability (theorem 2) and the local controllability (theorem 3) of the regular model (3a) have been established. It has been shown that for regular 2-D continuous-discrete linear systems with delays the local controllability is equivalent to their local reachability. The minimum energy control problem for the regular 2-D continuous-discrete linear systems with delays has been solved (theorem 4). An extension of the theorems for the regular model (1) satisfying condition (4) and n-D $(n > 2)$ case is straightforward. An extension of the above considerations for the singular model (1) which does not satisfy condition (4) is not easy and will be considered in subsequent paper.

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