Appl. Math. and Comp. Sci., 1996, vol.6, No.1, 27–32

STABILIZATION OF LINEAR DESCRIPTOR SYSTEMS BY STATE-FEEDBACK CONTROLLERS

TADEUSZ KACZOREK*

The necessary and sufficient conditions are established for the existence of a solution to the stabilization problem for linear continuous-time descriptor systems. A new procedure is presented for designing a state-feedback controller such that the closed-loop system is asymptotically stable and its response is impulse-free.

1. Introduction and Problem Formulation

The pole assignment, stabilization and regularisation of descriptor linear systems by state and output feedbacks have been considered in many papers and books (see References). Most results have been obtained for regular descriptor systems. In this note, the following stabilization problem for linear, not necessarily regular descriptor systems will be solved. Consider the linear continuous-time descriptor system

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the semistate vector, $u(t) \in \mathbb{R}^m$ is the control (input), $y(t) \in \mathbb{R}^p$ is the output, and $E,\ A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ with $\mathbb{R}^{p\times q}$ being the set of $p\times q$ real matrices. The problem is to find a state-feedback controller

$$u = -Kx, \qquad K \in \mathbb{R}^{m\times n} \tag{3}$$

such that

i) the response of the closed-loop system

$$E\dot{x}(t) = (A - BK)x(t) \tag{4}$$

is impulse-free, and

ii) $\lim_{t\to\infty} x(t) = 0$ for all initial conditions Ex_0.

The necessary and sufficient conditions for the existence of a solution to this problem will be established.

* Warsaw University of Technology, Institute of Control and Industrial Electronics, ul. Koszykowa 75, 00–662 Warsaw, Poland

Proof. Substituting

$$u = -Kx = -[K_1\, K_2]N^{-1}x = -[K_1\, K_2]\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

into (5) and (6) we obtain

$$\dot{z}_1 = \bar{A}_{11}z_1 + \bar{A}_{12}z_2 \tag{15}$$

$$0 = \bar{A}_{21}z_1 + \bar{A}_{22}z_2 \tag{16}$$

The response of (15)–(16) is impulse-free since $\bar{A}_{22}$ is non-singular (Cobb, 1983; Christodolou, 1988; Dai, 1989).

Substitution of $z_2 = -\bar{A}_{22}^{-1}\bar{A}_{21}z_1$ into (15) yields

$$\dot{z}_1 = (\hat{A}_{11} - \hat{B}_1K_1)z_1 \tag{17}$$

where $\hat{A}_{11}$ and $\hat{B}_1$ are defined by (11). Then $\lim_{t\to\infty} z_1(t) = 0$ and also $\lim_{t\to\infty} z_2(t) = 0$. Hence, by $x(t) = N\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$, we have $\lim_{t\to\infty} x(t) = 0$. ∎

Therefore we have proved the following result.

Theorem 3. *The stabilization problem has a solution if and only if the system (1)–(2) is stabilizable and condition (9) is satisfied.*

Remark 1. If the system (1)–(2) is regular $(\det[Es - A] \neq 0)$, then condition (9) is satisfied if and only if all its impulse modes are controllable (or, equivalently, the system is controllable at infinity) (Kaczorek, 1992).

Remark 2. From the above considerations it follows that there exist many matrices $K = [K_1, K_2]$ which are solutions to the stabilization problem (satisfying conditions (i) and (ii)) of the system (1)–(2). The freedom can be used to satisfy additional requirements imposed on the dynamics of the closed-loop system (4).

Example. *Consider eqn. (1) with*

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Note that the equation has already the form (5) and (6) with $r = 1$, $A_{11} = 1$, $A_{12} = A_{21} = 2$, $A_{22} = 0$, $B_1 = 2$, $B_2 = 1$.

Conditions (9) and (13) are satisfied since $\operatorname{rank} B_2 = 1$ and

$$\operatorname{rank}[Es - A, B] = \operatorname{rank}\begin{bmatrix} s-1 & -2 & 2 \\ -2 & 0 & 1 \end{bmatrix} = 2 \quad \text{for all} \quad \operatorname{Re}(s) \geq 0$$

Therefore the stabilization problem has a solution.

Choosing $K_2 = k_2 \neq 0$ we obtain $\bar{A}_{22} = A_2 - B_2K_2 = -k_2$, $\bar{A}_{12} = A_{12} - B_1K_2 = 2(1-k_2)$ and

$$\hat{A}_{11} = A_{11} + \bar{A}_{12}\bar{A}_{22}^{-1}A_{21} = 5 - \frac{4}{k_2}, \quad \hat{B}_1 = B_1 - \bar{A}_{12}\bar{A}_{22}^{-1}B_2 = \frac{2}{k_2}$$

Hence, for $K_1 = k_1$, we have $\hat{A}_{11} - \hat{B}_1K_1 = 5 - \frac{4}{k_2} - \frac{2k_1}{k_2} < 0$ for $k_1 > \frac{5}{2}k_2 - 2$ if $k_2 > 0$ and $k_1 < \frac{5}{2}k_2 - 2$ if $k_2 < 0$. The desired state-feedback controller has the form $u = -k_1x_1 - k_2x_2$.

3. Concluding Remarks

The necessary and sufficient conditions for the existence of a solution to the stabilization problem for linear continuous-time descriptor systems have been established. A new procedure has been presented for designing a state-feedback controller such that the closed-loop system is asymptotically stable and its response is impulse-free.

With slight modifications the presented approach can also be used for linear discrete-time descriptor systems.

References

Bender D.J. and Laub A.J. (1987): *The linear-quadratic optimal regulator for descriptor systems.* — IEEE Trans. Automat. Contr., v.AC-32, No.8, pp.672–688.

Bunse-Gerstner A., Mehrmann V. and Nichols N.K. (1992): *Regularization of descriptor system by derivative and proportional state feedback.* — SIAM J. Matrix Analysis Appl., v.13, No.1, pp.46–67.

Bunse-Gerstner A., Mehrmann V. and Nichols N. K. (1994): *Regularization of descriptor systems by output feedback.* — IEEE Trans. Automat. Contr., v.39, No.8, pp.1742–1748.

Cobb D. (1983): *Descriptor variable systems and optimal state regulation.* — IEEE Trans. Automat. Contr., v.AC-28, No.5, pp.601–611.

Christodolou M.A. (1988): *Pole placement for generalized MDFÆS.* — Kybernetika, v.24, No.2, pp.98–109.

Dai L. (1989): *Singular Control Systems*, (M. Thoma and A. Wyner, Eds.). — Lecture Notes in Control and Information. Sciences Berlin: Springer-Verlag.

Fletcher L.R., Kautsky J. and Nichols N.K. (1986): *Eigenstructure assignment in descriptor systems.* — IEEE Trans. Automat. Contr., v.AC-31, No.12, pp.1138–1141.

Kaczorek T. (1992): *Linear Control Systems, v.1.* — New York: Research Studies Press, J. Wiley.

Kautsky J., Nichols N.K. and Chu E.K.-W. (1989): *Robust pole assignment in singular control systems.* — Lin. Alg. Appl., v.121, No.1, pp.9–37.

Ozcaldiran K. and Lewis F.L. (1990): *On regularizability of singular systems.* — IEEE Trans. Automat. Contr., v.35, No.12, pp.1156–1160.

Varga V. (1993): *Computational methods for stabilization of descriptor systems.* — TR R103-93, DLR – Oberpffafenholen, Institute for Robotics and System Dynamics.

Verghese G.C., Levy B.C. and Kailath T. (1981): *A general state space for singular systems.* — IEEE Trans. Automat. Contr., v.AC-26, No.4, pp.811–831.

Xu H. and Mizukami K. (1992): *On linear-quadratic optimal regulator for continuous-time descriptor systems.* — Proc. 31st IEEE Conf. Dec. and Control, v.2, pp.987–988.

Xu H. and Mizukami K. (1993): *Hamilton-Jacobi equation for descriptor systems.* — System & Control Letters, v.21, No.2, pp.321–327.

Appendix

The matrix equation

$$A_{22} - B_2 K_2 = \bar{A}_{22} \tag{A.1}$$

has a solution $K_2 \in \mathbb{R}^{m\times(n-r)}$ *for given* $A_{22} \in \mathbb{R}^{(n-r)\times(n-r)}$ *and* $B_2 \in \mathbb{R}^{(n-r)\times m}$ *such that* $\bar{A}_{22}$ *is non-singular if and only if*

$$\operatorname{rank}[A_{22}, B_2] = n - r \tag{A.2}$$

Proof. From (A.1) rewritten as

$$A_{22} - B_2 K_2 = [A_{22}, B_2] \begin{bmatrix} I_{n-r} \\ -K_2 \end{bmatrix} \tag{A.3}$$

it follows that $\bar{A}_{22}$ is non-singular only if (A.2) holds.

To prove the sufficiency, we may assume without loss of generality that A_{22} and B_2 have been transformed to the form

$$A_{22} = \begin{bmatrix} A'_{22} & A''_{22} \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix} \tag{A.4}$$

where A'_{22} is square with $\det A'_{22} \neq 0$.

Choosing K_2 in the form $K_2 = [0\ K'_2]$ we obtain

$$A_{22} - B_2 K_2 = \begin{bmatrix} A'_{22} & A''_{22} - B_{21}K'_2 \\ 0 & B_{22}K'_2 \end{bmatrix}$$

Note that if (A.2) holds, then B_{22} has full row rank and there exists K'_2 such that $B_{22}K'_2$ is non-singular. ■

Received: June 6, 1995