Appl. Math. and Comp. Sci., *1996*, *vol.6*, *No.4*, *649–665*

EXTREMAL TRANSIENTS IN LINEAR CONTROL SYSTEMS†

HENRYK GÓRECKI*

The relation between extremal values of the error and the coefficients of its differential equations is one of the central problems of control systems in chemical industry, because extremal values of the error sometimes cause serious damages to the environment or to the system itself. Analytical formulae for the determination of these values are known only for the second-order systems. In this paper an approximate method which permits to determine extremal values of the error in higher-order systems is proposed.

1. Introduction

The most popular and possibly the simplest way of determining extremal transient errors in linear systems is as follows: Without loss of generality we start from the equation of the SISO system for the transient error $\varepsilon(t)$ of the output

$$\sum_{i=0}^{n} a_{n-1}\varepsilon^{(i)}(t) = 0 \tag{1}$$

where $\varepsilon^{(i)}(0)$, $i = 0, 1, \dots, n-1$ are known and $a_0 = 1$. The solution of (1) is determined by the formulae

$$\varepsilon(t) = \sum_{i=1}^{n} A_i e^{s_i t}, \quad s_i \neq s_j \quad \text{for} \quad i \neq j \tag{2}$$

where A_i can be calculated using the initial conditions and roots of the characteristic equation

$$\sum_{i=0}^{n} a_{n-i} s^i = 0 \tag{3}$$

At this point we meet the first obstacle, namely, there are no analytical formulae for the determination of the roots of eqn. (3), if its degree is higher than four. Even for equations of third or fourth degree the existing formulae are very complicated.

† This work was supported by the Polish State Committee of Scientific Research (grant No. 8T11A02709).

* Institute of Automatics, University of Mining and Metallurgy, al. Mickiewicza 30, 30–059 Kraków, Poland, e-mail: head@earth.ia.agh.edu.pl

For the determination of extrema it is first necessary to find the moments of time at which

$$\frac{d\varepsilon(t)}{dt} = \sum_{i=1}^{n} A_i s_i e^{s_i t} = 0 \tag{4}$$

It is possible to find the extremum points τ of eqn. (4) when this transcendental equation contains at most two different exponential terms, which corresponds to the second-order system described by (1).

The substitution of the values of extremum points τ into (2) gives the extremal values of the transient error. This way is impractical because we want to know the solution for higher-order systems in the form of an explicit relationship between the coefficients a_i representing the parameters of the plant and controller, and the extremum. For this reason, we look for another way directly giving the relations between the extremum of the error and coefficients a_i.

2. Overview of the Results

Let the transient error $\varepsilon(t)$ of a control system be described by the equation

$$\begin{cases} a_0 \dfrac{d^n\varepsilon(t)}{dt^n} + a_2 \dfrac{d^{n-1}\varepsilon(t)}{dt^{n-1}} + \cdots + a_{n-1}\dfrac{d\varepsilon(t)}{dt} + a_n\varepsilon(t) = 0 \\ c_\mu = \varepsilon^\mu(0) \neq 0 \quad \text{for} \quad \mu = 0, 1, \ldots n-1 \end{cases} \tag{5}$$

We assume that the characteristic equation (5) has m different real roots s_k and $2p$ different complex roots r_k such that

$$m + 2p = n \tag{6}$$

Write

$$\alpha_k + j\omega_k = r_k, \quad \alpha_k - j\omega_k = \bar{r}_k, \quad k = 1, \ldots, p \tag{7}$$

The solution of (5) takes the form

$$\varepsilon(t) = \sum_{k=1}^{m} A_k e^{s_k t} + \sum_{k=1}^{p} (B_k \cos\omega_k t + C_k \sin\omega_k t) e^{\alpha_k t} \tag{8}$$

where A_k, B_k, C_k, α_k, ω_k, and s_k are real numbers. The extremal values of the transient error occur at the points τ where $\left.\frac{d\varepsilon(t)}{dt}\right|_{t=\tau} = 0$ and

$$\varepsilon_e(\tau) = \sum_{k=1}^{m} A_k e^{s_k \tau} + \sum_{k=1}^{p} (B_k \cos\omega_k \tau + C_k \sin\omega_k \tau) e^{\alpha_k \tau} \tag{9}$$

In (Górecki and Turowicz, 1966a) two extremal problems were investigated:

1. When the extremal value $\varepsilon_e(\tau)$ is extremum with respect to the roots $s_k, \alpha_k + j\omega_k$;
2. When the time τ is extremum with respect to the roots $s_k, \alpha_k + j\omega_k$.

For the first problem it was found that the extremum of the extremal value of the transient error may occur when the following relation is fulfilled:

$$(-1)^p \tau^n \prod_{k=1}^{m} A_k \prod_{k=1}^{p} (B_k^2 + C_k^2) = 0 \tag{10}$$

For the other problem the extremum of the extremal value of time τ may occur when the following relation holds:

$$(-1)^p \prod_{k=1}^{m} A_k \prod_{k=1}^{p} (B_k^2 + C_k^2) \prod_{k=1}^{p} (\alpha_k^2 + \omega_k^2) \tau^{n-1} \left[\tau + \sum_{k=1}^{m} \frac{1}{s_k} + \sum_{k=1}^{p} \left(\frac{1}{r_k} + \frac{1}{\bar{r}_k} \right) \right] = 0 \tag{11}$$

From (10), (11) and the well-known Vieta's formulae we deduce that

$$\tau = 0 \quad \text{or} \quad \tau = -\left[\sum_{k=1}^{m} \frac{1}{s_k} + \sum_{k=1}^{p} \left(\frac{1}{r_k} + \frac{1}{\bar{r}_k} \right) \right] = \frac{a_{n-1}}{a_n} \tag{12}$$

In the case of one real root s_k of multiplicity n it was found that (12) is equivalent to

$$\tau = 0 \quad \text{or} \quad \tau = -\frac{n}{s_k} = \frac{a_{n-1}}{a_n} \tag{13}$$

and

$$\varepsilon_e = \left[A_n \left(\frac{a_{n-1}}{a_n} \right)^{n-1} + \cdots + A_2 \frac{a_{n-1}}{a_n} + A_1 \right] e^{-n} \tag{14}$$

The constants A_k, B_k, C_k are determined from the relations

$$\varepsilon^{(\mu)}(0) = \sum_{k=1}^{m} A_k s_k^{\mu} + \sum_{k=1}^{p} \left[B_k \mathrm{Re}\,(r_k^{\mu}) + C_k \,\mathrm{Im}\,(r_k^{\mu}) \right]_{\mu=0,1,\ldots,(n-1)} \tag{15}$$

In (Górecki, 1966a; Górecki and Turowicz, 1966b) it was proved that the extremum of the extremal value of τ determined by (12) from the necessary conditions exists only when the characteristic equation (5) has only real roots. It was shown that the extremum of the extremal value τ_k, where k denotes the k-th derivative of $\varepsilon(t)$, can be calculated from the equation

$$a_n \tau^n + \sum_{i=1}^{k} (-1)^i a_{n-i} \tau^{k-i} k(k-1) \cdots (k-i+1) = 0 \tag{16}$$

which is a generalization of (12) and (13). Taking these results into account the authors investigated the case when the characteristic equation had only real roots in (Kobayashi, 1993).

Let s_k be the roots of the equation

$$\sum_{k=0}^{n} a_k s^{n-k} = 0 \tag{17}$$

If eqn. (17) has n different real roots, then the general solution to eqn. (5) takes the form

$$\varepsilon(t) = \sum_{k=0}^{n} A_k e^{s_k t} \tag{18}$$

where

$$A_k = \sum_{j=1}^{n} c_{j-1} f_{n-j}(k)(-1)^{j-1} \prod_{\substack{v=1\\ v\neq k}} (s_v - s_k)^{-1}, \quad k = 1, 2, \dots, n \tag{19}$$

Here $f_i(k)$ for $i > 0$ denotes the i-th degree elementary symmetric function of $n-1$ variables $s_1, \dots, s_{k-1}, s_{k+1}, \dots, s_n$ and

$$\begin{cases} f_0(k) = 1 \\ f_i(k) = \sum_{j=1}^{i} (-1)^i a_{n-i+j}\, s_k^j, \quad i = 1, 2, \dots, n-1 \end{cases} \tag{20}$$

The relations (19) and (20) have already been communicated in (Szymkat, 1983).

If eqn. (17) has multiple roots, then the relations (18) and (19) should be transformed by the proper passing to the limit. In the particular case when $s_1 = s_2 = \dots = s_n = s$, we obtain

$$\varepsilon(t) = e^{st} \sum_{k=1}^{n} A_k t^{k-1} \tag{21}$$

and

$$A_k = \sum_{i=0}^{k-1} \frac{c_{k-i-1}(-1)^i s^i}{i!(k-i-1)!}, \quad k = 1, 2, \dots, n \tag{22}$$

The necessary condition for the existence of a local extremum of the solution (18) is of the form

$$\varepsilon'(t) = 0 \tag{23}$$

Let us assume that eqn. (23) has at least one nonnegative root. The set of all roots is at most countable. We denote by τ_i its elements and thus obtain

$$\varepsilon'(\tau_i) = 0 \tag{24}$$

Problem 1. The values $\varepsilon'(\tau_i)$ are functions of the variables $s_1, \dots, s_n$:

$$\varepsilon(\tau_i) = X_i(s_1, \dots, s_n) \tag{25}$$

and possess partial derivatives. We want to find the solution of the system of equations

$$\frac{\partial X_i}{\partial s_i} = 0, \quad j = 1, 2, \dots, n \tag{26}$$

Equation (26) represents a necessary condition for the existence of a local extremum of (25).

Problem 2. The values τ_i are functions of the variables $s_1, \dots, s_n$:

$$\tau_i = T_i(s_1, \dots, s_n) \tag{27}$$

and possess partial derivatives. We want to find the solution of the system of equations

$$\frac{\partial T_i}{\partial s_j} = 0, \quad j = 1, 2, \dots, n \tag{28}$$

Theorem 1. *Under the assumption $s_i \neq s_j$ for $i \neq j$ a necessary and sufficient condition for the existence of the solution to (26) is*

$$\tau_i \prod_{k=1}^{n} A_k = 0 \tag{29}$$

i.e. $\tau_i = 0$ or $A_k = 0$ for some k, $l \leq k \leq n$.

Proof. For the proof we refer the reader to (Górecki and Turowicz, 1966a) and (Kobayashi, 1993). ■

Theorem 2. *If $s_1 = s_2 = \ldots = s_n$, then a necessary and sufficient condition for the existence of the solution to (26) is*

$$\tau_i A_n = 0 \tag{30}$$

Proof. See (Górecki and Turowicz, 1966a) and (Kobayashi, 1993). ■

Remark. For Theorems 1 and 2, if $\tau_i = 0$, then $c_1 = 0$. Conversely, if $c_1 = 0$, then $\tau_i = 0$ for some i.

Theorem 3. *Under the assumption $s_i \neq s_j$ for $i \neq j$, a necessary and sufficient condition for the existence of the solution to (28) is*

$$\tau_i \prod_{k=1}^{n} A_k \prod_{k=1}^{n} s_k \Big(\tau_i + \sum_{k=1}^{n} \frac{1}{s_k}\Big) = 0 \tag{31}$$

where

$$-\sum_{k=1}^{n} \frac{1}{s_k} = \frac{a_{n-1}}{a_n}$$

i.e. $\tau_i = a_{n-1}/a_n$ *or* $A_k - 0$ *or* $\tau_i = 0$.

Proof. See (Górecki and Turowicz, 1966a) and (Kobayashi, 1993). ■

Taking into account that $s_k = -1/T_k$, where T_k represents a time constant, we conclude that the minimum extremal time equals the sum of time constants.

Theorem 4. *If $s_1 = s_2 = \ldots = s_n = s$, then a necessary and sufficient condition for the existence of the solution to (28) is*

$$A_n \tau_i (n + s\tau_i) = 0 \tag{32}$$

Theorem 5. *If the condition (31) is fulfilled due to*

$$\tau_i = \frac{a_{n-1}}{a_n} \tag{33}$$

then we have

$$\begin{vmatrix} c_0 & c_1 & c_2 & c_3 & c_4 & \cdots & c_{n-2} & c_{n-1} \\ a_{n-2} & -a_{n-1} & a_n & 0 & 0 & \cdots & 0 & 0 \\ a_{n-3} & 0 & -a_{n-1} & 2a_n & 0 & \cdots & 0 & 0 \\ a_{n-4} & 0 & 0 & -a_{n-1} & 3a_n & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_1 & 0 & 0 & 0 & 0 & \cdots & -a_{n-1} & (n-2)a_n \\ a_0 & 0 & 0 & 0 & 0 & \cdots & 0 & -a_{n-1} \end{vmatrix} = 0 \tag{34}$$

The determinant (34) has n rows. If we denote by a_{ik} its elements, then

$$a_{1k} = c_{k-1}, \qquad k = 1, \ldots, n$$

$$a_{i1} = a_{n-i}, \qquad i = 2, \ldots, n$$

$$a_{ii} = -a_{n-1}, \qquad i = 2, \ldots, n$$

$$a_{i,i+1} = (i-1)a_n, \qquad i = 2, \ldots, n-1$$

The other elements of the determinant (34) are equal to zero.

Proof. See (Kobayashi and Shimemura, 1981). ■

Theorem 6. *If $s_1 = s_2 = \dots = s_n = s$, and the condition (32) is fulfilled due to*

$$\tau_i = -\frac{n}{s} = -\frac{a_{n-1}}{a_n} \tag{35}$$

then we have

$$\sum_{i=0}^{n-1} \frac{c_i}{i!} \frac{a_n^{n-1-i}}{a_{n-1}^{n-1-i}} \sum_{j=0}^{n-i-1} \frac{n^{j-1}(n-j-i)}{j!} = 0 \tag{36}$$

Proof. The condition (36) has been proved in (Kobayashi, 1993) in a somewhat different form. It is possible to obtain it from (34) by using the relations

$$a_k = n^{n-k} \binom{n}{k} \left(\frac{a_n}{a_{n-1}}\right)^{n-k}, \quad k = 2, 3, \dots, n \tag{37}$$

which are true in this case. ■

Using (35)–(37) we may calculate

$$\varepsilon(\tau_i) = e^{-n} \sum_{k=1}^{n} \sum_{i=0}^{k-1} \frac{c_{k-i-1} n^i}{(k-i-1)! i!} \left(\frac{a_{n-1}}{a_n}\right)^{k-i-1} \tag{38}$$

The necessary and sufficient conditions for the existence of the solution to the system (26) is the zero value of a certain determinant whose elements are very complicated algebraic expressions. The same refers also to the system (28). Reducing these determinants to simple forms involves very long and tiresome algebraic calculations.

Because of the large volume of the paper (Górecki and Turowicz, 1966a) it was possible to publish it only in 1966, in spite of the fact that most of these results were obtained in 1960 and presented at the first IFAC Congress which was held in Moscow (cf. (Górecki and Turowicz, 1966a)). In (Kobayashi and Shimemura, 1981) it was shown that it was also possible to transform some determinant with rather complicated elements to the simple form (34).

In view of the mathematical results obtained in this paper, the following technical problems can be treated:

a) The problem of cancellation, as quickly as possible, of the transient error by using adaptive control systems (Kobayashi and Shimemura, 1981),

b) Determination of the standard optimal processes achieving extrema of the transient error at extremal time τ_i,

c) Determination of the shortest time at which the transient error attains its extremal value and

d) Optimization of the control system by reducing its order via the condition $A_k = 0$ (Kobayashi, 1993).

Further results concerning the relations between the extremum of the error $\varepsilon(\tau)$ and the value τ were obtained in (Górecki, 1966b).

Theorem 7. *The relation between the extremum of the error $\varepsilon(t)$ and the time of its occurence τ is as follows:*

$$e^{-a_1\tau}\prod_{k=1}^{n}\sum_{\mu=0}^{n-1}(-1)^n f_{n-\mu}^{(k)}c_\mu = \prod_{k=1}^{n}\sum_{\substack{\mu=0\\ \mu\neq 0}}^{n-1}(-1)^n f_{n-\mu}^{(k)}\varepsilon^{(n)}(\tau) \tag{39}$$

where $f_r^{(k)}$ denotes the fundamental symmetric function of the r-th order for $n-1$ variables $s_1,\dots,s_{k-1},s_{k+1},\dots,s_n$, $r=0,\dots,n-1$, and $f_0^{(k)}=1$.

Applying Vieta's formulae (40) to the relation (39)

$$\begin{cases} f_0^{(k)} = 1 \\ f_1^{(k)} = s_1 + s_2 + \dots + s_{k-1} + s_{k+1} + \dots + s_n = a_1 - s_k \\ f_2^{(k)} = s_1 s_2 + s_1 s_3 + \dots = a_2 - s_1 s_k - s_2 s_k \dots s_n s_k \\ \quad\vdots \end{cases} \tag{40}$$

it is possible to express (39) as a function of the coefficients a_i. For example, for $n=2$,

$$\varepsilon(\tau) = \pm\sqrt{c_0^2 + \frac{a_1}{a_2}c_0c_1 + \frac{1}{a_2}c_1^2}\; e^{-a_1\tau/2} \tag{41}$$

and for $n=3$

$$\begin{aligned}\Big(\varepsilon''(\tau)\Big)^3 &+ \Big(\varepsilon''(\tau)\Big)^2 a_2\varepsilon(\tau) + \varepsilon''(\tau)a_1a_3\varepsilon^2(\tau) + a_3^2\varepsilon^3(\tau)\\ &= \Big[c_2^3 + 2c_2^2c_1a_1 + c_2c_1^2(a_1^2+a_2) + c_2^2c_0a_2 + c_2c_0^2a_1a_3 + c_2c_1c_0(3a_3+a_1a_2)\\ &\quad + c_1^3(a_1a_2-a_3) + c_1^2c_0(a_2^2+a_1a_3) + 2c_0^2c_1a_2a_3 + c_0^3a_3^2\Big]e^{-a_1\tau}\,, \end{aligned} \tag{42}$$

3. Solution Method

In the papers (El-Khoury *et al.*, 1994; Górecki, 1965) it is shown that the linear system

$$\begin{cases} \dfrac{d\varepsilon(t)}{dt} = A\varepsilon(t), \quad t \geq 0 \\ \varepsilon(0) = c \end{cases} \tag{43}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & a_1 \end{bmatrix} \tag{44}$$

$$a_0 \triangleq 1$$

$$\varepsilon(0) = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix} \tag{45}$$

can be transformed to the nonlinear system

$$\begin{bmatrix} \varepsilon_2\frac{d}{d\varepsilon_1} + a_1 & a_2 & a_3 & \dots & a_{n-2} & a_{n-1} \\ -1 & \varepsilon_2\frac{d}{d\varepsilon_1} & 0 & \dots & 0 & 0 \\ 0 & -1 & \varepsilon_2\frac{d}{d\varepsilon_1} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -1 & \varepsilon_2\frac{d}{d\varepsilon_1} \end{bmatrix} \varepsilon_2 = -a_n\varepsilon_1 \tag{46}$$

which describes the relation between only two components of the vector $\varepsilon(t)$, namely the scalar function $\varepsilon(t) = \varepsilon_1$ and its first derivative $\varepsilon^{(1)}(t) = \varepsilon_2$.

Introducing the nonlinear operator

$$q = \varepsilon_2 \frac{d}{d\varepsilon_1} \tag{47}$$

we can rewrite (46) in the explicit form

$$\left[q^{n-1} + a_1 q^{n-2} + a_2 q^{n-3} + \dots + a_{n-2}q + a_{n-1}\right]\varepsilon_2 + a_n\varepsilon_1 = 0 \tag{48}$$

The explicit formula for the n-th power of the operator q is very complicated (Górecki and Turowicz, 1968, pp.228–230). It is easy to calculate this power directly for a particular n. The powers of the operator q for $n = 1, \dots, 5$ are respectively as follows:

$$q = \varepsilon_2 \frac{d\varepsilon_2}{d\varepsilon_1} \tag{49}$$

$$q^2 = \varepsilon_2 \frac{d}{d\varepsilon_1}\left[\varepsilon_2 \frac{d\varepsilon_2}{d\varepsilon_1}\right] = \varepsilon_2 \left[\varepsilon_2^{(1)}\right]^2 + \varepsilon_2^2 \varepsilon_2^{(2)} \tag{50}$$

$$q^3 = \varepsilon_2 \left[\varepsilon_2^{(1)}\right]^3 + 4\varepsilon_2^2 \varepsilon_2^{(1)} \varepsilon_2^{(2)} + \varepsilon_2^3 \varepsilon_2^{(3)} \tag{51}$$

$$q^4 = \varepsilon_2 \left[\varepsilon_2^{(1)}\right]^4 + 11\varepsilon_2^2 \left[\varepsilon_2^{(1)}\right]^2 \varepsilon_2^{(2)} + 7\varepsilon_2^3 \varepsilon_2^{(1)} \varepsilon_2^{(3)} + 4\varepsilon_2^3 \left[\varepsilon_2^{(2)}\right]^2 + \varepsilon_2^4 \varepsilon_2^{(4)} \tag{52}$$

$$\begin{aligned} q^5 = {} & \varepsilon_2 \left[\varepsilon_2^{(1)}\right]^5 + 26\varepsilon_2^2 \left[\varepsilon_2^{(1)}\right]^3 \varepsilon_2^{(2)} + 32\varepsilon_2^3 \left[\varepsilon_2^{(1)}\right]^2 \varepsilon_2^{(3)} + 34\varepsilon_2^3 \left[\varepsilon_2^{(2)}\right]^2 \varepsilon_2^{(1)} \\ & + 11\varepsilon_2^4 \varepsilon_2^{(1)} \varepsilon_2^{(4)} + 15\varepsilon_2^4 \varepsilon_2^{(2)} \varepsilon_2^{(3)} + \varepsilon_2^5 \varepsilon_2^{(5)} \end{aligned} \tag{53}$$

The analytical solution of the nonlinear equation (48) is known only for $n = 1$. The general solution for a degree higher than one is proposed using an approximation technique.

For $n = 1$, from the relation (46), we have

$$\varepsilon_2 \frac{d\varepsilon_2}{d\varepsilon_1} + a_1 \varepsilon_2 + a_2 \varepsilon_1 = 0 \tag{54}$$

with initial conditions ε_{10} and ε_{20}. Setting

$$\varepsilon_2 = y \varepsilon_1 \tag{55}$$

and substituting (55) into (54), we separate the variables and, after integration, obtain

$$\frac{\varepsilon_1}{\varepsilon_{10}} = \frac{\left| \dfrac{\dfrac{\varepsilon_2}{\varepsilon_1} - y_2}{\dfrac{\varepsilon_{10}}{\varepsilon_{20}} - y_2} \right|^{\frac{y_2}{y_1 - y_2}}}{\left| \dfrac{\dfrac{\varepsilon_2}{\varepsilon_1} - y_1}{\dfrac{\varepsilon_{10}}{\varepsilon_{20}} - y_1} \right|^{\frac{y_1}{y_1 - y_2}}} \tag{56}$$

where y_1 and y_2 are the roots of the equation

$$y^2 + a_1 y + a_2 = 0 \tag{57}$$

We deal with the extremum of the error ε_1 when

$$\varepsilon_2 = 0 \tag{58}$$

Taking into account (57) and (58) in the relation (56), we finally obtain that the extremal value of the error is

$$\varepsilon_e = \sqrt{\frac{a_2\varepsilon_{10}^2 + \varepsilon_{20}^2 + a_1\varepsilon_{10}\varepsilon_{20}}{a_2}} \left| \frac{2a_2\varepsilon_{10} + \left(a_1 + \sqrt{a_1^2 - 4a_2}\right)\varepsilon_{20}}{2a_2\varepsilon_{10} + \left(a_1 - \sqrt{a_1^2 - 4a_2}\right)\varepsilon_{20}} \right|^{-\frac{a_1}{2\sqrt{a_1^2-4a_2}}} \tag{59}$$

For $n > 1$, we will use a polynomial approximation whose unknown coefficients can be established on the basis of the agreement between the initial conditions of the exact and approximate solutions. We consider the following differential equation:

$$\frac{\mathrm{d}^n\varepsilon}{\mathrm{d}t^n} + a_1\frac{\mathrm{d}^{n-1}\varepsilon}{\mathrm{d}t^{n-1}} + \cdots + a_{n-1}\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} + a_n = 0 \tag{60}$$

which corresponds to eqn. (43) with the initial conditions (45). We need to calculate the initial conditions for eqn. (46) using the initial conditions (45).

The relation between the n-th derivative of eqn. (60) and the $(n-1)$-th derivative of eqn. (46) is given by the Leibniz formula. We observe that

$$\frac{\mathrm{d}\varepsilon_2}{\mathrm{d}\varepsilon_1} = \frac{\dfrac{\mathrm{d}\varepsilon_2}{\mathrm{d}t}}{\dfrac{\mathrm{d}\varepsilon_1}{\mathrm{d}t}} = \frac{\varepsilon_1^{(2)}}{\varepsilon_1^{(1)}} \tag{61}$$

$$\frac{\mathrm{d}^2\varepsilon_2}{\mathrm{d}\varepsilon_1^2} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon_1}\left(\frac{\mathrm{d}\varepsilon_2}{\mathrm{d}\varepsilon_1}\right) = \frac{\dfrac{\mathrm{d}}{\mathrm{d}t}\left[\dfrac{\varepsilon_1^{(2)}}{\varepsilon_1^{(1)}}\right]}{\dfrac{\mathrm{d}\varepsilon_1}{\mathrm{d}t}} = \frac{\varepsilon_1^{(3)}\varepsilon_1^{(1)} - \left[\varepsilon_1^{(2)}\right]^2}{\left[\varepsilon_1^{(1)}\right]^3} = \frac{\varepsilon_2^{(2)}\varepsilon_2 - \left[\varepsilon_1^{(1)}\right]^2}{\varepsilon_2^3} \tag{62}$$

$$\vdots$$

$$\begin{aligned}\frac{\mathrm{d}^n\varepsilon_2}{\mathrm{d}\varepsilon_1^n} &= \frac{1}{\varepsilon_2}\frac{\mathrm{d}^{n-1}\left[\varepsilon_2^{(1)}\varepsilon_2^{-1}\right]}{\mathrm{d}t^{n-1}} \\ &= \frac{1}{\varepsilon_2}\left[\varepsilon_2^{(1)}\frac{\mathrm{d}^{n-1}(\varepsilon_2^{-1})}{\mathrm{d}t^{n-1}} + \binom{n-1}{1}\frac{\mathrm{d}\varepsilon_2^{(1)}}{\mathrm{d}t}\frac{\mathrm{d}^{n-2}(\varepsilon_2^{-1})}{\mathrm{d}t^{n-2}}\right. \\ &\quad \left. + \binom{n-2}{2}\frac{\mathrm{d}^2\varepsilon_2^{(1)}}{\mathrm{d}t^2}\frac{\mathrm{d}^{n-3}(\varepsilon_2^{-1})}{\mathrm{d}t^{n-3}} + \cdots + \varepsilon_2^{-1}\frac{\mathrm{d}^{n-1}\varepsilon_2^{(1)}}{\mathrm{d}t^{n-1}}\right], \quad n \geq 2\end{aligned} \tag{63}$$

Using the relations (48) and (61)–(63) we can write the initial problem in the following vector form:

$$\begin{bmatrix} \varepsilon_2 \\ \varepsilon_2^{(1)} \\ \vdots \\ \varepsilon_2^{(n-2)} \\ \varepsilon_2^{n-1}\varepsilon_2^{(n-1)} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2/c_1 \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} \tag{64}$$

4. General Approximation Method

The approximate solution of eqn. (46) is proposed in the form

$$\varepsilon_{2a}(\varepsilon_1) = \alpha_0\varepsilon_1^{n-1} + \alpha_1\varepsilon_1^{n-2} + \cdots + \alpha_{n-2}\varepsilon_1 + \alpha_{n-1} \tag{65}$$

with the same initial conditions as for eqn. (46).

From the same initial conditions of eqns. (46) and (65) we obtain the set of linear equations to determine the unknown coefficients $\alpha_0, \ldots, \alpha_{n-1}$. Denoting by

$$\varepsilon^{(i)}(0) = c_i, \quad i = 0, 1, \ldots, n-1 \tag{66}$$

the initial conditions of (43), we can rewrite eqn. (65) in the following matrix form, taking into account the initial conditions (66):

$$\begin{bmatrix} \varepsilon_{2a} \\ \varepsilon_{2a}^{(1)} \\ \vdots \\ \varepsilon_{2a}^{(n-2)} \\ \varepsilon_{2a}^{n-1}\varepsilon_{2a}^{(n-1)} \end{bmatrix}_{c_0} = \begin{bmatrix} c_1^{n-1} & c_1^{n-2} & \ldots & c_1 & 1 \\ (n-1)c_1^{n-2} & (n-2)c_1^{n-3} & \ldots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (n-2)!c_1 & 1 & \ldots & 0 & 0 \\ (n-1)!c_1^{n-1} & 0 & \ldots & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{bmatrix} \tag{67}$$

Now we require that

$$\begin{bmatrix} \varepsilon_{2a} \\ \varepsilon_{2a}^{(1)} \\ \vdots \\ \vdots \\ \varepsilon_{2a}^{n-1}\varepsilon_{2a}^{(n-1)} \end{bmatrix}_{c_0} = \begin{bmatrix} \varepsilon_2 \\ \varepsilon_{2a}^{(1)} \\ \vdots \\ \vdots \\ \varepsilon_2^{n-1}\varepsilon_2^{(n-1)} \end{bmatrix}_{c_0} \tag{68}$$

By substitution of (64) and (67) into (68) we obtain the set of linear equations for unknown coefficients α_i. Setting $\varepsilon_{2a} = 0$, which is a necessary condition for the extremum, we obtain the extremal values of the error ε_{1a} (from the polynomial (65)).

In order to improve the accuracy, we can differentiate eqn. (46) and take as the approximation a polynomial of a higher degree than that in (65).

We illustrate this method with the example of a differential equation of the third order, which cannot be solved in exact analytical form.

5. Example

We consider the system (43) in the form

$$\begin{bmatrix} \dot{\varepsilon}_1 \\ \dot{\varepsilon}_2 \\ \dot{\varepsilon}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix}, \quad \begin{bmatrix} \varepsilon_1(0) \\ \varepsilon_2(0) \\ \varepsilon_3(0) \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} \tag{69}$$

Equation (46) takes the form

$$\begin{vmatrix} \varepsilon_2 \dfrac{\mathrm{d}}{\mathrm{d}\varepsilon_1} + a_1 & a_2 \\ -1 & \varepsilon_2 \dfrac{\mathrm{d}}{\mathrm{d}\varepsilon_1} \end{vmatrix} \varepsilon_2 = -a_3 \varepsilon_1 \tag{70}$$

which can be explicity written as

$$\begin{cases} \varepsilon_2^2 \dfrac{\mathrm{d}^2 \varepsilon_2}{\mathrm{d}\varepsilon_1^2} + \varepsilon_2 \left(\dfrac{\mathrm{d}\varepsilon_2}{\mathrm{d}\varepsilon_1} \right)^2 + a_1 \varepsilon_2 \dfrac{\mathrm{d}\varepsilon_2}{\mathrm{d}\varepsilon_1} + a_2 \varepsilon_2 + a_3 \varepsilon_1 = 0 \\ \varepsilon_2(c_0) = c_1 \\ \left. \dfrac{\mathrm{d}\varepsilon_2}{\mathrm{d}\varepsilon_1} \right|_{c_0} = \dfrac{c_2}{c_1} \end{cases} \tag{71}$$

As the first approximation we take the polynomial of the second order

$$\varepsilon_{2a} = \alpha_0 \varepsilon_{1a}^2 + \alpha_1 \varepsilon_{1a} \alpha_2 \tag{72}$$

with the same initial conditions as in eqn. (71). Taking into account (67) and (68), we can write

$$\begin{bmatrix} c_1 \\ \dfrac{c_2}{c_1} \\ -\dfrac{c_2^2}{c_1} - a_1 c_2 - a_2 c_1 - a_3 c_0 \end{bmatrix} = \begin{bmatrix} c_0^2 & c_0 & 1 \\ 2c_0 & 1 & 0 \\ 2c_1^2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} \tag{73}$$

From the set of equations (73) we obtain

$$\begin{cases} \alpha_0 = -\dfrac{1}{2c_1^2} \left(\dfrac{c_2^2}{c_1} + a_1 c_2 + a_2 c_1 + a_3 c_0 \right) \\ \alpha_1 = \dfrac{c_2}{c_1} + \dfrac{c_0}{c_1^2} \left(\dfrac{c_2^2}{c_1} + a_1 c_2 + a_2 c_1 + a_3 c_0 \right) \\ \alpha_2 = -\dfrac{c_0 c_2}{c_1} - \dfrac{c_0^2}{2c_1^2} \left(\dfrac{c_2^2}{c_1} + a_1 c_2 + a_2 c_1 + a_3 c_0 \right) \end{cases} \tag{74}$$

Setting $\varepsilon_{2a} = 0$ in (72) we can calculate the extremal values of error from the equation

$$\alpha_0 \varepsilon_{1a}^2 + \alpha_1 \varepsilon_{1a} + \alpha_2 = 0 \tag{75}$$

If we want to obtain a better accuracy, we can differentiate eqn. (71) with respect to ε_1 and obtain

$$\begin{cases} \varepsilon_2^2 \dfrac{\mathrm{d}^3\varepsilon_2}{\mathrm{d}\varepsilon_1^3} + 4\varepsilon_2 \dfrac{\mathrm{d}\varepsilon_2}{\mathrm{d}\varepsilon_1} \dfrac{\mathrm{d}^2\varepsilon_2}{\mathrm{d}\varepsilon_1^2} + a_1 \varepsilon_2 \dfrac{\mathrm{d}^2\varepsilon_2}{\mathrm{d}\varepsilon_1^2} + \left(\dfrac{\mathrm{d}\varepsilon_2}{\mathrm{d}\varepsilon_1}\right)^3 a_1 \left(\dfrac{\mathrm{d}\varepsilon_2}{\mathrm{d}\varepsilon_1}\right)^2 + a_2 \dfrac{\mathrm{d}\varepsilon_2}{\mathrm{d}\varepsilon_1} + a_3 = 0 \\ \varepsilon_2(c_0) = c_1 \\ \dfrac{\mathrm{d}\varepsilon_2}{\mathrm{d}\varepsilon_1} = \dfrac{c_2}{c_1} \\ \dfrac{\mathrm{d}^2\varepsilon_2}{\mathrm{d}\varepsilon_1^2} = \dfrac{-a_3 c_0 - a_2 c_1 - a_1 c_2 - \dfrac{c_2^2}{c_1}}{c_1^2} = p \end{cases} \tag{76}$$

As the polynomial approximation we assume

$$\varepsilon_{2a}(\varepsilon_1) = \alpha_0 \varepsilon_{1a}^3 + \alpha_1 \varepsilon_{1a}^2 + \alpha_2 \varepsilon_{1a} + \alpha_3 \tag{77}$$

The relation analogous to (73) takes the form

$$\begin{bmatrix} c_1 \\ c_2/c_1 \\ p \\ r \end{bmatrix} = \begin{bmatrix} c_0^3 & c_0^2 & c_0 & 1 \\ 3c_0^2 & 2c_0 & 1 & 0 \\ 6c_0 & 2 & 0 & 0 \\ 6c_1^2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \tag{78}$$

where

$$r = -\left(\frac{c_2}{c_1}\right)^3 - 4c_2 p - a_3 - a_2 \frac{c_2}{c_1} - a_1 \left(\frac{c_2}{c_1}\right)^2 - a_1 p c_1 \tag{79}$$

Setting $\varepsilon_{2a}(\varepsilon_1) = 0$ we can calculate approximate extremal values of the error ε_1.

In Fig. 1 the exact extremal values (A) of the equations of the third order and their approximate values (B), (C) obtained by means of the above method for different initial conditions are shown.

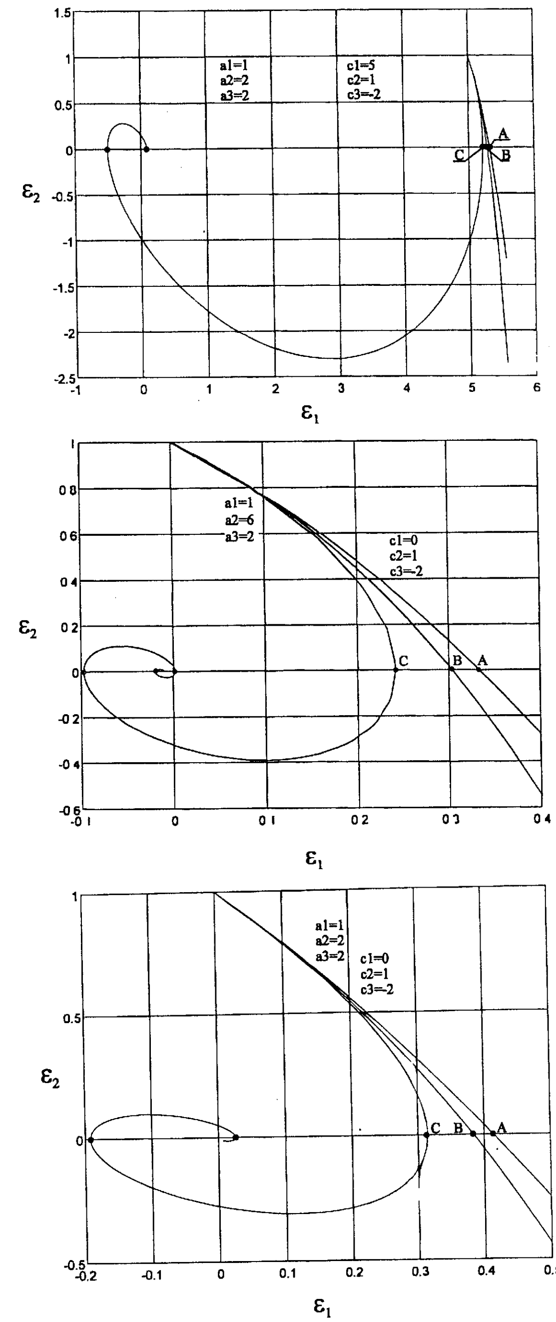

Fig. 1. The extremal values of error ε_1 for the third-order equations: (A) approximation of the extremal value by using the second-order equation, (B) approximation of the extremal value by using the the third-order equation, (C) the exact extremal value of the error ε_1.

6. Conclusions

In the paper the review of results concerning extremal values of the transient errors is given from a general point of view. The limits of extrema and the time of their occurrences are determined. A method of determining these extremal values is also proposed, which does not require the knowledge of the roots of the characteristic equation. Moreover, the knowledge of the time τ is not necessary, either.

The method gives directly a good approximate relation between extremal values and the coefficients of the differential equation in an analytical form. Some numerical examples are also presented.

Acknowledgement

The author wishes to express his gratitude to prof. P. Grabowski and Ms A. Sieniawska-Jakóbczak for calculation of the examples on a computer.

References

El-Khoury M., Oscard D., Longchamp R. (1994): *Discrete Transfer-Function Zeros and Step-Response Extrema.* — Chemical Eng. Dept., University of Florida, USA, v.3, pp.81–86.

Górecki H. (1960): *Optimum transient problem in linear automatic control systems.* — Automatic and Remote Control, Proc. 1st IFAC Congress, Moscow, v.1, pp.59–61.

Górecki H. (1965): *Overshootings in linear and nonlinear control systems.* — Archive of Automatics and Telemechanics, v.X, No.2, pp.95–102, (in Polish).

Górecki H. (1966a): *About some linear adaptive control systems.* — Proc. IX Int. Convention on *Automation and Instrumentation*, FAST, Milano, Italy, pp.103–123.

Górecki H. (1966b): *Extreme values of transients in linear and nonlinear control systems.* — Proc. 3rd IFAC Congress, London, Book 2, Session XXIII, v.1, pp.G1–G4.

Górecki H. and Turowicz A. (1966a): *Extremal transients in linear automatic control systems.* — Scientific Bulletins of the University of Mining and Metallurgy, Automatics, v.1, No.112, pp.11–67, (in Polish).

Górecki H. and Turowicz A. (1966b): *On the extremal transient processes in linear systems.* — Bulletin de l'Académie Polonaise des Sciences, Série des sciences math., astr. et phys.- v.XIV, No.11, pp.609–613.

Górecki H. and Turowicz A. (1968): *Determination of the dependence of the maximal deviation of the controlled quantity and the time of deviation on the parameters in linear control systems.* — Avtomatika and Telemekhanika, No.6, pp.179–181, (in Russian).

Kobayashi H. (1993): *Output overshoot and pole-zero configuration.* — Proc. 12-th IFAC Congress, Sydney, v.3, pp.73–76.

Kobayashi H. and Shimemura E. (1981): *Some properties of optimal regulators and their applications.* — Int. J. Control, v.33, No.4, pp.587–599.

Sędziwy S. (1967): *Extremal transient problems in linear systems.* — Bulletin de l'Académie Polonaise des Sciences, Série des sciences math., astr. et phys. v.XV, No.8, pp.557–561.

Sędziwy S. (1969): *On extremal transient in linear systems.* — Bulletin de l'Académie Polonaise des Sciences, Séries des sciences math., astr. et phys. v.XVII, No.3, pp.141–145.

Szymkat M. (1983): *Form of the n-th iteration of the operator* $q = f\frac{\mathrm{d}}{\mathrm{d}x}$. — Applied Mathematics, v.XXII, seria III, pp.227–249, (in Polish).

Received: June 6, 1996