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ROBUSTNESS ANALYSIS FOR LINEAR TIME-INVARIANT SYSTEMS WITH STRUCTURED INCREMENTALLY SECTOR BOUNDED FEEDBACK NONLINEARITIES

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This paper addresses the stability analysis of a negative feedback interconnection of a multivariable linear time-invariant system and a structured time-invariant incrementally sector bounded nonlinearity. The classic Zames-Falb multiplier is extended to the multivariable case and is approximated by linear matrix inequalities. The problem of finding the multiplier that provides the largest stability bound then becomes a convex optimization problem over state space parameters. The method is also applied to symmetric incrementally sector bounded structured nonlinearities and provides an upper bound for the generalized structured singular value. Numerical examples are provided to demonstrate the effectiveness of this method.

1. Introduction

The structured singular value (SSV or μ) was introduced in (Doyle, 1982) to characterize robust stability of a linear time invariant (LTI) system subject to complex LTI structured uncertainties. As the complex uncertainty description is in general too conservative for real parametric uncertainties (Fan *et al.*, 1991), the μ framework was extended to the mixed real and complex uncertainties (the mixed μ problem) (Fan *et al.*, 1991; Young, 1993). The μ theory was further generalized in (Krause *et al.*, 1988) to allow for nonlinear/time-varying (NLTV) nominal systems and uncertainties; this result has, in particular, been applied to the case of nominally LTI systems subject to $\mathcal{L}_2$ induced norm bounded NLTV uncertainties. An upper robustness bound can be computed by using constant diagonal scalings. However, as in the linear case, norm bounded nonlinear/time varying perturbations are often a conservative characterization.

In this paper, we consider the stability of a linear multivariate time invariant system, T, connected to a real, structured (i.e., block diagonal) nonlinear time invariant (NLTI) uncertainty, Δ, in a negative feedback configuration. Each diagonal

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block in Δ is assumed to be incrementally sector bounded and possibly odd. In the single-input/single-output (SISO) case, stability condition of such systems can be obtained via graphical criteria such as the Popov criterion, circle criterion, and off-axis circle criterion (Narendra and Taylor, 1973). However, these criteria often give conservative stability conditions. Using an operator approach, Zames and Falb (1965) introduced a much more general class of multipliers which includes the graphical criteria mentioned above. In (Safonov and Wyetzner, 1987), an optimal multiplier method was first proposed as an infinite-dimensional linear programming problem. An optimal multiplier method involving a sequence of approximate finite dimensional optimization was proposed in (Wen and Chen, 1990), but the optimization problem is not convex. A concave nonlinear programming approach was proposed in (Gapski and Geromel, 1994). Robust stability for multivariable systems with diagonal differentiable monotone and odd monotone nonlinearities has been studied using a combination of Popov and RC multipliers (Haddad *et al.*, 1992; How, 1993; Narendra and Neuman, 1967). This approach results in a nonconvex parameterization and no systematic method has been given to find the best robust stability bound within this class. Currently, no efficient computational method exists for the stability analysis of structured monotone (or odd monotone) nonlinearities.

Noting the recent numerical advance in solving convex optimization over linear matrix inequalities (LMIs), we show in this paper that a subclass of the Zames-Falb multipliers, and the generalization to the multivariable case, can be characterized by linear matrix inequalities (LMIs). Convex optimization can then be applied to fine the optimal multiplier (the one that provides the best robustness bound) within this class. This method can be further extended to the odd monotone case.

In the special case of symmetric incrementally sector bounded structured monotone and odd monotone nonlinearities, our method also produces a less conservative upper bound for the generalized structured singular value (GSSV) (Krause *et al.*, 1988). Since our approach is state space based and no frequency domain searching is involved, the method is computationally very attractive and can be combined with recently developed LMI based optimal multiplier methods for linear complex and real structured uncertainties (Balakrishnan *et al.*, 1994; Ly *et al.*, 1994) and norm bounded NLTV perturbations (Balakrishnan *et al.*, 1994) to provide a unified state space LMI optimization framework for robustness analysis of structured uncertainties.

The rest of the paper is organized as follows. Section 2 reviews and generalizes the multiplier class considered in (Zames and Falb, 1968) to system with block diagonal monotone or odd monotone nonlinearities. In Section 3, we parameterize a subclass of multipliers for monotone or odd monotone nonlinearities based on LMIs. We also show that in the case of monotone nonlinearities, the LMI formulation can be made arbitrarily close to the general multiplier class. In Section 4, we formulate the problem of finding the optimal multiplier and the largest robust stability bound as a convex optimization problem. Numerical examples are provided in Section 5.

2. Robustness Analysis

2.1. Review of SISO Case

The multiplier method was first developed in the 60's for scalar systems connected with a feedback nonlinearity (Zames and Falb, 1968). As shown in Fig. 1, the forward system, denoted by T, is linear time invariant, and the feedback, Δ, is a time-invariant nonlinearity and belongs to an incrementally sector bounded $[\alpha, \beta+\alpha]$, which means, $\Delta(0) = 0$ and

$$\alpha \leq \frac{\Delta(y_1) - \Delta(y_2)}{y_1 - y_2} \leq \beta + \alpha, \qquad \forall y_1, y_2 \in \mathbb{R}$$

If $\alpha = 0$, Δ is called a monotone nonlinearity. If Δ is monotone and satisfies $\Delta(-y) = -\Delta(y)$, $\forall y \in \mathbb{R}$, it is called an odd monotone nonlinearity.

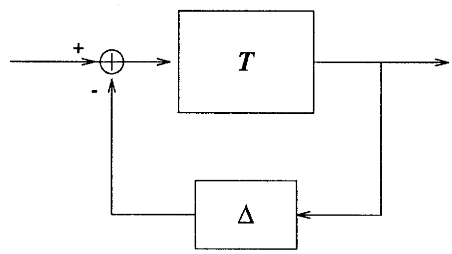

Fig. 1. Interconnected system.

Using a loop transformation (Vidyasagar, 1993), $\Delta \in \text{sector}[\alpha, \beta]$ can be converted to $\widehat{\Delta} \triangleq (I - (\Delta - \alpha)(\frac{1}{\beta}))^{-1}(\Delta - \alpha)$ which is in the $[0, \infty)$ sector. With the loop transformation, a sufficient condition for the global asymptotic stability of the zero equilibrium of the interconnected system is that the transformed forward system

$$\hat{T} \triangleq \frac{T}{1 + \alpha T} + \frac{1}{\beta}$$

is strictly positive real (Desoer and Vidyasagar, 1975). The graphical interpretation of this condition leads to the circle criterion.

The feedforward system and feedback systems can be further modified, without affecting the overall interconnections, by linear, but possibly noncausal, systems Z and $1/Z$, respectively. If Z is chosen such that $\widehat{\Delta}(\frac{1}{Z} \cdot y)$ remains monotone, then $Z \cdot \hat{T}$ being strictly positive real is a sufficient condition for stability. Such Z's are called multipliers (Desoer and Vidyasagar, 1975). Several well-known graphical stability criteria, listed below, were obtained using this approach (Safonov and Wyetzner, 1987)

1. Circle criterion: $Z = 1$;
2. Popov criterion: $Z = 1 + j\omega q$, $q > 0$;
3. Off-axis circle criterion: $Z = e^{j\theta}$, $\theta \in (-\pi/2, \pi/2)$.

In (Zames, 1968; Zames and Falb, 1968), the authors showed that there exists a much broader class of multipliers for the case of monotone feedback uncertainty $\widehat{\Delta}$:

$$Z(j\omega) = z_0 - \int_{-\infty}^{\infty} z(t)e^{-j\omega t}\,\mathrm{d}t \tag{1}$$

$$\int_{-\infty}^{\infty} |z(t)|\,\mathrm{d}t < z_0 \tag{2}$$

$$z(t) \geq 0 \quad \text{for all } t \in \mathbb{R} \quad \text{or } \widehat{\Delta}(\cdot) \text{ is odd} \tag{3}$$

where z_0 is any positive constant.

2.2. Extension to MIMO Case

We now generalize the classic multiplier approach to multivariable systems with structured nonlinearities from the following class:

$$\begin{aligned}\mathbf{\Delta} = \Big\{ & \Delta : \Delta = \mathrm{diag}\{\Delta^{(1)}, \Delta^{(2)}, \cdots, \Delta^{(p)}\}, \Delta^{(k)} : \mathbb{R}^{m_k} \to \mathbb{R}^{m_k}, \\ & \left\|\Delta^{(k)}(y)\right\| \leq M\,\|y\| \text{ for some } M > 0 \text{ and for all } y \in \mathbb{R}^{m_k} \\ & \Delta^{(k)}(0) = 0, \left(\Delta^{(k)}(y_1) - \Delta^{(k)}(y_2)\right)'(y_1 - y_2) \geq 0 \quad \text{for all } y_1, y_2 \in \mathbb{R}^{m_k} \Big\}\end{aligned} \tag{4}$$

If, in addition, each diagonal $\Delta^{(k)}$ is odd, $\Delta^{(k)}(-y) = -\Delta^{(k)}(y)$, then we say $\Delta \in \mathbf{\Delta}_o$. Note that by definition, Δ is of finite L_2 gain.

Define the following classes of multivariable multipliers:

Definition 1. Given $(m_1, \ldots, m_k)$, an LTI transfer matrix Z belongs to the multiplier class $\mathcal{Z}$ if $Z = \mathrm{diag}\{Z^{(1)}I_{m_1}, \cdots, Z^{(p)}I_{m_p}\}$, $Z^{(k)}$ is a scalar LTI transfer function that satisfies

1. $$Z^{(k)}(j\omega) = z_0^{(k)} - \int_{-\infty}^{\infty} z^{(k)}(t)e^{-j\omega t}\,\mathrm{d}t \tag{5}$$

2. $$\int_{-\infty}^{\infty} \left|z^{(k)}(t)\right|\,\mathrm{d}t < z_0^{(k)} \tag{6}$$

3. $$z^{(k)}(t) \geq 0 \qquad \text{for all } t \in \mathbb{R} \tag{7}$$

An LTI transfer matrix Z belongs to the multiplier class $\mathcal{Z}_o$, if Z satisfies conditions 1–2 above.

As in the SISO case, $\mathcal{Z}$ is applicable to the case where $\Delta \in \mathbf{\Delta}$ and $\mathcal{Z}_o$ is applicable to the case where $\Delta \in \mathbf{\Delta}_o$.

We also need the following definition related to positive realness.

Definition 2. An exponentially stable LTI transfer matrix T is extendedly strictly positive real (ESPR) if there exists $\eta > 0$ such that for all $\omega \in \mathbb{R}$,

$$T(j\omega) + T'(-j\omega) \geq \eta I$$

The following theorem summarizes the generalization of the multiplier stability analysis criterion to MIMO systems.

Theorem 1. *Consider the interconnected system in Fig. 1 where T is an exponentially stable LTI transfer matrix with a minimal state space realization (A, B, C, D):*

$$\dot{x} = Ax + Bu, \qquad x(0) = x_0$$

$$y = Cx + Du$$

$$u = -\Delta(y)$$

and $x \in \mathbb{R}^n$, u and y belong to $\mathbb{R}^m$. Assume that a unique solution exists for all t.

Suppose that one of the following conditions is true:

1. $\Delta \in \mathbf{\Delta}$ *and there exists $Z \in \mathcal{Z}$ such that ZT is ESPR,*
2. $\Delta \in \mathbf{\Delta}_o$ *and there exists $Z \in \mathcal{Z}_o$ such that ZT is ESPR.*

Then the zero equilibrium of the interconnected system is globally asymptotically stable.

Following the standard argument on the existence and uniqueness of solutions of ordinary differential equations, a sufficient condition for the existence of a unique solution is that Δ is globally Lipschitz with Lipschitz constant L, and $L\|D\| < 1$.

To prove the theorem, we need the following lemma which shows that cascading Δ with a multiplier Z remains passive. Note that $\langle \cdot, \cdot \rangle$ denotes the L_2 inner product. The proof is given in Appendix A.

Lemma 1. *If $\Delta \in \mathbf{\Delta}$ and $Z \in \mathcal{Z}$, or $\Delta \in \mathbf{\Delta}_o$ and $Z \in \mathcal{Z}_o$, then*

$$\langle v, \Delta Z^{-1} v \rangle \geq 0, \quad \forall v \in \mathcal{L}_2^m(\mathbb{R}), \qquad m = \sum_{k=1}^{p} m_k \tag{8}$$

Proof of Theorem 1. Given $Z \in \mathcal{Z}$ or $\mathcal{Z}_o$, there exist $Z_+ = \mathrm{diag}(Z_+^{(1)}, \cdots, Z_+^{(p)})$ and $Z_- = \mathrm{diag}(Z_-^{(1)}, \cdots, Z_-^{(p)})$, with $Z_+^{(k)}, (Z_+^{(k)})^{-1}, Z_-^{(k)^*}, (Z_-^{(k)^*})^{-1}$ causal and of finite gain, such that $Z = Z_- Z_+$ (Desoer and Vidyasagar, 1975; Zames and Falb, 1968). By Lemma 1, ΔZ^{-1} is passive, and by assumption, ZT is ESPR. It follows that $Z_-^* \Delta Z_+^{-1}$ is passive and $Z_+ T (Z_-^*)^{-1}$ is ESPR and finite gain. By the Passivity Theorem (Desoer and Vidyasagar, 1975), the interconnected system is L_2 stable. With no exogenous input, the output y belongs to L_2 (from L_2 stability). Since Δ is of finite L_2 gain, u also belongs to L_2. It follows from the exponential stability of T that $x(t) \to 0$ as $t \to \infty$. ■

The following corollary strengthens the result to exponential stability if Z is finite-dimensional (i.e., $Z(s)$ is a rational transfer function).

Lemma 3. (Szego, 1975) *If $f(t) \in \mathcal{L}_1[0,\infty)$, then for every $\epsilon > 0$, there exists a vector $(a_0, a_1, \cdots, a_N) \in \mathbb{R}^{N+1}$ such that*

$$\int_0^\infty \left| f(t) - \sum_{i=0}^N a_i e^{-t} t^{(i+\alpha)} \right| dt < \epsilon$$

where $\alpha > -1$.

Replacing t by $-t$, the same result also holds for $\mathcal{L}_1(-\infty, 0]$.

By choosing $\alpha = 0$, we obtain a basis for the approximation: $e_i^+(t) \triangleq e^{-t}t^i$, $t \geq 0$, (zero for $t \leq 0$), $e_i^-(t) \triangleq e^t t^i$, $t \leq 0$ (zero for $t \geq 0$). Then an N-th order approximation of an $\mathcal{L}_1$ function $z(t)$ is

$$z_N(t) = \sum_{i=0}^N \left(a_i e_i^+(t) + b_i e_i^-(t) \right) \tag{11}$$

There are infinitely many choices of α that would render the Laplace transform of z_N rational. We make the arbitrary choice of $\alpha = 0$.

Define $\mathcal{Z}$ and $\mathcal{Z}_o$ subject to the above N-th order approximation as $\mathcal{Z}_N$ and $\mathcal{Z}_{oN}$, respectively. From Lemma 3, it is clear that $\mathcal{Z}_{oN}$ can be made arbitrarily close to $\mathcal{Z}_o$ by choosing N sufficiently large. However, in $\mathcal{Z}$, $z^{(k)}(t)$ is required to be non-negative in addition to being L_1. The approximate $z_N^{(k)}$ would be close to $z^{(k)}$ in the L_1 norm but the pointwise non-negativity may be violated. Therefore, in using $\mathcal{Z}_N$, the result may be conservative even for large N.

3.3. Monotone Nonlinearities

In this section, we show that $\mathcal{Z}_N$ can be equivalently parameterized by LMIs. The basic strategy is to simply substitute (11) into each $z^{(k)}(t)$ in (5)–(7) and show the resulting expression is linear in the constants a_i's and b_i's.

First replace $z^{(k)}(t)$ by its N-th order approximation:

$$z^{(k)}(t) = \sum_{i=0}^N \left(a_i^{(k)} e_i^+(t) + b_i^{(k)} e_i^-(t) \right) \tag{12}$$

where $a_i^{(k)}, b_i^{(k)} \in \mathbb{R}$, $k = 1, \cdots, p$.

The first condition in Definition 1, (5), can be written as a linear function of constants $z_0^{(k)}$, $a_i^{(k)}$'s and $b_i^{(k)}$'s:

$$Z^{(k)}(j\omega) = z_0^{(k)} - \sum_{i=0}^N \left(\frac{a_i^{(k)}}{(j\omega+1)^{i+1}} - \frac{b_i^{(k)}}{(j\omega-1)^{i+1}} \right) i! \tag{13}$$

The second condition in Definition 1, (6), can be written as an LMI by direct integration:

$$\sum_{i=0}^{N} \left(a_i^{(k)} + (-1)^i b_i^{(k)} \right) i! < z_0^{(k)} \tag{14}$$

Note that the absolute value in (6) can be removed since $z^{(k)}(t) \geq 0$ by (7).

The last condition in Definition 1, (7), can be written as

$$\sum_{i=0}^{N} a_i^{(k)} e^{-t} t^i \geq 0, \quad \forall t \geq 0 \quad \text{and} \quad \sum_{i=0}^{N} b_i^{(k)} e^{t} t^i \geq 0, \quad \forall t \leq 0$$

$$\iff \sum_{i=0}^{N} a_i^{(k)} t^i \geq 0 \quad \text{and} \quad \sum_{i=0}^{N} (-1)^i b_i^{(k)} t^i \geq 0, \quad \forall t \geq 0$$

$$\iff \sum_{i=0}^{N} a_i^{(k)} \omega^{2i} \geq 0 \quad \text{and} \quad \sum_{i=0}^{N} (-1)^i b_i^{(k)} \omega^{2i} \geq 0, \quad \forall \omega \in \mathbb{R}$$

$$\iff \sum_{i=0}^{N} a_i^{(k)} (-1)^i s^{2i} \geq 0 \quad \text{and} \quad \sum_{i=0}^{N} b_i^{(k)} s^{2i} \geq 0, \quad \forall s = j\omega$$

$$\iff \frac{\sum_{i=0}^{N} a_i^{(k)} (-1)^i s^{2i}}{(-s+1)^N (s+1)^N} \geq 0 \quad \text{and} \quad \frac{\sum_{i=0}^{N} b_i^{(k)} s^{2i}}{(-s+1)^N (s+1)^N} \geq 0, \ \forall s = j\omega$$

Finding conditions on a_i to ensure $\sum_{i=0}^{N} a_i t^i \geq 0$, $\forall t \geq 0$, is a classic problem. For example, the case of $N = 3$ is completely worked out in (Jury, 1974, pp.149–152). The resulting condition is rather involved and the approach becomes intractable for $N > 3$. In (Siljak, 1989), a general procedure involving the Modified Routh Array is proposed. This result also leads to complicated algebraic conditions even for a small N. In contrast, the approach here leads to a simple state space LMI test. In addition to the application to the multiplier problem, this result is important in its own right.

Based on the above, we now have an equivalent LMI characterization of $\mathcal{Z}_N$:

Theorem 2. *Given a transfer matrix*

$$Z(j\omega) = \operatorname{diag}\left\{ Z^{(1)}(j\omega) I_{m_1} \cdots, Z^{(p)}(j\omega) I_{m_p} \right\}$$

For the case where T is SISO, we can set $z_0 = 1$ (dividing (1)–(3) by z_0). Let $\lambda = 1/\beta$, then C_{ZT} and D_{ZT} are affine in λ. We can now minimize λ subject to the LMIs in Theorem 3 directly and the problem becomes a linear objective problem over LMIs (Boyd *et al.*, 1993; Gahinet and Nemirovskii, 1993). In this case, no iteration on β is needed. Theorem 3 can be applied repeatedly for increasing N's until β converges.

4.2. Nonlinearity with Symmetric Sector Bounds

We now consider the symmetric sector bound case, i.e., $-\alpha_k = \beta_k + \alpha_k = \beta$, for $k = 1, \cdots, p$. A loop transformation $\widehat{\Delta} = (\beta I + \Delta)(\beta I - \Delta)^{-1}$ converts Δ to the $[0, \infty)$ sector and $\widehat{\Delta}$ remains block diagonal and monotone (or odd monotone if Δ is odd). The transformed forward system then becomes

$$\widetilde{T} = (I + \beta T)(I - \beta T)^{-1} \tag{29}$$

The condition for closed loop global exponential stability becomes: $\widetilde{T}$ is exponentially stable and

$$Z(j\omega)\widetilde{T}(j\omega) \text{ is ESPR} \tag{30}$$

for some $Z \in \mathcal{Z}_N$ (or $Z \in \mathcal{Z}_{oN}$ if Δ is odd).

Write a controllable state space realization of the N-th order approximation of Z as in (28) and a minimal realization for $\widetilde{T}$ as

$$\widetilde{T}(s) \sim \left[\begin{array}{c|c} A_T & B_T \\ \hline C_T & D_T \end{array}\right] = \left[\begin{array}{c|c} A + \beta B(I - \beta D)^{-1}C & 2B(I - \beta D)^{-1} \\ \hline (I - \beta D)^{-1}\beta C & (I + \beta D)(I - \beta D)^{-1} \end{array}\right] \tag{31}$$

where (A, B, C, D) is a minimal realization for $T(s)$. Then a controllable realization for $Z(s)\widetilde{T}(s)$ is

$$Z(s)\widetilde{T}(s) \sim \left[\begin{array}{c|c} A_{ZT} & B_{ZT} \\ \hline C_{ZT} & D_{ZT} \end{array}\right] = \left[\begin{array}{cc|c} A_T & 0 & B_T \\ B_Z C_T & A_Z & B_Z D_T \\ \hline D_Z C_T & C_Z & D_Z D_T \end{array}\right] \tag{32}$$

Note that as in Section 4.1, D_{ZT} is an affine function of $z_0^{(k)}$, and C_{ZT} is an affine function of $a_i^{(k)}$ and $b_i^{(k)}$ if $Z \in \mathcal{Z}_N$, or an affine function of $a_i^{(k)}$, $b_i^{(k)}$, $c_i^{(k)}$, $d_i^{(k)}$ if $Z \in \mathcal{Z}_N$, $k = 1, \cdots, p$, $i = 0, 1, \cdots, N$.

Now for a given β, we can check the closed loop stability by applying Theorem 3 with $(A_{ZT}, B_{ZT}, C_{ZT}, D_{ZT})$ as defined in (32). Again, to find the largest β, a line search algorithm such as golden search or bisection search is also needed.

We can further show that the sector bound obtained with the multiplier method above provides an upper bound for the generalized structured singular value (GSSV).

For norm bounded nonlinear and time varying (NLTV) uncertainties, GSSV is defined by (Krause *et al.*, 1988)

$$\mu(T) \triangleq \left\{ \inf_{\Delta \in X} \|\Delta\| \;\middle|\; I + T\Delta \text{ is not invertible} \right\}^{-1}$$

where $X = \{\Delta \mid \Delta = \text{diag}(\Delta_1, \cdots, \Delta_p), \|\Delta_i\|_2 \leq 1, i = 1, \cdots, p\}$ and $\|\cdot\|_2$ is the induced $\mathcal{L}_2$ norm. An upper bound is given by

$$\mu(T) \leq \gamma \triangleq \inf_{D \in \boldsymbol{D}} \left\{ \sup_{\omega} \bar{\sigma}(DTD^{-1}) \right\} \tag{33}$$

where $\boldsymbol{D} = \text{diag}(d_1 I_{m_1}, \cdots, d_p I_{m_p})$ (Krause *et al.*, 1988). To relate GSSV to incrementally bounded nonlinearities, note that $\Delta \in \text{sector}[-\beta, \beta]$ implies that Δ is L_2 norm bounded by β (Zames, 1966).

The following theorem shows that the sector bound obtained based on the multiplier method is less conservative than γ.

Theorem 4. *Define*

$$\gamma_N = \inf \left\{ \frac{1}{\beta} \;\middle|\; Z\widetilde{T} \textit{ is ESPR}, \textit{ where } \widetilde{T} \textit{ is given by (29) and } Z \in \mathcal{Z}_N \right\}$$

and

$$\gamma_{N_o} = \inf \left\{ \frac{1}{\beta} \;\middle|\; Z\widetilde{T} \textit{ is ESPR}, \textit{ where } \widetilde{T} \textit{ is given by (29) and } Z \in \mathcal{Z}_{oN} \right\}$$

Then

$$\mu(T) \leq \gamma_{N_o} \leq \gamma_N \leq \gamma \qquad \forall\, N$$

Proof. Consider any β_N for which there exists a $Z \in \mathcal{Z}_{oN}$ such that $Z\widetilde{T}$ is ESPR. By Lemma 1, $\tilde{\Delta} Z^{-1}$ is in $\text{sector}[0, \infty)$ (equivalently, passive). As in the proof of Theorem 1, Z can be decomposed as $Z = Z_- Z_+$ where $Z_+ = \text{diag}(Z_+^{(1)}, \cdots, Z_+^{(p)})$, $Z_- = \text{diag}(Z_-^{(1)}, \cdots, Z_-^{(p)})$, Z_+, Z_+^{-1}, Z_-^*, $(Z_-^*)^{-1}$ are causal and have finite gains. It follows that $Z_+ \widetilde{T} (Z_-^*)^{-1}$ is ESPR and $Z_-^* \tilde{\Delta} Z_+^{-1}$ is passive. Since Z_+, $\widetilde{T}$, $(Z_-^*)^{-1}$ are all of finite gain, $Z_+ \widetilde{T} (Z_-^*)^{-1}$ also has finite gain. By the Passivity theorem, $(I + Z_+ \widetilde{T} (Z_-^*)^{-1} Z_-^* \tilde{\Delta} Z_+^{-1})^{-1}$ has a bounded induced $\mathcal{L}_2$ norm. Hence, $I + \widetilde{T}\tilde{\Delta}$ is invertible. This implies that $I + T\Delta$ is invertible since $I + \widetilde{T}\tilde{\Delta} = 2\beta(I - \beta T)^{-1}(I + T\Delta)(\beta I - \Delta)^{-1}$. From the definition of GSSV, it follows that $\mu(T) \leq \inf_{Z \in \mathcal{Z}_{oN}} \frac{1}{\beta_N} = \gamma_{N_o}$ for all N.

The second inequality, $\gamma_{N_o} \leq \gamma_N$, follows directly from the fact that $\mathcal{Z}_N \subset \mathcal{Z}_{oN}$.

It remains to show $\gamma_N \leq \gamma$. Note that γ can be rewritten in the following form:

$$\begin{aligned}\gamma &= \inf\left\{\gamma_1 \;\middle|\; (DTD^{-1})(DTD^{-1})^* \leq \gamma_1^2 I \text{ for some diagonal } D \text{ and all } \omega \in \mathbb{R}\right\}\\ &= \inf\left\{\gamma_1 \;\middle|\; TDD^*T^* \leq \gamma_1^2 DD^* \text{ for some diagonal } D \text{ and all } \omega \in \mathbb{R}\right\}\\ &= \inf\left\{\gamma_1 \;\middle|\; TPT^* \leq \gamma_1^2 P \text{ for some diagonal } P > 0 \text{ and all } \omega \in \mathbb{R}\right\} \qquad (34)\end{aligned}$$

From the definition of γ_N, for any $Z \in \mathcal{Z}_N$, $\gamma_N \leq \frac{1}{\beta_N}$ where $Z(I+\beta_N T)(I-\beta_N T)^{-1}$ is ESPR. In particular, Z may be chosen as Z_o, any constant positive definite diagonal matrix. Then the ESPR condition reduces to the condition that $TZ_oT \leq \frac{1}{\beta^2}Z_o$. This condition is the same as in (34). Hence, $\gamma_N \leq \gamma$. This completes the proof. ■

5. Numerical Examples

Example 1. A summary of some examples from (Wen and Chen, 1990) by using our optimal multiplier method (with specified lower bound) is shown in Tab. 1. For comparison, we have also included results obtained by using the Off-Axis Circle Criterion and Popov Criterion. In all of the examples, our method gives the least conservative upper bound. Examples 4 and 5 are known counterexamples to Aizerman's Conjecture as shown in (Willems, 1971). This fact is also confirmed by the multiplier method. In Tab. 1, N denotes the order of approximation when no appreciable improvement in β occurs. The corresponding $Z(j\omega)$ is of order $N+1$. In all of the chosen examples, N is very small.

Tab. 1. Results of optimal multiplier algorithm 1.

	Transfer Functions	α	Popov	Off-Axis	monotone	odd	Nyquist	N
1	$\frac{3(s+1)}{s^2(s^2+s+25)}$	1	3.70	7.25	8.00	8.00	8.00	1
2	$\frac{(s+1)}{s(s+0.1)(s^2+0.5s+9)}$	10^{-3}	2.95	3.13	4.23	4.23	4.23	1
3	$\frac{7500}{(s^3+34.5s^2+7500s)}$	10^{-3}	30.00	33.13	33.19	33.29	34.50	2
4	$\frac{s^2}{(s^2+1)(s^2+9)+10^{-4}(3s^3+21s)}$	0	2.3×10^{-4}	—	7.6×10^{-4}	7.6×10^{-4}	∞	2
5	$\frac{s^2}{(s^2+1)(s^2+9)+10^{-6}(2s^3+16s)}$	0	1.1×10^{-6}	—	2.5×10^{-6}	2.5×10^{-6}	∞	2

Tab. 2. Comparison of upper bounds for GSSV in Example 2.

	real parametric	odd sector bounded	sector bounded	norm bounded
$\bar{\mu}$	4.0988	4.1000	4.1000	4.8031
$1/\bar{\mu}$	0.2440	0.2439	0.2439	0.2082
N	—	2	2	—

Example 2. Consider the following nominal transfer function (Balakrishnan *et al.*, 1994)

$$T(s) = \begin{pmatrix} 2 & \dfrac{-10s-8}{5(s+1)} \\ \dfrac{-2s+8}{s+1} & 2 \end{pmatrix}$$

with $\Delta = \mathrm{diag}(\delta_1, \delta_2)$. The upper bound for norm bounded NLTV case are taken from (Balakrishnan *et al.*, 1994) where constant diagonal multipliers were used. The upper bound for linear real parametric uncertainty is calculated by MUSOL4 software (Fan, 1994) which computes the upper bound of structured singular value reported in (Fan *et al.*, 1991). Note that for this particular example, the upper bound obtained by MUSOL4 is less conservative than the one obtained by using the multiplier approach in (Balakrishnan *et al.*, 1994). The upper bounds for symmetric incrementally sector bounded NLTI uncertainties are obtained using our optimal multiplier method. From Tab. 2, it is seen that our optimal multiplier method gives almost the same upper bound as the upper bound for linear real parametric uncertainties.

Example 3. The following example is taken from (Doyle, 1982) with $\Delta = \mathrm{diag}(\Delta_1, \Delta_2)$.

$$T = \begin{pmatrix} (I+KP)^{-1}KP & (I+KP)^{-1}K \\ -(I+PK)^{-1}P & (I+PK)^{-1}PK \end{pmatrix}$$

where

$$P(s) = \begin{pmatrix} \dfrac{9}{s+1} & \dfrac{-10}{s+1} \\ \dfrac{-8}{s+2} & \dfrac{9}{s+2} \end{pmatrix}, \qquad K(s) = \frac{1}{0.0159s}\begin{pmatrix} 9(s+1) & 10(s+2) \\ 8(s+1) & 9(s+2) \end{pmatrix}$$

The results are shown in Tab. 3. Again, our multiplier based bounds are less conservative than the norm bounded NLTV case. In the case of odd nonlinearity, our bound is even better than the upper bound of the real μ case.

Tab. 3. Comparison of upper bounds for GSSV in Example 3.

	real parametric	odd sector bounded	sector bounded	norm bounded
$\bar{\mu}$	42.7350	34.4828	62.5000	554.5389
$1/\bar{\mu}$	0.0234	0.0290	0.0160	0.0018033
N	—	4	4	—

6. Conclusion

In this paper, we have developed a class of multipliers parameterized by linear matrix inequalities for structured time invariant sector bounded nonlinearities. Convex optimization techniques are used to generate optimal multipliers over this class. The result can also be specialized to symmetric incrementally sector bounded nonlinear uncertainties to provide a less conservative upper bound for the generalized structured singular value. Since this method is state space based, it is computationally very effective as confirmed by our numerical experience. Results presented here can be easily applied to the robustness analysis of LTI systems with mixed linear and nonlinear structured uncertainties to provide a unified computational framework for linear and nonlinear μ analysis theory.

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Appendices

A. Proof of Lemma 1

For the proof, we need the following result that extends some integral inequalities, shown for the SISO case in (Desoer and Vidyasagar, 1975), to the MIMO case.

Lemma 5. *If $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ is monotone, then, for all $x(\cdot) \in \mathcal{L}_2^m(\mathbb{R})$ and all $\tau \in \mathbb{R}$,*

$$\int_{-\infty}^{\infty} \Phi'[x(t)]x(t+\tau)\,\mathrm{d}t \le \int_{-\infty}^{\infty} \Phi'[x(t)]x(t)\,\mathrm{d}t$$

If, in addition, Φ is odd, then

$$\left|\int_{-\infty}^{\infty} \Phi'[x(t)]x(t+\tau)\,\mathrm{d}t\right| \le \int_{-\infty}^{\infty} \Phi'[x(t)]x(t)\,\mathrm{d}t$$

Proof. The monotonicity of Φ implies that for all $x_1,\ x_2 \in \mathbb{R}^m$,

$$\Big(\Phi(x_2) - \Phi(x_1)\Big)'(x_2 - x_1) \geq 0 \tag{A1}$$

Denote the j-th component of Φ by Φ_j. Then from (A1), we have

$$\Phi_j\Big(x_1, \dots, x_{j-1}, x_j + k_j \frac{y_j - x_j}{\ell_j}, x_{j+1}, \dots, x_m\Big)\frac{(y_j - x_j)}{\ell_j}$$

$$-\Phi_j(x_1, \dots, x_j, \dots, x_m)\frac{(y_j - x_j)}{\ell_j} \geq 0$$

Sum k_j from 0 to ℓ_j and let $\ell_j \to \infty$, we have

$$\int_{x_j}^{y_j} \Phi_j(x_1, \dots, \xi_j, \dots, x_m)\, \mathrm{d}\xi_j - \Phi_j(x_1, \dots, x_m)(y_j - x_j) \geq 0 \tag{A2}$$

Setting $y_j = 0$, (A2) becomes

$$\Phi_j(x_1, \dots, x_m)x_j \geq \int_0^{x_j} \Phi_j(x_1, \dots, \xi_j, \dots, x_m)\, \mathrm{d}\xi_j \tag{A3}$$

Setting $x_j = 0$, (A2) becomes

$$-\Phi_j(x_1, \dots, x_m)y_j \geq -\int_0^{y_j} \Phi_j(x_1, \dots, \xi_j, \dots, x_m)\, \mathrm{d}\xi_j \tag{A4}$$

By using (A3) and (A4), we obtain

$$\int_{-\infty}^{\infty} \Phi_j(x(t))x_j(t)\, \mathrm{d}t - \int_{-\infty}^{\infty} \Phi_j(x(t))x_j(t+\tau)\, \mathrm{d}t$$

$$\geq \int_{-\infty}^{\infty}\int_0^{x_j(t)} \Phi_j(x_1, \dots, \xi_j, \dots, x_m)\, \mathrm{d}\xi_j\, \mathrm{d}t$$

$$-\int_{-\infty}^{\infty}\int_0^{x_j(t+\tau)} \Phi_j(x_1, \dots, \xi_j, \dots, x_m)\, \mathrm{d}\xi_j\, \mathrm{d}t = 0$$

If Φ is odd, then we also have

$$\int_{-\infty}^{\infty} \Phi_j(x(t))x_j(t)\, \mathrm{d}t + \int_{-\infty}^{\infty} \Phi_j(x(t))x_j(t+\tau)\, \mathrm{d}t$$

$$= \int_{-\infty}^{\infty} \Phi_j(x(t))x_j(t)\, \mathrm{d}t - \int_{-\infty}^{\infty} \Phi_j(x(t))(-x_j(t+\tau))\, \mathrm{d}t$$

$$\geq \int_{-\infty}^{\infty}\int_0^{x_j(t)} \Phi_j(x_1, \dots, \xi_j, \dots, x_m)\, \mathrm{d}\xi_j\, \mathrm{d}t$$

$$- \int_{-\infty}^{\infty} \int_{0}^{-x_j(t+\tau)} \Phi_j(x_1, \ldots, \xi_j, \ldots, x_m) \, d\xi_j \, dt$$

$$= \int_{-\infty}^{\infty} \int_{0}^{x_j(t)} \Phi_j(x_1, \ldots, \xi_j, \ldots, x_m) \, d\xi_j \, dt$$

(since Φ is odd)

$$- \int_{-\infty}^{\infty} \int_{0}^{x_j(t+\tau)} \Phi_j(x_1, \ldots, \xi_j, \ldots, x_m) \, d\xi_j \, dt = 0 \, .$$

Summing j from 1 to m, we have the desired result. ■

Proof of Lemma 1. Let $v = Zx$ where $v = (v_1', \cdots, v_p')'$, $x = (x_1', \cdots, x_p')'$. It follows from (5)–(7) that $v \in \mathcal{L}_2^m(\mathbb{R})$ if and only if $x \in \mathcal{L}_2^m(\mathbb{R})$. To prove the lemma, it is sufficient to show that (8) holds for each $\Delta^{(k)}$, $k = 1, \cdots, p$, since $\Delta^{(k)}(Z^{(k)})^{-1}$'s are decoupled.

$$\left\langle v_k, (\Delta^{(k)}) {Z^{(k)}}^{-1} v_k \right\rangle = \left\langle Z^{(k)} x_k, \Delta^{(k)} x_k \right\rangle$$

$$= \int_{-\infty}^{\infty} \Delta^{(k)'}\Big(x(t)\Big) z_0^{(k)} x(t) \, dt - \int_{-\infty}^{\infty} \Delta^{(k)'}\Big(x(t)\Big) (z^{(k)} * x)(t) \, dt$$

$$= z_0^{(k)} \int_{-\infty}^{\infty} \Delta^{(k)'}\Big(x(t)\Big) x(t) \, dt - \int_{-\infty}^{\infty} \Delta^{(k)'}\Big(x(t)\Big) \int_{-\infty}^{\infty} z^{(k)}(\tau) x(t-\tau) \, d\tau \, dt$$

$$= z_0^{(k)} \int_{-\infty}^{\infty} \Delta^{(k)'}\Big(x(t)\Big) x(t) \, dt - \int_{-\infty}^{\infty} z^{(k)}(\tau) \int_{-\infty}^{\infty} \Delta^{(k)'}\Big(x(t)\Big) x(t-\tau) \, dt \, d\tau$$

$$\geq z_0^{(k)} \int_{-\infty}^{\infty} \Delta^{(k)'}\Big(x(t)\Big) x(t) \, dt - \int_{-\infty}^{\infty} z^{(k)}(\tau) \, d\tau \int_{-\infty}^{\infty} \Delta^{(k)'}\Big(x(t)\Big) x(t) \, dt$$

by Lemma 5 if $\Delta^{(k)} \in \mathbf{\Delta}$

$$\geq z_0^{(k)} \int_{-\infty}^{\infty} \Delta^{(k)'}\Big(x(t)\Big) x(t) \, dt - \int_{-\infty}^{\infty} \left| z^{(k)}(\tau) \right| \, d\tau \int_{-\infty}^{\infty} \Delta^{(k)'}\Big(x(t)\Big) x(t) \, dt$$

by Lemma 5 if $\Delta^{(k)} \in \mathbf{\Delta}_o$

$$\geq 0 \qquad \text{by (6)}$$

■

B. State Space Realizations

The controllable canonical realizations for the transfer functions in Condition 2 in Theorem 2 are:

$$A_a^{(k)} = A_b^{(k)} = A_c^{(k)} = A_d^{(k)} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ (-1)^{N-1} & 0 & (-1)^{N-2}N & 0 & \cdots & N & 0 \end{bmatrix}$$

$$B_a^{(k)} = B_b^{(k)} = B_c^{(k)} = B_d^{(k)} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C_a^{(k)} = \begin{bmatrix} (-1)^N(a_0^{(k)} - 1) & 0 & (-1)^{N-1}(a_1^{(k)} - N) & 0 & \cdots & -(a_{N-1}^{(k)} - N) & 0 \end{bmatrix}$$

$$C_b^{(k)} = \begin{bmatrix} (-1)^N(b_0^{(k)} - 1) & 0 & (-1)^{N-1}(b_1^{(k)} - N) & 0 & \cdots & -(b_{N-1}^{(k)} - N) & 0 \end{bmatrix}$$

$$C_c^{(k)} = \begin{bmatrix} (-1)^N(c_0^{(k)} - 1) & 0 & (-1)^{N-1}(c_1^{(k)} - N) & 0 & \cdots & -(c_{N-1}^{(k)} - N) & 0 \end{bmatrix}$$

$$C_d^{(k)} = \begin{bmatrix} (-1)^N(d_0^{(k)} - 1) & 0 & (-1)^{N-1}(d_1^{(k)} - N) & 0 & \cdots & -(d_{N-1}^{(k)} - N) & 0 \end{bmatrix}$$

$$D_a^{(k)} = a_N^{(k)}, \qquad D_b^{(k)} = b_N^{(k)}, \qquad D_c^{(k)} = c_N^{(k)}, \qquad D_d^{(k)} = d_N^{(k)}$$

A controllable realization for Z is

$$A_Z = \mathrm{diag}\Big(A_{Z^{(1)}}, A_{Z^{(2)}}, \cdots A_{Z^{(p)}}\Big)$$

$$B_Z = \mathrm{diag}\Big(B_{Z^{(1)}}, B_{Z^{(2)}}, \cdots B_{Z^{(p)}}\Big)$$

$$C_Z = \mathrm{diag}\Big(C_{Z^{(1)}}, C_{Z^{(2)}}, \cdots C_{Z^{(p)}}\Big)$$

$$D_Z = \mathrm{diag}\Big(D_{Z^{(1)}}, D_{Z^{(2)}}, \cdots D_{Z^{(p)}}\Big)$$

where

$$A_{Z^{(k)}} = \begin{bmatrix} A_{ac}^{(k)} & 0 \\ 0 & A_{bd}^{(k)} \end{bmatrix}, \qquad B_{Z^{(k)}} = \begin{bmatrix} B_{ac}^{(k)} \\ B_{bd}^{(k)} \end{bmatrix}$$

$$C_{Z^{(k)}} = -C_{ac}^{(k)} + C_{bd}^{(k)}, \qquad D_{Z^{(k)}} = z_0^{(k)}$$

and

$$A_{ac}^{(k)} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -1 & -(N+1) & -\frac{(N+1)N}{2} & \cdots & -(N+1) \end{bmatrix}$$

$$B_{ac}^{(k)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$\begin{aligned} C_{ac}^{(k)} = (a_0^{(k)} - c_0^{(k)}) & \begin{bmatrix} 1 & N & \cdots & \frac{N(N-1)}{2} & N \end{bmatrix} \\ +(a_1^{(k)} - c_1^{(k)}) & \begin{bmatrix} 1 & N-1 & \cdots & N-1 & 0 \end{bmatrix} \\ + \cdots + (a_N^{(k)} - c_N^{(k)}) & \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \end{aligned}$$

$$D_{ac}^{(k)} = 0$$

$$A_{bd}^{(k)} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ (-1)^N & (-1)^{N-1}(N+1) & (-1)^{N-2}\frac{(N+1)N}{2} & \cdots & (N+1) \end{bmatrix}$$

$$B_{bd}^{(k)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$\begin{aligned} C_{bd}^{(k)} = (b_0^{(k)} - d_0^{(k)}) & \begin{bmatrix} (-1)^N & (-1)^{N-1}N & \cdots & (-1)^2\frac{N(N-1)}{2} & -N \end{bmatrix} \\ & +(b_1^{(k)} - d_1^{(k)}) \begin{bmatrix} (-1)^{N-1} & (-1)^{N-2}(N-1) & \cdots & -(N-1) & 0 \end{bmatrix} \\ & + \cdots + (b_N^{(k)} - d_N^{(k)}) \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \end{aligned}$$

$$D_{bd}^{(k)} = 0$$

where $k = 1, \cdots, p$. Note that for the monotone nonlinearity case (without the assumption that Δ is odd), $c_i^{(k)} = 0$, $d_i^{(k)} = 0$ for $i = 0, \cdots N$, $k = 1, \cdots, p$. ■

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