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ON THE EXISTENCE OF NASH EQUILIBRIUM IN A NON-COOPERATIVE, N-PERSON, LINEAR DIFFERENTIAL GAMES WITH MEASURES AS COEFFICIENTS

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In this paper, the existence of ε-equilibrium in the sense of Nash for linear differential games with measures as coefficients and with fixed time duration is studied. The strategies of the players are understood in a sense similar to Varaiya-Lin strategies. Two cases of payoffs are considered: payoffs dependent on the whole trajectories and terminal payoffs.

1. Introduction

This paper is devoted to the study of the existence of Nash ε-equilibrium for linear differential games of fixed time duration with measures as coefficients. In such games the trajectories are functions of locally bounded variation, which are piecewise continuous only (i.e. not necessarily continuous).

At first, some facts from the theory of linear differential equations with measures as coefficients are presented without proofs.

The main component of the paper consists of two parts. In the first part, the payoff functions depend on the whole trajectories, whereas in the other, the terminal payoffs are considered. The existence of ε-equilibrium is proved in both games by construction of their multistep and discrete approximations being games with complete information. The strategies in the original games and in their approximations are understood in a sense similar to the Varaiya-Lin strategies (Varaiya and Lin, 1969).

This paper generalizes some results included in the papers (Malafyeev, 1974; 1978; 1979; Wyderka and Malafyeev, 1985; 1986; Zaremba, 1982; 1983), where two-person games were studied, and in the papers (Malafyeev, 1980; 1982; Wyderka and Malafyeev, 1991; Zaremba, 1982; 1983), to the case when the coefficients of linear differential equations describing the dynamics of the game are measures. It is also related to the paper (Wyderka and Malafyeev, 1991). Similar problems for games with dynamics described by the generalized dynamical systems were studied e.g. in (Elliott and Kalton, 1972; Malafyeev, 1974; 1978).

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2. Linear Differential Equations with Measures as Coefficients

Here and subsequently, $BV_{loc}(a,b)$ denotes the space of all the right-continuous functions of locally bounded variation in some open interval (a,b). We write $A(\cdot) = \mathrm{d}\mathcal{A}(\cdot)$ for the Lebesgue-Stieltjes measure generated by a function $\mathcal{A}(\cdot) \in BV_{loc}(a,b)$. Let us consider the following linear differential equation

$$\dot{x} = A(t)x + f(t), \qquad x(t_0) = x_0, \qquad x \in \mathbb{R}^n, \quad t \in (a,b) \tag{1}$$

and the corresponding homogeneous equation

$$\dot{x} = A(t)x, \qquad x(t_0) = x_0, \qquad x \in \mathbb{R}^n, \quad t \in (a,b) \tag{2}$$

where the elements of the matrix $A(\cdot)$ are measures and the free term $f(\cdot)$ belongs to $L^1_{loc}(a,b)$.

By the solution of eqn. (1) we mean a function $x(\cdot) \in BV_{loc}(a,b)$ which satisfies the following Lebesgue-Stieltjes integral equation:

$$x(t) = x_0 + \int_{t_0}^{t} \mathrm{d}\mathcal{A}(s)x(s) + \int_{t_0}^{t} f(s)\,\mathrm{d}s, \qquad t \in (t_0, b) \tag{3}$$

or, in the homogeneous case,

$$x(t) = x_0 + \int_{t_0}^{t} \mathrm{d}\mathcal{A}(s)x(s), \qquad t \in (t_0, b) \tag{4}$$

Here and subsequently, the integral $\int_c^d \mathrm{d}\mathcal{A}(s)h(s)$ stands for the integral $\int_{(c,d]} \mathrm{d}\mathcal{A}(s)h(s)$.

By the Lebesgue decomposition theorem, the measure $A(\cdot)$ may be expressed as

$$A(t) = \hat{A}(t) + \sum_{k=1}^{\infty} C_k \delta(t - t_k)$$

where $\hat{A}(\cdot) = \mathrm{d}\hat{\mathcal{A}}(\cdot)$ is the continuous part of the measure $A(\cdot)$, i.e. $\hat{\mathcal{A}}(\cdot) \in BV_{loc}(a,b) \cap C^0(a,b)$ is a continuous function of locally bounded variation, C_k are some $n \times n$ real matrices and $\delta(\cdot)$ denotes Dirac's delta measure.

We will make the following assumptions:

H$_1$) The sequence $\{t_k\}$ of atomic points of the measure $A(\cdot)$ is ordered: $a < t_0 < t_1 < \ldots < t_n < \ldots < b$ and the unique accumulation point of this sequence may be b.

H$_2$) $\det\,(E - C_k) \neq 0$ for $k = 1, 2, \ldots$; E is the identity matrix.

Under the above hypotheses, eqn. (1) has the unique solution $x(\cdot) \in BV_{loc}(a,b)$ and the following Cauchy formula holds:

$$x(t) = \phi(t)x_0 + \phi(t)\int_{t_0}^{t} \phi^{-1}(s)f(s)\,\mathrm{d}s \tag{5}$$

where $\phi(\cdot) \in BV_{loc}(a,b)$ denotes the fundamental matrix of system (2), normed at t_0.

The auxilliary equation

$$\dot{x} = \hat{A}(t)x, \qquad x(t_0) = x_0 \tag{6}$$

has the solution

$$\hat{x}(t) = \hat{\phi}(t)x_0, \qquad \left(\hat{\phi}(t_0) = E\right) \tag{7}$$

such that $\hat{x}(\cdot) \in BV_{loc}(a,b) \cap C^0(a,b)$.

Let us denote by s_k (resp. by ε_k) the value (resp. the jump) of the solution $x(\cdot)$ of eqn. (2) at t_k. The following formulae hold:

$$(E - C_k)s_k = \hat{\phi}(t_k)\hat{\phi}^{-1}(t_{k-1})s_{k-1}, \qquad k = 1,2,\ldots, \qquad s_0 = x_0 \tag{8}$$

$$s_k = (E - C_k)^{-1}x(t_k-), \qquad k = 1,2,\ldots \tag{9}$$

$$\varepsilon_k = (E - C_k)^{-1}C_k x(t_k-), \qquad k = 1,2,\ldots \tag{10}$$

In each interval $[t_{k-1}, t_k)$, the solution $x(\cdot)$ of eqn. (1) is a continuous function which may be written as

$$x(t) = \hat{\phi}(t)\hat{\phi}^{-1}(t_{k-1})s_{k-1}, \qquad t \in [t_{k-1}, t_k) \tag{11}$$

Moreover, if $\hat{A}(\cdot) \equiv 0$, then this solution is a piecewise constant function:

$$x(t) \equiv s_{k-1}, \qquad t \in [t_{k-1}, t_k)$$

The solution of eqn. (1) may be rewritten as

$$x(t) = \hat{\phi}(t)\hat{\phi}^{-1}(t_0)x_0 + \sum_{k:t_k \le t} \hat{\phi}(t)\hat{\phi}^{-1}(t_k)\varepsilon_k H(t - t_k)$$

where $H(\cdot)$ denotes the Heaviside function. If we compute all ε_k as functions of x_0 (by (8) and (10)), and next eliminate x_0 outside the bracket, we obtain construction of the matrix $\phi(\cdot)$.

Summarizing, the solution $x(\cdot)$ of eqn. (1) is a piecewise continuous function of locally bounded variation (therefore $x(\cdot)$ is bounded), which depends continuously on the initial data (t_0, x_0).

Now let us assume that the free term in (1) contains the control parameter $u(\cdot)$, i.e. consider the equation

$$\dot{x} = A(t)x + f(t,u), \quad x(t_0) = x_0, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad t \in (a,b) \tag{12}$$

We make the following assumptions:

C_1) The elements of the matrix $A(\cdot)$ are measures for which the hypotheses H_1 and H_2 are fulfilled.

C_2) The function $f(\cdot,\cdot)$ is of the Caratheodory type, i.e. $f(\cdot,u)$ is measurable and $f(t,\cdot)$ is continuous.

C_3) The set of admissible controls

$$\mathcal{U} = \Big\{ u(\cdot) : \ (a,b) \to U(t) : \ u(\cdot) \text{ is measurable} \Big\}$$

where $U(t)$ is a non-empty, compact set for all $t \in (a,b)$ and the multifunction $t \to U(t)$ is measurable, i.e. for any closed subset $D \subset \mathbb{R}^m$ the set $\{t \in (a,b) : U(t) \cap D \neq \emptyset\}$ is measurable.

C_4) There exists a measurable vector function $\mu(\cdot) \in L^1_{loc}(a,b)$ such that for every $u(\cdot) \in \mathcal{U}$, $|f_i(t,u(t))| \leq \mu_i(t)$ for $t \in (a,b)$, a.e. $i = 1,\ldots,n$.

Let us fix $T > t_0$ and consider the attainable set $\mathcal{K}(T,\mathcal{U})$ for system (12) at the time moment T

$$\mathcal{K}(T,\mathcal{U}) = \phi(T) \left[x_0 + \int_{t_0}^{T} \phi^{-1}(s) f(s,U(s)) \,\mathrm{d}s \right] \tag{13}$$

where the last integral is understood in the sense of Aumann (1965). In (Wyderka, 1980) the following properties of the attainable set were proved.

Theorem 1. *$\mathcal{K}(T,\mathcal{U})$ is a non-empty, compact and convex set in $\mathbb{R}^n$, which continuously depends on x_0 and the attainability multifunction $T \to \mathcal{K}(T,\mathcal{U})$ is continuous from the right in the Hausdorff metric ϱ_H, i.e. if $T_n \to \overline{T}$, $T_n > \overline{T}$, then $\varrho_H(\mathcal{K}(T_n,\mathcal{U}),\mathcal{K}(\overline{T},\mathcal{U})) \to 0$. More precisely, if $\overline{T} \neq t_k$ (for $k = 1,2,\ldots$), then $\mathcal{K}(\overline{T},\mathcal{U})$ is continuous at $\overline{T}$* ∎

For a deeper discussion we refer the Reader to (Wyderka, 1989; 1994).

3. Differential Games with Payoffs Dependent on the Whole Trajectories

Let us consider a differential game of n players, $\{1,\ldots,n\} = I$, in which the dynamics of the i-th player is described by the system of differential equations

$$\begin{aligned} &\dot{x}_i = A_i(t)x_i + f_i(t,u_i), && x_i(t_0) = x_i^0, \quad x_i \in \mathbb{R}^{n_i} \\ &u_i(t) \in U_i(t) \subset \mathbb{R}^{m_i}, && i \in I, \quad t,t_0 \in (a,b) \end{aligned} \tag{14}$$

where the matrices $A_i(\cdot)$ are measures.

Let us assume that, for all $i \in I$, the measures $A_i(\cdot)$, the functions $f_i(\cdot,\cdot)$ and the set $U_i(t)$ satisfy all the assumptions of Section 2.

At each time instant $t \in (a,b)$, all the players know the initial instant t_0 and the final instant T from (a,b), all the positions $\{x_i(t)\}_{i\in I}$ and the dynamics of the game. At time T, each player obtains the payoff

$$H_i(x_1(\cdot),\ldots,x_n(\cdot)), \qquad i \in I \tag{15}$$

where $x_i(\cdot)$ is the trajectory of the system (14) which was realized in the game process by the i-th player $(i \in I)$ and $H_i(\cdot)$, $i \in I$, are given, continuous functionals. The aim of each player is maximization of his payoff. The equilibrium is understood in the sense of Nash (see e.g. Vorobiov, 1984).

It is assumed that the players choose their strategies as similar in spirit to those in the sense of Varaiya-Lin (see e.g. Varaiya and Lin, 1967; 1969; Zaremba, 1982; 1983). Let us denote by $\mathcal{T}_i = \{t_i^1,\ldots,t_i^{k_i}\}$ the set of all atomic points of the measure $A_i(\cdot)$ in the interval $[t_0,T]$ and set

$$\mathcal{T} = \bigcup_{i\in I} \mathcal{T}_i$$

Let us fix some finite partition $\sigma : \{t_0 < t_1 < t_2 < \ldots < t_N = T\}$ of the interval $[t_0,T]$ with the diameter $\sigma = \max_k[t_k - t_{k-1}]$ such that $\mathcal{T} \subset \delta$ and denote by $F_i(x_i^0,T)$ the set of all trajectories $x_i(\cdot)$ of the system (14) on $[t_0,T]$. Moreover, some permutation $P = \{i_1,\ldots,i_n\}$ of the players' set I is also fixed.

By the strategy of the i-th player in the auxilliary game $\Gamma_P^\delta(x_0, T-t_0)$ we mean a non-anticipating operator $\phi_{P,i}^\delta$ of the type

$$\phi_{P,i}^\delta : F_i^*(\cdot) := X_{\substack{j\in I\\ j\neq i}} F_j(\cdot) \to F_i(\cdot)$$

with the following property: if $x_i^*, x_i^{*'} \in F_i^*(\cdot)$ are such that

$$x_{i_l}(\tau) = x'_{i_l}(\tau) \qquad \text{for} \quad \tau \in [t_0,t_k],\ i_l < i$$

and

$$x_{i_l}(\tau) = x'_{i_l}(\tau) \qquad \text{for} \quad \tau \in [t_0,t_{k-1}],\ i_l > i,\ k = 1,\ldots,N$$

then

$$\phi_{P,i}^\delta(x_i^*(\tau)) = \phi_{P,i}^\delta(x_i^{*'}(\tau)) \qquad \text{for} \quad \tau \in [t_0,t_k],\ k = 1,\ldots,N$$

(The next partition σ' is obtained from σ by division of each subinterval $[t_j,t_{j+1}]$ into two equal parts). Let $\varphi_{P,i}^\delta$ be the set of all strategies for the i-th player. It is clear that each situation $\phi_P^\delta = (\phi_{P,i_1}^\delta,\ldots,\phi_{P,i_n}^\delta) \in \varphi_P^\delta := X_{i\in I}\varphi_{P,i}^\delta$ determines uniquely a trajectory $x_I(\phi_P^\delta)$ in the game $\Gamma_P^\delta(\cdot)$ as its outcome (see Varaiya and Lin, 1967; 1969; Zaremba, 1982; 1983). The payoffs in the auxilliary game $\Gamma_P^\delta(\cdot)$ are now defined in a natural way by (15).

Lemma 1. *In the game $\Gamma_P^\delta(\cdot)$, there exists an ε-Nash-equilibrium for every $\varepsilon > 0$, i.e. there exists such a situation $\overline{\phi}_P^\delta = (\overline{\phi}_{P,i_1}^\delta, \ldots, \overline{\phi}_{P,i_n}^\delta)$ for which the following inequalities hold:*

$$H_{i_l}(\overline{\phi}_P^\delta) \geq H_{i_l}(\overline{\phi}_{P,i_1}^\delta, \ldots, \phi_{P,i_l}^\delta, \ldots, \overline{\phi}_{P,i_n}^\delta) - \varepsilon$$

for all $i_l \in I$ and for $\phi_{P,i_l}^\delta \neq \overline{\phi}_{P,i_l}^\delta$.

Proof. The procedure is similar to that of the Zermelo theorem for finite positional games with complete information, because $\Gamma_P^\delta(\cdot)$ is a game with complete information. The proof of the Zermelo theorem may be found in (Parthasarathy and Raghavan, 1971, Thm. 2.5.1) in the case of two-person, finite games and may be simply generalized for n-person games. Next the transfinite induction must be used. ■

Lemma 2. *For a sufficiently small $\eta > 0$ there exists an operator $\Pi_{i,\eta} : F_i(\cdot) \to F_i(\cdot)$, $i \in I$, such that if*

$$x_i, x_i' \in F_i(\cdot), \quad x_i(\tau) = x_i'(\tau) \qquad \text{for} \quad \tau \in [t_0, t]$$

then

$$\Pi_{i,\eta}(x_i)(\tau) = \Pi_{i,\eta}(x_i')(\tau) \qquad \text{for} \quad \tau \in [t_0, t+\eta]$$

and

$$\varepsilon'(\eta) := \sup_{x_i \in F_i(\cdot)} \|x_i - \Pi_{i,\eta}(x_i)\|_i \xrightarrow[\eta \to 0]{} 0$$

where $\|\cdot\|_i$ denotes the norm in the space $BV([0,T) : \mathbb{R}^{n_i})$ — see e.g. (Natanson, 1974).

Proof. The proof is based on the following facts:

a) All trajectories are piecewise-continuous functions of bounded variation which depend continuously on the initial condition $x_i(t_k) = x_i^k$ in the interval $[t_k, t_{k+1})$, independently of whether t_k is or is not an atomic point of the measure $A_i(\cdot)$,

b) $BV([0,T], \mathbb{R}^{n_i})$ is a Banach space and $F_i(\cdot)$ is a compact set in this space. (It is essential that the jump points for every trajectory $x_i(\cdot) \in F_i(\cdot)$ are the same.)

For example, as $\Pi_{i,\eta}(\cdot)$ we may simply choose a translation operator along the trajectory of (14). ■

A simple consequence of Lemma 2 is the following result.

Lemma 3. *For every strategy $\phi_{P,i}^\delta$ such that $i \neq i_1$, $\Pi_{i,\delta}(\phi_{P,i}^\delta)$ is a strategy of the i-th player in the game $\Gamma_{P(i)}^\delta(\cdot)$, where $P(i) = (i, P_{i-1})$ and P_{i-1} denotes a permutation of the set $I \setminus \{i\}$. Moreover,*

$$\sup_{\substack{\phi_{P,i}^\delta \in \varphi_{P,i}^\delta \\ x_i^* \in F_i^*(\cdot)}} \|\phi_{P,i}^\delta(x_i^*) - \Pi_{i,\delta}(\phi_{P,i}^\delta)(x_i^*)\|_i \leq \varepsilon'(\delta) \xrightarrow[\delta \to 0]{} 0$$

and if $\delta_1 < \delta_2$, *then* $\varphi^{\delta_1}_{P,i} \supset \varphi^{\delta_1}_{P(i),i} \supset \varphi^{\delta_2}_{P(i),i}$.

The following lemma is quite obvious.

Lemma 4. *Let a non-cooperative, n-person game in the normal form*

$$\Gamma'_{H'} = \Big\{I, \{\varphi_i'\}_{i\in I}, \quad \{H_i'\}_{i\in I}\Big\}$$

be the image of a game

$$\Gamma_H = \Big\{I, \{\varphi_i\}_{i\in I}, \quad \{H_i\}_{i\in I}\Big\}$$

under some epimorphism

$$\alpha = (\alpha_1, \ldots, \alpha_n), \qquad \alpha_i : \varphi_i \to \varphi_i', \quad i \in I$$

such that $|H_i(\phi) = H_i'(\phi)| \leq \varepsilon$ *for every* $\phi_i \in \alpha_i^{-1}(\phi_i')$. *If there exists an* ε*-equilibrium situation in the game* Γ_H, *then in the game* $\Gamma_{H'}$ *there exists a* 2ε*-equilibrium situation.*

Proof. The proof follows immediately from the definition of the ε-equilibrium and from the assumptions of the lemma. ■

Let us consider now the game

$$\Gamma^\delta(\cdot) = \Big\{I, \{\phi_i = \Pi_{i,\delta}(\varphi^\delta_{P,i})\}_{i\in I}, \quad \{K_i\}_{i\in I}\Big\}$$

where P is a permutation of the set I and the payoff K_1 for the i-th player in the situation

$$\phi^\delta = (\phi_1^\delta, \ldots, \phi_n^\delta) \in \varphi^\delta : X_{i\in I}\varphi_i^\delta$$

is defined in the usual way by (15) as a function of the trajectory $x(\phi^\delta)$ of the game in the situation ϕ^δ with the use of the function $H_i(\cdot)$ — see (15). From Lemmas 2 to 4 we have as a corollary the following result.

Lemma 5. *For every* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that in the game* $\Gamma^\delta(\cdot)$ *the* ε*-equilibrium situation exists and this situation is the image of the same situation for the game* $\Gamma^\delta_P(\cdot)$ *under the operator* $\Pi_\delta := (\Pi_{i_1,\delta}, \ldots, \Pi_{i_n,\delta}) : \varphi^\delta_P \to \varphi^\delta_P$. ■

To return to the differential game of fixed duration, we define this game in the normal form:

$$\Gamma(x^0, T - t_0) = \Big\{I, \{\varphi_i\}_{i\in I}, \quad \{E_i\}_{i\in I}\Big\}$$

The strategy ϕ_i of the i-th player is the pair $(\delta_i, \phi^{\delta_i}_{P(i),i})$ and the payoff for the i-th player in the situation $\phi = (\phi_1, \ldots, \phi_n)$ is equal to

$$E_i(\phi) = H_i\Big(x_I(\phi^{\delta_1}_{P(1),1}, \ldots, \phi^{\delta_n}_{P(n),n})\Big)$$

where $x_I(\cdot) = (x_1(\cdot), \dots, x_n(\cdot))$ is the vector of the players' trajectories, which is an outcome of the situation $(\phi^{\delta_i}_{P(i),i})_{i \in I}$. From Lemma 5 and from the definition of the game $\Gamma(\cdot)$ we obtain the following result.

Theorem 2. *For every $\varepsilon > 0$ in the game $\Gamma(\cdot)$ there exists an ε-equilibrium situation in the class of non-anticipating strategies.* ■

4. Differential Games with Terminal Payoffs

In this section, we study differential games with the same dynamics as previously, but now the payoffs are terminal: at a fixed instant $T > t_0$ the i-th player obtains

$$G_i(x_1(T), \dots, x_n(T))$$

where $G_i(\cdot)$ is a continuous, real function defined on the product $X_{i \in I} \mathcal{K}_i(T, \mathcal{U}_i)$ of the attainable sets and $x(T) = (x_1(T), \dots, x_n(T))$ are the right ends at T of all the trajectories of systems (14) which were realized in the game process. The equilibrium is understood in the sense of Nash, too. The dynamics of the game is described by eqns. (14) and all the previous conditions concerning $A_i(\cdot), f_i(\cdot, \cdot)$, and $\mathcal{U}_i$ are assumed to be fulfilled. The information accessible to each player is also the same as previously.

The strategies are now defined in a slightly different way. Let us introduce the set $\mathcal{T}$ as above, the initial partition $\sigma \subset \mathcal{T}$ with diameter $\delta > 0$ (next partitions are constructed as in the previous game) and some permutation P of the set I; the symbols $F_i(x_i^0, T), F_I(\cdot)$ have the same sense as in Section 3.

By the strategy of the i-th player we mean a non-anticipating operator $\psi^{\delta}_{P,i} : F_i^*(\cdot) \to F_i(\cdot)$ such that if $x^*, x^{*\prime} \in F_i^*(\cdot)$, $x_{i_l}(t_j) = x_{i_l}{}'(t_j)$ for $j = 1, \dots, k$, $k \le N$, $i_l < i$ and if

$$x_{i_l}(t_j) = x_{i_l}{}'(t_j) \qquad \text{for} \quad j = 1, \dots, k-1, \quad k \le N, \quad i_l > i$$

then

$$\psi^{\delta}_{P,i}(x_i^*(\tau)) = \psi^{\delta}_{P,i}(x_i^*(\tau)) \qquad \text{for} \quad \tau \in [t_{k-1}, t_k]$$

Let us denote by $\Psi^{\delta}_{P,i}$ the set of all strategies of the i-th player and define the n-person multistep game with terminal payoff in the normal form:

$$\widetilde{\Gamma}^{\delta}_{P}(x^0, T - t_0) = \left\{ I, \{\Psi^{\delta}_{P,i}\}_{i \in I}, \{G^{\delta}_{P,i}\}_{i \in I} \right\}$$

where $G^{\delta}_{P,i}(\cdot)$ is constructed in the usual way based on the original payoff function $G_i(\cdot)$. The trajectory in the game $\widetilde{\Gamma}_P(\cdot)$ is constructed in the same way as the game $\Gamma^{\delta}_P(\cdot)$, namely, as an outcome of the situation $(\psi^{\delta}_{P,i_1}, \dots, \psi^{\delta}_{P,i_n})$. In contrast to the previous game, the operators $\Pi_{i,\delta}$ for $i \in I$ will be fixed.

We can now state the analogue of Lemma 3.

Lemma 6. *For every strategy $\psi^{\delta}_{P,i}$ such that $i \neq i_1$, the image $\Pi_{i,\delta}(\psi^{\delta}_{P,i})$ is a strategy of the i-th player (in the game $\widetilde{\Gamma}^{\delta}_{P(i)}(\cdot)$) from the set $\Psi^{\delta}_{P(i),i} := \Psi^{\delta}_{i}$. Moreover,*

$$\sup_{\substack{\psi^{\delta}_{P,i} \in \Psi^{\delta}_{P,i} \\ x^*_i \in F^*_i(\cdot)}} \|\psi^{\delta}_{p,i}(x^*_i) - \Pi_{i,\delta}(\psi^{\delta}_{P,i}(x^*_i))\|_i \leq \varepsilon'(\delta) \underset{\delta \to 0}{\longrightarrow} 0$$

and $\Psi^{\delta_1}_{P,i} \supset \Psi^{\delta_1}_{P(i),i} \supset \Psi^{\delta_2}_{P(i),i}$ *if* $\delta_1 < \delta_2$. ■

(The corresponding lemmas, analogous to Lemmas 1, 4 and 5 may be formulated in a similar way and therefore are omitted.)

The original differential game with the terminal payoff $\widetilde{\Gamma}(x^0, T - t_0)$ in the class of non-anticipating strategies is now defined in the following way. The strategy ψ_i of the i-th player is the pair $(\delta_i, \psi^{\delta_i}_{P(i),i})$ and the payoff for the i-th player in the situation $\psi = (\psi_1, \ldots, \psi_n)$ is equal to

$$G_i(\psi) = G_i\left(x(\psi^{\delta_1}_{P(1),1}, \ldots, \psi^{\delta_n}_{P(n),n})\right)$$

where $x(\cdot)$ is the trajectory of the game which is the outcome of the situation ψ.

From definition of the game $\widetilde{\Gamma}(\cdot)$, using a method similar to that used in the previous section we obtain the following result.

Theorem 3. *For every $\varepsilon > 0$ in the game $\widetilde{\Gamma}(\cdot)$ there exists an ε-equilibrium situation in the class of non-anticipating strategies.* ■

Remark. For the approximation $\widetilde{\Gamma}^{\delta}_{P}(\cdot)$ of the game $\widetilde{\Gamma}(\cdot)$ the following reccurence relations of dynamic programming hold in the equilibrium situation:

$$\mathrm{val}\left(\widetilde{\Gamma}^{\delta}_{P}(x^0, T)\right) = \mathrm{val}^P\left[\ \mathrm{val}(\widetilde{\Gamma}^{\delta}_{P}(x^1, T - t_1))\right]$$

$$\vdots$$

$$\mathrm{val}\left(\widetilde{\Gamma}^{\delta}_{P}(x^{N-1}, T - t_{N-1})\right) = \mathrm{val}^P G\left(x^0(T)\right)$$

Here the operator $\mathrm{val}^P W(\cdot)$ denotes the vector of the payoffs for all the players ordered by the permutation P in the game with the payoff vector function $W(\cdot)$ in the equilibrium situation, in which each player chooses the alternative maximizing his payoff.

5. Example

In this section, we illustrate our main results by one example concerning a three-person differential game in a one-dimensional state space with terminal payoffs. Let us consider three one-dimensional control systems which describe the dynamics of the game:

$$\dot{x} = 1.2\delta(t-1)x + u, \qquad x(0) = 0, \quad -1 \leq u \leq 3 \tag{X}$$

$$\dot{y} = -\delta(t-1.5)y + v, \qquad y(0) = 4, \quad -4 \le v \le 0 \tag{Y}$$

$$\dot{z} = 1.25\delta(t-1.9)z + w, \qquad z(0) = -2, \quad 0 \le w \le 5 \tag{Z}$$

for $t \in [0,2]$.

The payoff functions are given, respectively, by

$$H_x\Big(x(2), y(2), z(2)\Big) = |x(2) - y(2)|$$

$$H_y\Big(x(2), y(2), z(2)\Big) = |y(2) - 4|$$

$$H_z\Big(x(2), y(2), z(2)\Big) = |x(2) - z(2)| + |y(2) - z(2)|$$

Each player tends to maximize his payoff.

The emission zones of the systems under consideration are bounded by the following curves:

$$x_{-1}(t) = \begin{cases} -t & \text{for } t < 1 \\ 6 - t & \text{for } t \ge 1 \end{cases}$$

$$x_3(t) = \begin{cases} 3t & \text{for } t < 1 \\ 3t - 18 & \text{for } t \ge 1 \end{cases}$$

$$y_{-4}(t) = 3 - 3t \qquad \text{for } t \in [0,2]$$

$$y_0(t) = \begin{cases} 4 & \text{for } t < 1.5 \\ 2 & \text{for } t \ge 1.5 \end{cases}$$

$$z_0(t) = \begin{cases} -2 & \text{for } t < 1.9 \\ 0.8 & \text{for } t \ge 1.9 \end{cases}$$

$$z_5(t) = \begin{cases} 5t - 2 & \text{for } t < 1.9 \\ 5t - 17 & \text{for } t \ge 1.9 \end{cases}$$

Therefore the corresponding attainable sets are as follows:

$$\mathcal{K}_x(2) = [-12, 4], \quad \mathcal{K}_y(2) = [-3, 4], \quad \mathcal{K}_z(2) = [-7, 0.8]$$

For the above problem there exists Nash equilibrium (not only an ε-equilibrium) point. The optimal (programmed) strategies are given by

$$\overline{u}(\cdot) \equiv 3, \qquad \overline{v}(\cdot) \equiv -3, \qquad \overline{w}(\cdot) \equiv 0$$

They steer the corresponding systems to the following optimal final states:

$$\overline{x}(2) = -12, \qquad \overline{y}(2) = -3, \qquad \overline{z}(2) = 0.8$$

Computation of this Nash equilibrium point is straightforward using the geometric approach. In fact, we note that it is possible to calculate the attainable sets for the corresponding players (for details see (Wyderka, 1994)).

Moreover, the above problem may simply be treated as a hierarchical optimal control one with three dynamical systems and with vector-valued performance index. Indeed, the problem of the (Y)-player is simply an optimal control one consisting in maximization of the reach. The problem of the (X)-player may be studied as another problem of optimal control, namely that of maximization of the final distance to the (Y). Finally, the (Z)-player maximizes the sum of the distances to (X) and to (Y). As is easy to prove, the corresponding optimal final states satisfy these three inequalities which define the Nash equilibrium point.

This example also illustrates some "pathological" situations we meet in control problems for linear systems with measures as coefficients. For example, if the (X)-player tends to maximization of the final distance from the (Y)-player, he must come nearly to (Y) within the initial time interval.

6. Conclusions

In the paper, the existence of Nash ε-equilibrium point for n-person differential games has been proved. The payoffs dependent on the whole trajectories and the terminal ones have been considered. The paper generalizes some well-known results to the case of the games with discontinuous trajectories and with more than two players. Unfortunately, the proof of the existence theorems gives no information about computing Nash ε-equilibrium situations.

The presented example illustrates some "pathological" situations which are "normal" in the study of control or game problems for linear systems with measures as coefficients rather than the method of proving the main theorems, which (in our opinion) is impossible due to the non-constructivity of these proofs.

Such differential game models may be employed in the analysis of situations in a port, when a few ships (e.g. merchantmen or fishing boats) are to go to the jetty with maximal safety, or similar situations in an airport or on the aircraft carrier. Unfortunately, the models are described by differential equations with smooth, continuous or integrable coefficients.

Differential systems with measures as coefficients describe some practical problems, among others, in radiation of electrical or magnetical waves in two different media with a common part of boundary (Friedmann, 1956), in classical (Dzyra and Ishchuk, 1976) or quantum mechanics (Babikov, 1968; Gotfried, 1966) or in optimization of cosmic manoeuvres (Ivashkin, 1975); this last domain of possible applications with a great number of participants is close to the problems studied in this paper.

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