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# TWO-WAY RECURSIVE DECOMPOSITION OF THE FREE-FLYING ROBOT MASS MATRIX

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This paper shows that the mass matrix for free-flying $(N+1)$-link serial robots can be expressed as $\mathcal{M} = H[r + C_f\widetilde{\phi}_f r_f + C_b\widetilde{\phi}_b r_b]H^*$, which corresponds to the optimal combination of the covariance of two independent spatially recursive random processes, one going from left-to-right along the spatial span of the system, and the other going in the opposite direction. The term 2-way recursive decomposition is used here to describe the corresponding mass matrix that is synthesized by the two opposite recursions. The combination is optimal in a statistical sense analogous to situations in which two independent estimates, with known estimation error covariances, are combined. The coefficient matrices $C_f$ and $C_b$ add up to the unit matrix so that $C_f + C_b = I$. These coefficients are expressed as $C_f = r_b(r_f + r_b)^{-1}$ and $C_b = r_f(r_f + r_b)^{-1}$ in terms of what are referred to respectively as the "forward" composite body mass matrix kernel $r_f$ and the "backward" composite body mass matrix kernel $r_b$. The $6N \times 6N$ matrix $r_f = \operatorname{diag}[r_f(1), \ldots, r_f(N)]$ is a block-diagonal matrix whose typical diagonal block $r_f(k)$ is the 6-dimensional mass associated with the composite body formed by collecting all of the links of the system from the left tip to the joint $k$. Similarly, $r_b = \operatorname{diag}[r_b(1), \ldots, r_b(N)]$ is a block-diagonal matrix whose diagonal block $r_b(k)$ is the 6-dimensional spatial mass of the composite body formed by all of the links from joint $k$ to the right-end of the system. The forward and backward mass matrices are combined by the equation $r^{-1} = r_f^{-1} + r_b^{-1}$, which is quite analogous to the way two masses are combined in classical mechanics to obtain what is typically referred to as a "reduced" mass. The spatial operators $\widetilde{\phi}_f$ and $\widetilde{\phi}_b$ are defined in terms of the vectors that link one joint to the next joint. They represent respectively forward and backward spatial recursions along the span of the system. This expression for the mass matrix leads to a spatially recursive inverse dynamics algorithm obtained by summing two independent algorithms going in opposite direction to each other. The computations necessary to evaluate the mass matrix, as well as the corresponding inverse dynamics algorithm, are all implemented by means of the spatially recursive operations advanced by the authors in recent years.

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# 1. Introduction

This is an additional chapter in a story which the authors have been uncovering in recent years (Jain and Rodriguez, 1995a; 1995b; Rodriguez, 1987; 1990; Rodriguez and Kreutz-Delgado, 1992), which draws from the Kalman filtering and smoothing techniques of estimation theory, to solve robot dynamics problems by spatially recursive methods. The primary problems addressed in the paper are those associated with the dynamics of free-flying robots projected for future space applications (Bejczy *et al.*, 1993; Mukherjee and Nakamura, 1992). The space robots have a unique characteristic: they are free to rotate and translate with respect to inertial coordinates, without being constrained to be immobile at their base. This lack of a constraint at the base results in distinct robot dynamics characteristics that are typically not present in more traditional base-attached robotic manipulators. The main goal of this paper is to investigate these unique characteristics, particularly as they are reflected in the free-flying robot mass matrix.

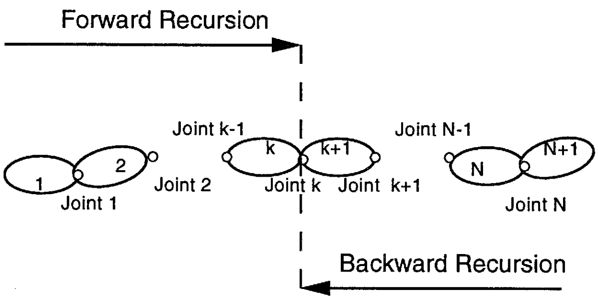


Fig. 1. $(N+1)$-link free-flying mechanical system.

The configuration analyzed in this paper is illustrated in Fig. 1, and it consists of a series of $N+1$ links connected together by $N$ rotational joints, with each joint having a single degree-of-freedom in rotation. The extension to other type of joints is easy and has been addressed previously (Jain and Rodriguez, 1995a; 1995b; Rodriguez, 1987; 1990; Rodriguez and Kreutz-Delgado, 1992). This configuration can move with a rigid-body motion, in which all of the joints are locked at a fixed angle, and the resulting "rigidized" system moves in 6 dimensions with respect to inertial coordinates. In addition to this rigidized-body motion, the system can also articulate as a result of articulation at its internal joints.

## 1.1. Forward and Backward Recursions

The type of dynamics solutions investigated here are also illustrated in Fig. 1. First, the dynamics algorithms process the various dynamical quantities one at a time, in two independent sequences. One of the two sequences starts at one end, and the other starts at the opposite end. The two independent sequences go in opposite directions. The forward sequence, illustrated by the top arrow in the figure, starts at the left end, moves from left to right, and terminates at the right end of the system. Similarly, the backward sequence starts at the right end, moves from right to left, and terminates at the right end. Each of the sequences represents a spatially recursive algorithm of the type developed in (Jain and Rodriguez, 1995a; 1995b; Rodriguez, 1987; 1990; Rodriguez and Kreutz-Delgado, 1992) to solve robot dynamics problems.

The forward algorithm uses only "past" information, in the sense that its output at any given body in the sequence depends only on dynamical quantities, such as link masses, associated with bodies to the left of the given body. Similarly, the backward algorithm uses only "future" information associated with bodies to the right of the given body. Together, these two algorithms use precisely all of the available information. The algorithms are independent in the sense that their outputs are uncorrelated to each other. Due to this independence, the outputs of the two algorithms can be combined in an optimal sense, using the by now classical result (Fraser and Potter, 1969; Gelb, 1974; Liebelt, 1967) of the optimal combination of two uncorrelated state estimates. The optimal combination turns out to be an optimally weighted sum. It is this optimal weighted sum that is used to determine the dynamics solutions advanced in this paper, and the corresponding expression for the free-flying robot mass matrix.

### 1.2. Two-Way Recursive Decomposition of the Mass Matrix

The main result of the paper is summarized by Table 1. The decomposition for the mass matrix shown in the table is a 2-way recursive decomposition, in the sense that it can be synthesized by two independent recursive algorithms going in opposite directions, one moving in the forward direction and the other moving in the backward direction. These algorithms are characterized respectively by the rigid transition operators $\widetilde{\phi}_f$ and $\widetilde{\phi}_b$. The operator $\widetilde{\phi}_f$ is a rigid transition because it is used to transfer forces and velocities from one link to the next one in a forward direction from left to right, starting at the extreme left end of the multilink system, with the multilink system rigidized at all its joints. The operator $\widetilde{\phi}_b$ is a rigid transition in a similar sense, except that this operator corresponds to a backward recursion from the right extreme to the left extreme of the multilink system. The operator $\widetilde{\phi}_f$ is strictly lower-triangular, whereas $\widetilde{\phi}_b$ is strictly upper-triangular.

Table 1. The main result of the paper.

| | |
|---|---|
| $\mathcal{M}$ | $H[r + C_f\widetilde{\phi}_f r_f + C_b\widetilde{\phi}_b r_b]H^*$ |
| $r_f$ | Forward Composite Mass |
| $r_b$ | Backward Composite Mass |
| $r$ | Reduced Composite Mass |
| $\widetilde{\phi}_f$ | Forward Rigid Transition Operator |
| $\widetilde{\phi}_b$ | Backward Rigid Transition Operator |
| $H$ | Joint Projection Operator |

The mass matrix decomposition therefore decomposes the mass matrix as the sum of a diagonal matrix $HrH^*$, a strictly lower-triangular matrix characterized by the forward transition operator $\widetilde{\phi}_f$, and a strictly upper-triangular matrix characterized by the backward transition operator $\widetilde{\phi}_b$.

The 2-way recursive decomposition for the mass matrix depends on two key quantities: the forward composite mass $r_f$ and the backward composite mass $r_b$. The forward composite mass $r_f(k)$ at any given joint represents the total spatial mass (Jain and Rodriguez, 1995a; 1995b; Rodriguez, 1987; 1990; Rodriguez and Kreutz-Delgado, 1992) of the collection of bodies to the left of this joint, assuming that all of the joints are "rigidized" in the sense that no rotation is allowed. The composite mass $r_f$ does not allow joint articulation. Similarly, the backward composite mass $r_b(k)$ represents the total mass of the collection of bodies to the right of the given joint, assuming that all of the joints are locked.

### 1.3. Reduced Composite Mass

The block-diagonal matrix $r = r_f(r_f + r_b)^{-1}r_b$ appearing in Table 1 is computed by combining the "forward" composite spatial mass matrix $r_f$ with the "backward" composite mass matrix $r_b$. The $k$-th diagonal element $r_f(k)$ of the block-diagonal matrix $r_f = \text{diag}\,[r_f(1), \ldots, r_f(N)]$ represents the spatial mass of the composite body formed by putting together all of the manipulator links outboard, toward the tip, of joint $k$. Similarly, $r_b$ is a block-diagonal matrix, whose generic element $r_b(k)$ equals the spatial mass of the composite body formed by putting together all of the links inboard, toward the base, of joint $k$. The combined mass matrix $r$ is analogous to the scalar "reduced mass" obtained in classical mechanics (Goldstein, 1950) when combining two scalar masses in translational motion.

### 1.4. Two-Way Recursive Algorithms for Inverse Dynamics

A by-product of the recursive decomposition for the mass matrix is the specification of inverse dynamics algorithms, in which two independent recursions, moving in opposite directions, are combined optimally to arrive at the inverse dynamics solution for the free-flying robotic system.

### 1.5. New Results Drawn from the Interplay of Classical Mechanics and Estimation Theory

The paper arrives at new results by continuing to investigate the interplay between classical mechanics and estimation theory, which has been set forth in (Jain and Rodriguez, 1995a; 1995b; Rodriguez, 1987; 1990; Rodriguez and Kreutz-Delgado, 1992). It begins conceptually from the result of (Fraser and Potter, 1969), in which the optimal smoother is obtained by combining two independent linear filters, and extends this idea from the time-domain to the very complex situation of spatial relationships associated with articulated multibody systems. To achieve this extension, the paper draws heavily on previous results of the authors on the application of spatially recursive and operator methods to robot dynamics. Once the extension to mechanical systems of the fundamental idea in (Fraser and Potter, 1969) is achieved, previously unrecognized architectural characteristics of the mass matrix are revealed, which would have been very difficult, if not impossible to obtain using other methods.

The 2-way recursive solutions developed here have an interesting physical interpretation, as the generalization to multibody systems of classical results in the combination of two masses in dynamical motion. In doing this, the paper presents one more circle of ideas at the intersection of estimation theory and mechanics.

## 2. Spatial Operator Model

The analysis contained in this paper is based on the spatial operator notation explained in substantial detail in (Jain and Rodriguez, 1995a; 1995b; Rodriguez, 1987; 1990; Rodriguez and Kreutz-Delgado, 1992), appropriately extended to account for the 2-way symmetric formulation to be developed here.

The "forward" spatial operators, corresponding to algorithms that go from left-to-right of the system, are essentially identical to those developed previously in these references. The "backward" operators, those associated with algorithms that go from right-to-left, are quite similar to the forward algorithms, except that they go in the opposite direction.

This section provides a brief summary of this spatial operator notation, and its extension to the 2-way symmetric formulation. To illustrate ideas, at a few places in the paper, an example with 4 bodies and 3 joints is considered. The bodies are labeled 1, 2, 3, 4 starting at one of the two ends. There is no generality lost in focusing on this 4-body system, as the essential characteristics of all of the spatial operators defined for this system are identical to those of the more general system with an arbitrary number of links and joints.

### 2.1. Forward and Backward Shift Operators

Let us define the operators

$$\mathcal{E}_{\phi f} = \begin{pmatrix} 0 & 0 & 0 \\ \phi(2,1) & 0 & 0 \\ 0 & \phi(3,2) & 0 \end{pmatrix}, \quad \mathcal{E}_{\phi b} = \begin{pmatrix} 0 & \phi(1,2) & 0 \\ 0 & 0 & \phi(2,3) \\ 0 & 0 & 0 \end{pmatrix} \tag{1}$$

They are related by the identities

$$\mathcal{E}_{\phi f}\mathcal{E}_{\phi b} = I - \pi_b^* \pi_b, \qquad \mathcal{E}_{\phi b}\mathcal{E}_{\phi f} = I - \pi_f^* \pi_f \tag{2}$$

where $\pi_f = [0, 0, I]$ and $\pi_b = [I, 0, 0]$ are "pick-off" operators. To illustrate the identity $\mathcal{E}_{\phi f}\mathcal{E}_{\phi b} = I - \pi_b^* \pi_b$, observe that

$$\begin{pmatrix} 0 & 0 & 0 \\ \phi(2,1) & 0 & 0 \\ 0 & \phi(3,2) & 0 \end{pmatrix} \begin{pmatrix} 0 & \phi(1,2) & 0 \\ 0 & 0 & \phi(2,3) \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \tag{3}$$

The rigid forward-shift operator $\mathcal{E}_{\phi f}$ is used to transfer spatial forces from one joint to the next joint in a forward direction. References (Jain and Rodriguez, 1995a; 1995b; Rodriguez, 1987; 1990; Rodriguez and Kreutz-Delgado, 1992) contain a more complete discussion of the physical interpretation of this operator. The rigid backward-shift operator $\mathcal{E}_{\phi b}$ is used to transfer spatial forces in the opposite direction.

### 2.2. Rigid Transition Operators

The rigid transition operators $\phi_f$ and $\phi_b$ can be obtained from the corresponding rigid shift operators $\mathcal{E}_{\phi f}$ and $\mathcal{E}_{\phi b}$:

$$\phi_f = (I - \mathcal{E}_{\phi f})^{-1}, \qquad \phi_b = (I - \mathcal{E}_{\phi b})^{-1} \tag{4}$$

which imply

$$\phi_f = \begin{pmatrix} I & 0 & 0 \\ \phi(2,1) & I & 0 \\ \phi(3,1) & \phi(3,2) & I \end{pmatrix}, \qquad \phi_b = \begin{pmatrix} I & \phi(1,2) & \phi(1,3) \\ 0 & I & \phi(2,3) \\ 0 & 0 & I \end{pmatrix} \tag{5}$$

Observe the relationships

$$\tilde{\phi}_f = \mathcal{E}_{\phi f}\phi_f = \phi_f - I, \qquad \tilde{\phi}_b = \mathcal{E}_{\phi b}\phi_b = \phi_b - I \tag{6}$$

and

$$\tilde{\phi}_f = \begin{pmatrix} 0 & 0 & 0 \\ \phi(2,1) & 0 & 0 \\ \phi(3,1) & \phi(3,2) & 0 \end{pmatrix}, \qquad \tilde{\phi}_b = \begin{pmatrix} 0 & \phi(1,2) & \phi(1,3) \\ 0 & 0 & \phi(2,3) \\ 0 & 0 & 0 \end{pmatrix} \tag{7}$$

The decomposition for the mass matrix is a 2-way recursive decomposition, in the sense that it can be synthesized by two indepedent recursive algorithms characterized respectively by the rigid transition operators $\tilde{\phi}_f$ and $\tilde{\phi}_b$. The operator $\tilde{\phi}_f$ is a rigid transition because it is used to transfer forces and velocities from one link to the next one in a forward direction from left to right, starting at the extreme left end of the multilink system, with the multilink system rigidized at all its joints. The operator $\tilde{\phi}_b$ is a rigid transition in a similar sense, except that this operator corresponds to a backward recursion from the right extreme to the left extreme of the multilink system. The operator $\tilde{\phi}_f$ is strictly lower-triangular, whereas $\tilde{\phi}_b$ is strictly upper-triangular.

### 2.3. Forward and Backward Spatial Masses for Each Link

Associated with each link $k$, there is a "forward" spatial matrix $M_f(k)$ representing the $6 \times 6$ mass matrix of the link about joint $k$. This can be computed by the volume integral

$$M_f(k) = \int \phi(k,x) B \rho(x) B^* \phi^*(k,x)\,\mathrm{d}x = \begin{pmatrix} \mathcal{I}_f(k) & m(k)\tilde{c}_f(k) \\ -m(k)\tilde{c}_f(k) & m(k)U \end{pmatrix} \tag{8}$$

where $B^*$ is defined as the $3 \times 6$ matrix $B^* = [0, I]$, and $\rho(x)$ is the scalar mass density at the point $x$, an arbitrary point inside the $k$-th link. The integral is performed over the volume occupied by this link. The spatial mass of the link is characterized by 10 scalar quantities (Rodriguez, 1987): 6 scalars for the rotational inertia $\mathcal{I}_f(k)$ about joint $k$; 3 scalars for the vector $c_f(k)$ from the joint $k$ to the link mass center; and 1 scalar for the total mass of the link. Similarly, the "backward" spatial mass $M_b(k-1)$ for link $k$ reflected at the joint $k-1$ is the $6 \times 6$ matrix

$$M_b(k-1) = \int \phi(k-1,x)B\rho(x)B^*\phi^*(k-1,x)\,\mathrm{d}x \tag{9}$$

represents the spatial inertia of link $k$ about link $k-1$. These two matrices are related by the identities

$$M_b(k-1) = \phi(k-1,k)M_f(k)\phi^*(k-1,k) \tag{10}$$

$$M_f(k) = \phi(k,k-1)M_b(k-1)\phi^*(k,k-1) \tag{11}$$

In terms of the rigid-shift operators $\mathcal{E}_{\phi f}$ and $\mathcal{E}_{\phi b}$, these identities are:

$$M_b = \mathcal{E}_{\phi b} M_f \mathcal{E}_{\phi b}^*, \qquad M_f = \mathcal{E}_{\phi f} M_b \mathcal{E}_{\phi f}^* \tag{12}$$

where $M_f = \operatorname{diag}[M_f(1), \ldots, M_f(N)]$ and $M_b = \operatorname{diag}[M_b(1), \ldots, M_b(N)]$ are $6N \times 6N$ block-diagonal matrices.

### 2.4. Forward and Backward Composite Masses

The "forward" composite mass at any given joint is the spatial mass matrix (Jain and Rodriguez, 1995a; 1995b; Rodriguez, 1987; 1990; Rodriguez and Kreutz-Delgado, 1992) of the collection of bodies to the left of this joint, assuming that all of the system joints are rigidized. The joints are not allowed any rotation. Similarly, the "backward" composite mass at any given joint is the spatial mass matrix of the collection of bodies to the right of the joint, assuming all of the joints in this collection are locked. An illustration of the forward and backward composite bodies is shown in Fig. 2.

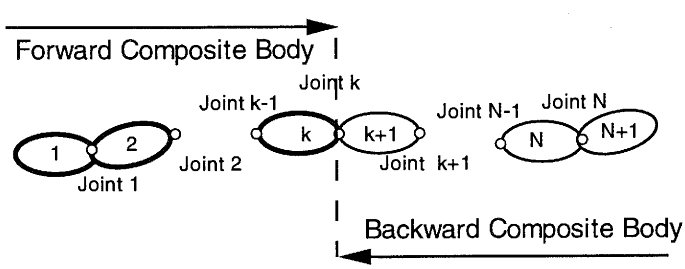


Fig. 2. Forward and backward composite bodies.

Table 2 summarizes algorithms to evaluate the forward and backward spatial masses, by 2 independent recursions going in opposite directions. The forward algorithm has been discussed at length in previous work by the authors (Jain and

Rodriguez, 1995a; 1995b; Rodriguez, 1987; 1990; Rodriguez and Kreutz-Delgado, 1992). The backward algorithm, presented here for the first time, is a relatively simple extension of the forward algorithm.

Table 2. Recursions to compute composite masses.

| FORWARD | BACKWARD |
|---|---|
| $r_f = \mathcal{E}_{\phi f} r_f \mathcal{E}_{\phi f}^* + M_f$ | $r_b = \mathcal{E}_{\phi b} r_b \mathcal{E}_{\phi b}^* + M_b$ |
| $r_f(0) = 0$ | $r_b(N+1) = 0$ |
| $k = 1, \ldots, N$<br>$r_f(k) = \phi(k, k-1) r_f(k-1) \phi^*(k, k-1)$<br>$+M_f(k)$ | $k = N, \ldots, 1$<br>$r_b(k) = \phi(k, k+1) r_b(k+1) \phi^*(k, k+1)$<br>$+M_b(k)$ |

These two independent recursions compute the "forward" composite mass matrix $r_f = \text{diag}\,[r_f(1), \ldots, r_f(N)]$, and the "left" composite mass matrix $r_b = \text{diag}\,[r_b(1), \ldots, r_b(N)]$. In the following section, these two matrices are combined to arrive at the "reduced" mass matrix kernel $r = r_b(r_f + r_b)^{-1} r_f$, from which the diagonal elements of the free-flying manipulator mass matrix can be readily computed.

## 2.5. Reduced Composite Mass

The reduced mass matrix kernel $r$ is defined as

$$r = r_b(r_f + r_b)^{-1} r_f = r_f(r_f + r_b)^{-1} r_b \tag{13}$$

in terms of the FORWARD composite mass matrix $r_f$ and the BACKWARD composite mass matrix $r_b$. This equation states that the reduced kernel $r$ can be computed by taking the product of the two component matrices $r_f$ and $r_b$, and dividing by the sum $r_f + r_b$ of these two matrices. In this last statement, division is interpreted to mean matrix inversion, and the word product is interpreted to mean that the matrix inverse is "sandwiched" by the operator $r_f[\,\cdot\,]r_b$ which appears in the numerator of $r = r_f(r_f + r_b)^{-1} r_b$. Alternatively,

$$r^{-1} = r_f^{-1} + r_b^{-1} \tag{14}$$

The formulas in eqns. (13) and (14) for combining composite spatial masses are analogous to the well-known formula in classical mechanics for analyzing the motion of 2 scalar masses in translation. They can be viewed as generalizations to linked, multibody systems defined in 6 spatial dimensions, 3 for translation and 3 for rotation, of the classical results for scalar masses. For example, Goldstein (1950) discusses how the concept of a reduced scalar mass is used to analyze the problem of two bodies under attraction by a central force. The central force motion of two bodies about their center of mass can always be reduced to an equivalent one-body problem, where the scalar mass of the single equivalent body is the reduced scalar mass.

The reduced mass matrix $r = r_f(r_f + r_b)^{-1} r_b$ has been defined for the entire system, in that it is a block-diagonal matrix $r = \text{diag}\,[r(1), \ldots, r(N)]$ defined over the entire spatial system configuration. Alternatively, $r^{-1} = r_f^{-1} + r_b^{-1}$. These are two formulas that determine the reduced system mass matrix from the FORWARD and BACKWARD composite mass matrices $r_f$ and $r_b$, respectively. The diagonal elements $r(k)$ of this reduced mass matrix can be computed by very similar formulas:

$$r(k) = r_f(k)[r_f(k) + r_b(k)]^{-1} r_b(k), \quad r^{-1}(k) = r_f^{-1}(k) + r_b^{-1}(k) \tag{15}$$

These last two equations apply to any particular joint in the system. As illustrated in Fig. 2, for any arbitrary joint $k$, there are two independent collection of bodies: (1) a FORWARD composite body consisting of all of the links to the left of the given joint, with a corresponding FORWARD composite mass matrix $r_f$, and (2) a BACKWARD composite body consisting of all of the links to the right of the given joint, with a corresponding BACKWARD composite mass matrix $r_b$.

## 3. Two-Way Recursive Mass Matrix Decomposition

The foregoing operator notation provides all of the necessary ingredients to describe the following 2-way recursive factorization of the system mass matrix. The mass matrix is formed by combining two independent fixed-base manipulator recursions, going in opposite directions to each other.

**Identity 1.** *The mass matrix for a "free-flying" serial manipulator, one which is unconstrained at both ends, can be expressed as* $\mathcal{M} = H\overline{R}H^*$ *with*

$$\overline{R} = r + C_f \tilde{\phi}_f r_f + C_b \tilde{\phi}_b r_b \tag{16}$$

*where* $r = r_f(r_f + r_b)^{-1} r_b$ *is the reduced composite mass. The weighting matrices*

$$C_f = r_b(r_f + r_b)^{-1} = r r_f^{-1}, \qquad C_b = r_f(r_f + r_b)^{-1} = r r_b^{-1} \tag{17}$$

*add up to the identity so that* $C_f + C_b = I$.

The lower-triangular matrix $C_f \tilde{\phi}_f r_f$, involving the weighting coefficient $C_f = r_b(r_f + r_b)^{-1}$, is determined by combining the forward and backward composite mass matrices $r_f$ and $r_b$, respectively. This weighting coefficient multiplies the term $\tilde{\phi}_f r_f$ which appears in the forward composite mass matrix representation of the "attached-base" manipulator mass matrix. The upper-triangular matrix $C_b \tilde{\phi}_b r_b$ involves the weighting coefficient matrix $C_b = r_f(r_f + r_b)^{-1}$. This matrix multiplies the term $\tilde{\phi}_b r_b$ which appears in the backward composite mass matrix representation of the "attached-tip" manipulator mass matrix.

### 3.1. Two-Way Recursive Algorithm to Evaluate the Mass Matrix

The operator decomposition described in Identity 1 leads to a corresponding algorithm to evaluate the mass matrix recursively. To describe this algorithm, it is convenient to refer to Table 2, which specifies spatial recursions to compute the forward and backward composite masses $r_f$ and $r_b$. It is assumed in this section that these two recursions are used to compute $r_f$ and $r_b$, as well as to evaluate the resulting reduced composite mass $r = r_f(r_f + r_b)^{-1}r_b$, and the forward weighting coefficient $C_f = r_b[r_f + r_b]^{-1}$. This is done here only to make it easier to describe the recursive algorithm presented below to evaluate the mass matrix. However, in practice, this algorithm can easily be combined with the algorithms in Table 2 which compute the essential quantities $r_f$ and $r_b$. Since the mass matrix is symmetric, it is sufficient to evaluate its elements $\mathcal{M}(k,m)$ in the lower triangle $m \le k \le N$.

**Algorithm 1.** *The mass matrix element $\mathcal{M}(k,m)$ in the lower-triangle $m \le k \le N$ can be computed by*
**loop** $m = 1, \ldots, N$

$$X(m) = r_f(m)H^*(m), \qquad \mathcal{M}(m,m) = H(m)X(m) \tag{18}$$

**loop** $k = m+1, \ldots, N$

$$X(k) = \phi(k,k-1)X(k-1), \qquad \mathcal{M}(k,m) = H(k)C_f(k)X(k) \tag{19}$$

**end** $k$ **loop**
**end** $m$ **loop**

### 3.2. Two-Way Recursive Inverse Dynamics

**Algorithm 2.** *The "computed" moments $T(k)$ which result from the specified joint accelerations $\alpha(k)$ are provided by the weighted sum of two independent inverse dynamics recursions given in Table 3.*

Table 3. Two-way recursive inverse dynamics.

| FORWARD | BACKWARD |
|---|---|
| $X_f(0) = 0$ | $X_b(N+1) = 0$ |
| $k = 1, \ldots, N$ | $k = N, \ldots, 1$ |
| $X_f(k) \to \phi(k,k-1)X_f(k-1)$ | $X_b(k) \to \phi(k,k+1)X_b(k+1)$ |
| $X_f(k) \to X_f(k) + r_f(k)H^*(k)\alpha(k)$ | $X_b(k) \to X_b(k) + r_b(k)H^*(k)\alpha(k)$ |
| $X(k) = C_f(k)X_f(k) + C_b(k)X_b(k)$ | |
| $T(k) = H(k)X(k)$ | |

The algorithm assumes that the composite masses $r_f$ and $r_b$, and the corresponding weighting coefficients $C_f = r_b(r_f + r_b)^{-1}$ and $C_b = r_f(r_f + r_b)^{-1}$, come from the recursive algorithms in Table 2, which could of course be run concurrently with the present algorithm.

Algorithm 2 is based on the equation $T = \mathcal{M}\alpha$, where $\mathcal{M}$ is the mass matrix, $\alpha = [\alpha(1), \ldots, \alpha(N)]$ is a vector of desired angular accelerations at the joints, and the vector of applied joint moments $T = [T(1), \ldots, T(N)]$ has to be applied in order to achieve the desired joint accelerations. It is a generalization of the traditional (Luh *et al.*, 1980) inverse dynamics algorithm for fixed-base manipulators. It is a generalization in two important respects: (1) it is based on composite masses instead of the masses of the individual links, and (2) it is based on two independent recursions which are implemented in parallel, instead of consisting of a forward recursion followed sequentially by a backward recursion. Nonetheless, because it solves the inverse dynamics problem using spatial recursions, it can be viewed as being in the same spirit as the traditional (Luh *et al.*, 1980) inverse dynamics solutions for fixed-base systems.

## 4. Derivation of the Two-Way Recursive Decompositions

No proofs have been presented up to here. Instead, the preceding sections have focused on stating the 2-way recursive decomposition for the mass matrix, on outlining the corresponding spatially recurive algorithms for inverse dynamics, and on physical interpretation. This section fills this gap by focusing on the analytical derivation of the recursive decomposition results summarized in the previous sections. The proofs begin by first decomposing the total kinetic energy as the sum of two terms: (1) an energy term due to rigid body motion in inertial space of the "rigidized" multibody system, and (2) a term due to internal articulation at the system joints. This energy decomposition is achieved in the following subsection. After this, a series of spatial operator identities are established leading incrementally to the desired 2-way recursive decompositions set forth in the previous sections. In addition to having value in their own right, the derivation of the recursive decompositions lead to additional physical insights concerning the overall system kinetic energy and the multibody system mass matrix.

### 4.1. Kinetic Energy due to Rigid and Articulated Motion

A good place to start in analyzing the dynamics of free-flying systems is the decomposition of the total system kinetic energy as the sum of energy due to rigid body motion plus the energy due to internal articulated motion. This is the approach taken in this paper. To this end, the total kinetic energy can be expressed as

$$J(V_{N+1},\dot{\theta}) = K.E. = \frac{1}{2}V^* M_f V + \frac{1}{2}V^*(N+1)M_f(N+1)V(N+1) \tag{20}$$

where the first term on the right side represents the kinetic energy due to links $1,\ldots,N$, and the second term is the energy due to the last link $N+1$. The "stacked" spatial velocity $V = [V(1),\ldots,V(N)]$ is a composite vector, with $V(k)$ being the spatial velocity at joint $k$. This "stacked" spatial velocity is

$$V = \phi_f^* H^* \dot{\theta} + \phi_f^* C^* V_{N+1} \tag{21}$$

and $C = [0,0,\ldots,\phi(N+1,N)]$. Observe that $V_{N+1} = V(N+1)$ is the velocity of the base of the robot. Typically, this base is immobile, so the corresponding spatial velocity $V(N+1)$ vanishes. However, in the case of the free-flying robot under investigation here, the base velocity may be non-zero. Substitution of (21) into (20) leads to

$$J(V_{N+1},\dot{\theta}) = \frac{1}{2}\begin{pmatrix} V_{N+1}^* & \dot{\theta}^* \end{pmatrix}\begin{pmatrix} r_f(N+1) & CRH^* \\ HRC^* & HRH^* \end{pmatrix}\begin{pmatrix} V_{N+1} \\ \dot{\theta} \end{pmatrix} \tag{22}$$

where to simplify eqn. (22), the symbols $V_{N+1}$ has been used instead of $V(N+1)$.

The kernel $R = \phi_f M_f \phi_f^*$ is the central kernel of the "fixed-base" manipulator mass matrix $HRH^*$ corresponding to an $N+1$ link manipulator in which the last link corresponds to a fixed base which is not moving. The kernel $R$ has been extensively studied by the authors (Jain and Rodriguez, 1995a; 1995b; Rodriguez, 1987; 1990; Rodriguez and Kreutz-Delgado, 1992) in analyzing the dynamics of fixed-base manipulators. Here the objective is to observe that this fixed-base mass matrix kernel plays a significant role in arriving at a related mass matrix kernel for the free-flying systems analyzed in this paper.

### 4.2. Minimal Kinetic Energy: No Rigid Body Motion

Now, the goal is to find a relationship between the base velocity $V(N+1)$ and the joint angle rates $\dot{\theta}$, so that the total value of the kinetic energy is minimized. This is done by minimizing the kinetic energy in eqn. (22) with respect to $V(N+1)$, under the condition that joint angle rates $\alpha$ are prescribed. This would correspond to a physical state of the manipulator, in which all of the motion is "internal" in the sense that it is due only to the joint angle rates. Motion of the composite mass center of the complete system, which could be interpreted as external motion, is not included in the resulting minimal value of the kinetic energy. The base "WIGGLE" velocity $V_{WIGGLE}(N+1)$, that results at the base link $N+1$ due to inner joint articulation, is provided in the following identity.

**Identity 2.** *The WIGGLE velocity $V_{WIGGLE}(N+1)$ due to internal joint articulation, and which thereby minimizes the kinetic energy, is*

$$V_{WIGGLE}(N+1) = -r_f^{-1}(N+1)CRH^*\dot{\theta} \tag{23}$$

*The corresponding minimal value of the kinetic energy is*

$$J_{MIN}(\dot{\theta}) = \frac{1}{2}\dot{\theta}^* H\overline{R}H^*\dot{\theta} \tag{24}$$

This result is established by minimizing the kinetic energy in eqn. (22) with respect to $V_{N+1}$ with $\dot{\theta}$ held constant. The matrix $r_f(N+1)$ that needs to be inverted is the forward composite mass matrix consisting of all of the links $1,\ldots,N+1$ of the manipulator.

The base velocity determined in the above identity is that due only to articulation of the internal system joints. There is no rigid body motion, in the sense that the system is not translating or rotating with respect to inertial coordinates. However, even though there is no rigid-body motion, the base of the system still moves due to the internal system motion at the articulated joints. It is possible to think about this base motion, induced by joint articulation, as a "wiggle", which is present at the system base, even though there is no rigid-body motion of the entire system. The amount of motion at the base, due to articulated motion, is reflected in the velocity $V_{WIGGLE}(N+1)$ given by eqn. (23) in terms of the joint angle rates $\dot{\theta}$.

### 4.3. RIGID + WIGGLE Base Velocity Decomposition

An arbitrary velocity $V(N+1)$ at the base link can be decomposed as the sum of a RIGID base velocity component $V_{RIGID}(N+1)$ due to overall rigid body motion plus a WIGGLE base velocity component due to articulation

$$V(N+1) = V_{RIGID}(N+1) + V_{WIGGLE}(N+1) \tag{25}$$

Substitution of eqn. (25) into eqn. (22) leads to the following decomposition of the kinetic energy as the sum of energy due to rigid-body motion plus energy due to articulation.

### 4.4. RIGID + ARTICULATED Energy Decomposition

**Identity 3.** *The total kinetic energy*

$$J\Big[V(N+1), \dot{\theta}\Big] = \frac{1}{2} V^*_{RIGID}(N+1) r_f(N+1) V_{RIGID}(N+1) + \frac{1}{2}\dot{\theta}^* H\overline{R}H^*\dot{\theta} \tag{26}$$

*is the sum of two terms: the kinetic energy due to the rigid-body motion of the complete system, plus the kinetic energy due to internal articulation.*

The first term $(1/2)V^*_{RIGID}(N+1)r_f(N+1)V_{RIGID}(N+1)$ on the right side of eqn. (26) is that due to the RIGID BODY MOTION of the complete system, involving translation and rotation with respect to inertial coordinates. There are a total of 6 independent degrees-of-freedom associated with this motion, 3 for rotation and 3 for translation. The energy is expressed in terms of the composite body mass $r_f(N+1)$ at the base of the system. This quantity represents the total composite mass of the system, as it is seen at its base. It is also expressed in terms of the spatial velocity, due to rigid body motion and not to articulated motion, at the base of the system.

The second term on the right side of eqn. (26) is $(1/2)\dot{\theta}^*\overline{R}\dot{\theta}$, and it represents the kinetic energy due to ARTICULATED MOTION. This term depends on the "reduced" mass matrix kernel $\overline{R}$, which leads to the mass matrix $\mathcal{M} = H\overline{R}H^*$. It is this mass matrix that is the focus of the present paper. It is the mass matrix associated with articulated motion due to rotation at the joints only.

### 4.5. The Mass Matrix Associated with Energy due to Articulation

**Identity 4.** *The mass matrix $\mathcal{M}$ which determines the kinetic energy $\frac{1}{2}\dot{\theta}\mathcal{M}\dot{\theta}$ due to joint articulation is*

$$\mathcal{M} = H\overline{R}H^* \tag{27}$$

*where*

$$\overline{R} = R - RC^* r_f^{-1}(N+1)CR \tag{28}$$

*and where $r_f(N+1)$ is the forward composite mass at the $N+1$ link tip, and $R = \phi_f M_f \phi_f^*$ is the mass matrix kernel of the fixed-base manipulator mass matrix $HRH^*$.*

This result essentially states that the mass matrix $\mathcal{M} = H\overline{R}H^*$ for a free-base manipulator can be obtained from the fixed-base manipulator mass matrix $HRH^*$ by modifying the kernel $R = \phi_f M_f \phi_f^*$ by means of eqn. (28). Understanding this kernel $\overline{R}$ better is therefore the key to obtaining the 2-way recursive algorithms for the free-base manipulator mass matrix presented in earlier sections. The paper now turns toward developing such an understanding. In particular, the objective is to find an alternative expression to eqn. (28) that evaluates the modified kernel $\overline{R}$ as the weighted-sum of 2 spatial recursions going in opposite directions.

### 4.6. Two-Way Recursive Decomposition of the Mass Matrix Kernel

The goal of this subsection is to determine the following identity, which represents the desired 2-way recursive decomposition for the mass matrix.

**Identity 5.**

$$\overline{R} = r[I + r_f^{-1}\widetilde{\phi}_f r_f + r_b^{-1}\widetilde{\phi}_b r_b] \tag{29}$$

*where $r = r_f(r_f + r_b)^{-1} r_b$ is the reduced composite mass.*

To this end, it is convenient to first establish a sequence of four preliminary identities, which together imply the desired result.

**Identity 6.**

$$\overline{R} = (r_f - r_f q r_f) + (I - r_f q)\widetilde{\phi}_f r_f + r_f \widetilde{\phi}_f^*(I - q r_f) \tag{30}$$

*where $q = \text{diag}\,[q(1), \ldots, q(N)]$ is a $6N \times 6N$ block-diagonal matrix which satisfies the backward recursion*

$$q = \mathcal{E}_{\phi f}^* q \mathcal{E}_{\phi f} + C^* r_f^{-1}(N+1)C \tag{31}$$

To establish this identity, recall (Jain and Rodriguez, 1995a) that the fixed-base mass matrix kernel $R$ can be decomposed as $R = r_f + \tilde{\phi}_f r_f + r_f \tilde{\phi}_f^*$. Observe also that $CR = C\phi_f r_f$, since $C r_f \tilde{\phi}_f^* = 0$. However, $\phi_f^* C^* r^{-1}(N+1) C\phi_f = q + q\tilde{\phi}_f + \tilde{\phi}_f^* q$. Substitution of these combined results into the reduced mass kernel in eqn. (28) leads to the desired identity.

**Identity 7.**

$$\overline{R} = r + C_f \tilde{\phi}_f r_f + r_f \tilde{\phi}_f^* C_f^* \tag{32}$$

*where* $C_f = r_b(r_f + r_b)^{-1}$ *is the "weighting matrix" associated with the forward composite mass* $r_f$.

To establish this identity, define the backward composite mass $r_b$ as $r_b = q^{-1} - r_f$. Substitute this into (30) to obtain the desired identity. Observe that the block-diagonal matrix $q = (r_f + r_b)^{-1}$ appearing in the Identity 7 is the inverse of the sum of the forward and backward composite masses.

**Identity 8.** *The transpose of the lower-triangular terms* $C_f \tilde{\phi}_f r_f$, *which are computed by the forward recursive algorithm, are the upper-triangular terms* $C_b \tilde{\phi}_b r_b$ *computed by the backward recursive algorithm, i.e.,*

$$r_f \tilde{\phi}_f^* C_f^* = C_b \tilde{\phi}_b r_b \tag{33}$$

*where* $C_b = r_f(r_f + r_b)^{-1}$ *is the "backward" weighting matrix associated with the backward composite mass* $r_b$.

To establish this identity, the first step is to show the simpler and more fundamental identity $\tilde{\phi}_f^* q = q\tilde{\phi}_b$. Observe from eqn. (31) that $q(k) = \phi^*(k+1,k) q(k+1) \phi(k+1,k)$ which implies $q(k)\phi(k,k+1) = \phi^*(k+1,k) q(k+1)$. In operator notation, $q\mathcal{E}_{\phi b} = \mathcal{E}_{\phi f}^* q$. Hence, $q(I - \phi_b^{-1}) = (I - \phi_f^{-1})^* q$. This implies $\tilde{\phi}_f^* q = q\tilde{\phi}_b$. Premultiply by $r_f$, postmultiply by $r_b$, and recall that $q = (r_f + r_b)^{-1}$ to obtain the desired identity in eqn. (33).

The last remaining step, toward establishing the desired 2-way recursive decomposition of the free-base manipulator mass matrix kernel, is to develop a backward recursive algorithm to compute the backward composite mass $r_b$ as is done in the following result.

**Identity 9.** *The backward composite mass* $r_b$ *satisfies the backward spatial recursion*

$$r_b = \mathcal{E}_{\phi b} r_b \mathcal{E}_{\phi b}^* + M_b \tag{34}$$

To establish this identity, observe from eqn. (31) that $q(k)\phi(k,k+1) = \phi^*(k+1,k)$ $q(k+1)$. Invert this equation to obtain that $\phi(k+1,k)[r_f(k) + r_b(k)] = [r_f(k+1) + r_b(k+1)]\phi^*(k,k+1)$. Post-multiply this by $\phi^*(k+1,k)$, and recall that $r_f(k+1) = \phi(k+1,k) r_f(k) \phi^*(k+1,k) + M_f(k+1)$ to obtain that $r_b(k+1) + M_f(k+1) = \phi(k+1,k) r_b(k) \phi^*(k+1,k)$. Finally, premultiply by $\phi(k,k+1)$, post-multiply by $\phi^*(k,k+1)$, and recall that the forward and backward link masses $M_f$ and $M_b$ are related to each other by eqn. (12).

## 5. Analogy with the Brownian Bridge

In this section, an analogy is developed which allows interpretation of the preceding results within the context of a very easy to understand one-dimensional analogy. The Brownian bridge of length $\ell$ can be thought of (Hida, 1980) as a random process generated by the equation

$$\dot{X} = W \tag{35}$$

with the boundary conditions

$$X(0) = X(\ell) = 0 \tag{36}$$

The random process $W$ is a zero-mean white-noise process, with covariance

$$E(WW^*) = \rho I \tag{37}$$

with $\rho$ being the mass density of the bridge. In relatively imprecise terms, the Brownian bridge is the integral of white noise, subject to boundary conditions, similar to those of an actual physical bridge: the bridge does not translate at its two end-points. The Brownian bridge is therefore Brownian motion with fixed conditions at both ends. The equation $\dot{X} = W$ that governs the Brownian bridge process within the spatial interval is identical to that of the Brownian motion process.

The random process $X(x)$ is "sampled" at the finite number $N$ of discrete locations, corresponding to the locations of the $N$ joints of the manipulator, which the bridge is analogous to. At each of these sample locations $x = k$, there is defined a corresponding random variable $X(k)$. The covariance of this sampled random process $X = [X(1), \ldots, X(N)]$ is analogous to the kernel $\overline{R}$ associated with the manipulator mass matrix $\mathcal{M} = H\overline{R}H^*$ as discussed below.

### 5.1. Covariance of the Brownian Bridge Process

**Identity 10.**

$$\overline{R} = E(XX^*) = r + C_f\phi_f r_f + C_b\phi_b r_b \tag{38}$$

Here $r_f$ is the covariance of a "forward" Brownian motion process that starts at the left and moves forward in space toward the right, with zero initial condition at the left end of the bridge. Similarly, $r_b$ is the covariance of a "backward" Brownian motion process that moves backward in space toward the left, with zero terminal condition at the right end of the bridge. The covariance $r_f(x)$ at a point $x$ represents the composite mass of that portion of the bridge to the left of this point. The covariance $r_b(x)$ at a point $x$ is the composite mass of that portion of the bridge to the right of this point. The forward and backward covariances are:

$$r_f(x) = \rho x, \qquad r_b(x) = \rho(\ell - x) \tag{39}$$

where $\rho$ is the mass density of the bridge. The reduced composite mass $r(x) = r_f(x)[r_f(x) + r_b(x)]^{-1}r_b(x)$ is obtained by combining the composite masses $r_f(x)$

and $r_b(x)$, in the same sense as in computing the reduced mass of two bodies in classical mechanics (Goldstein, 1950).

Since the Brownian bridge process $X$ is sampled at the finite number of discrete locations $N$, the corresponding covariance kernel $\overline{R}$ is a matrix. To simplify notation, consider the case in which there are 3 equally spaced discrete locations, separated by the distance $\Delta$, and which are inside the spatial interval $\ell$ representing the overall length of the bridge. In this case, the covariance kernel $\overline{R}$ is

$$\overline{R} = \rho\Delta \begin{pmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{1} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{pmatrix} \tag{40}$$

The 2-way recursive decomposition corresponding to this matrix is provided by eqn. (38) with the various matrices involved in the right side of this equation being defined as:

$$r = \rho\Delta \begin{pmatrix} \frac{3}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{3}{4} \end{pmatrix}, \quad r_f = \rho\Delta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad \tilde{\phi}_f = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \tag{41}$$

and

$$r_b = \rho\Delta \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C_f = \begin{pmatrix} \frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}, \quad C_b = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{3}{4} \end{pmatrix} \tag{42}$$

## 6. Extension to Tree Configurations

While the results of previous sections have been illustrated by using a simple serial multilink system, it is relatively easy to extend these results to arbitrary tree-like configurations. To this end, and without loss of generality, consider the 9-link example shown in Fig. 3.

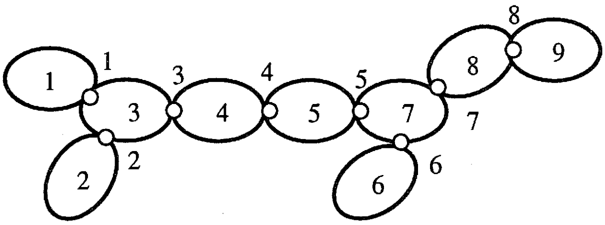


Fig. 3. Nine-link tree configuration.

Table 4. Forward tree at each joint.

| Joint | Forward Tree |
|---|---|
| 1 | 1 |
| 2 | 2 |
| 3 | 1,2,3 |
| 4 | 1,2,3,4 |
| 5 | 1,2,3,4,5 |
| 6 | 6 |
| 7 | 1,2,3,4,5,6,7 |
| 8 | 1,2,3,4,5,6,7,8 |

The general idea in arriving at the desired extension is to view each of the joints in the system as "partitioning" the overall system into two subsystems, a forward tree and a backward tree. In the example above, for joint 1, the forward tree consists only of the first link, while the backward tree consists of all of the other links. Similarly, the forward tree for joint 2 consists of link 2, while the backward tree consists of all of the other links. As the joint number increases, the number of links in the forward tree increases, and the corresponding number of links in the backward tree decreases. At the last joint, labeled No. 8 in Fig. 3, only link No. 9 remains in the backward tree, and all of the other links are in the forward tree. While the above procedure for constructing the forward and backward trees for each joint has been explained in the context of a specific example, similar results can be achieved for arbitrary tree-like configurations.

## 7. Concluding Remarks

This paper has developed a new 2-way recursive decomposition for the free-flying robot mass matrix. It has been done by drawing substantially from the spatial operator methods advanced by the authors in previous publications. The decomposition reveals structural properties of the system mass matrix which are not be easy to detect using other methods. The decomposition has a very elegant interpretation in terms of both classical mechanics and estimation theory.

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