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BOUNDARY STABILIZATION OF COUPLED WAVE EQUATIONS

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The qualitative behavior of the coupled wave equations in bounded domains are addressed. Here it has been shown that uniform exponential stability can be achieved provided both equations are exposed to velocity feedback controllers. However, if one of the controllers is removed from one equation and placed in another one, only strong stability can be guaranteed. Numerical computations are presented to match the theoretical results.

1. Introduction

Stability is a very desirable property for an elastic system, particularly if the rate of decay is exponential. To this end, the energy multiplier or energy perturbation methods (Bensoussan *et al.*, 1992; Chen, 1979; Lunardi, 1991) have been successfully applied to reach this goal for various partial differential equations and boundary conditions. Stabilization properties of serially connected vibrating strings or beams have been examined by several authors (Chen and Wang, 1989; Ho, 1993). They suggested that uniform stabilization can be achieved if we employ dissipative boundary conditions at one end. If instead, one damper is located at the mid-span joint of two vibrating strings coupled in series, the uniform stabilization property holds if c_1/c_2 (wave speeds) has certain rational values. In this paper, we will investigate the stabilization properties of vibrating strings in parallel whose energy will be damped out by boundary velocity feedback controllers for various boundary conditions.

The governing equation of such a system is described by the following system of wave equations (mixed initial-boundary value problems):

$$\left.\begin{aligned} u_{tt} - c^2 u_{xx} &= \alpha(v-u) \\ v_{tt} - c^2 v_{xx} &= \alpha(u-v) \end{aligned}\right\} \quad \text{in } \Omega \times (0,\infty) \tag{1}$$

where $c \in \mathbb{R}^+$, $\alpha \in \mathbb{R}^+$, and $\Omega = (0,1)$. The initial conditions are:

$$\left.\begin{aligned} u(x,0) &= u_0, \quad u_t(x,0) = u_1 \\ u(x,0) &= v_0, \quad v_t(x,0) = v_1 \end{aligned}\right\} \quad \text{in } 0 \le x \le 1 \tag{2}$$

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The following cases are under consideration:

Case 1. (one sided boundary controllers for both equations)

$$u(0,t) = 0, \quad u_x(1,t) = -\beta_1 u_t(1,t) \tag{3}$$

$$v(0,t) = 0, \quad v_x(1,t) = -\beta_2 v_t(1,t), \quad t \geq 0 \tag{4}$$

Case 2. (two sided boundary controllers for one equation)

$$u(0,t) = 0, \quad u(1,t) = 0 \tag{5}$$

$$v_x(0,t) = \beta_2 v_t(0,t), \quad v_x(1,t) = -\beta_2 v_t(1,t), \quad t \geq 0 \tag{6}$$

Here t and x are the time and space variable, respectively, and u and v are the deflections of the strings from the equilibrium positions. The wave speed c and α (spring constant) are the system parameters and the damping coefficients $\beta_i > 0$ $(i = 1, 2)$ depend on the control devices. These parameters play an important role in the physical behaviour of the system. Generally, this boundary control corresponds to a control mechanism which monitors u_t and v_t at $x = 1$ or at $x = 0$. This phenomenon takes place if the system is exposed to external forces or by (2). This problem is motivated by an analogous problem in ordinary differential equations for coupled oscillators, and has a potential application in oscillation of objects from outside disturbances.

Associated with each solution of (1) is its total natural energy at time t:

$$E(t) = \frac{1}{2}\int_0^1 \left\{|u_t|^2 + c^2|u_x|^2 + |v_t|^2 + c^2|v_x|^2 + \alpha|u - v|^2\right\} \mathrm{d}x \tag{7}$$

In Section 2, we discuss the well-posedness of the problem via semigroup theory. Section 3 is devoted to the exponential stability of the system for Case 1. We observe in Section 4 that the point spectra $\sigma_p(A)$ are approaching the imaginary axis (Theorem 2) asymptotically if we apply boundary conditions (5) and (6), hence the rate of decay of the solution is not uniformly exponential for Case 2. But, it is shown that the solutions tend to zero strongly as $t \to \infty$ (strong stability). That is to say, if u and v are the solutions to the above system with $E(u, v, 0) < +\infty$, then

$$E(u, v, \infty) \doteq \lim_{t\to\infty} E(u, v, t) = 0 \tag{8}$$

Section 5 is a convincing part of this paper due to a comparison of the theoretical results with numerical computations for the system (1)–(6). There, we apply the finite difference method to solve the system (1) and (2) along with boundary conditions (3)–(6), respectively. The results are illustrated in solutions u and v versus spatial dimension x and also energy versus time t. One can find the overall result of this paper in Section 6. The Appendix is for the proofs of lemmas in Sections 3 and 4.

2. Notations and Preliminaries

We shall use the notations

$$(\dot{\ }) = \frac{\mathrm{d}}{\mathrm{d}t}, \quad ({}') = \frac{\mathrm{d}}{\mathrm{d}x}, \quad f_t = \frac{\partial f}{\partial t}, \quad f_x = \frac{\partial f}{\partial x}, \quad f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad \partial_x^2 = \frac{\partial^2}{\partial x^2}$$

We also need to adapt $L^2 = L^2([0,1])$, $\|\cdot\| = \|\cdot\|_{L^2}$, and the standard Sobolev space:

$$H^m([0,1]) = H^m = \left\{ f : [0,1] \to \mathbb{R} \mid \|f\|^2_{H^m(0,1)} = \sum_{k=0}^{m} \int_0^1 |f^{(k)}(x)|^2 \,\mathrm{d}x < \infty \right\}$$

for $m \in \mathbb{N}$. Moreover, $H^m_{\{0\}} = \{f \in H^m \mid f(0) = 0\}$. Our proper function space $H = H^1_{\{0\}} \times L^2 \times H^1_{\{0\}} \times L^2$ is the set of all quadruplets $U = (u, z, v, w)^\top$, $\top$ is the transpose, equipped with the norm:

$$\|U\|_H^2 = \int_0^1 \left\{ |z|^2 + c^2|u_x|^2 + |w|^2 + c^2|v_x|^2 + \alpha|u - v|^2 \right\} \mathrm{d}x \tag{9}$$

Remark 1. The norm defined by (9) cannot be utilized in Case 2 since it would be an illegitimate application of the Poincaré inequality:

$$\int_0^1 |v|^2 \,\mathrm{d}x \le \gamma \int_0^1 |v_x|^2 \,\mathrm{d}x, \quad \gamma > 0 \tag{10}$$

due to left boundary conditions (6). However, if we consider the quotient space generated by the class of functions $[f] = \{g \mid g = f + \text{constant}\}$, then we are able to use (9).

The unbounded linear operator A associated with (1) is given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ c^2\partial_x^2 - \alpha & 0 & \alpha & 0 \\ 0 & 0 & 0 & 1 \\ \alpha & 0 & c^2\partial_x^2 - \alpha & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ c^2\partial_x^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & c^2\partial_x^2 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\alpha & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & 0 & -\alpha & 0 \end{bmatrix} = \widehat{A} + \alpha\mathcal{B} \tag{11}$$

The domain of A with respect to each boundary condition (3)–(6) is as follows:

$$\mathcal{D}_1(A) = \left\{ U \in \widehat{H} \mid \begin{array}{l} u(0) = 0, \;\; c^2 u_x(1) = -\beta_1 z(1) \\ v(0) = 0, \;\; c^2 v_x(1) = -\beta_2 w(1) \end{array} \right\} \tag{12}$$

$$\mathcal{D}_2(A) = \left\{ U \in \widehat{H} \mid \begin{array}{ll} u(0) = 0, & u(1) = 0 \\ c^2 v_x(0) = \beta_2 w(0), & c^2 v_x(1) = -\beta_2 w(1) \end{array} \right\} \tag{13}$$

where $\widehat{H} = H^2_{\{0\}} \times H^1_{\{0\}} \times H^2_{\{0\}} \times H^1_{\{0\}}$ is dense in the Hilbert space H.

In Lemma 1 below, we summarize some important functional properties of the operator A in H, which also leads us to the well-posedness of the problem (1), (2).

Lemma 1. *The unbounded linear operator A satisfies the following properties:*

(i) A is a densely defined, closed, dissipative linear operator in H;

(ii) A has compact resolvent and consequently $\sigma(A)$ consists entirely of isolated eigenvalues;

(iii) A is the infinitesimal generator of a strongly continuous semigroup $S(t)$ of contractions on H.

Proof.

(i) It is a routine procedure to verify that A is a densely defined, closed and linear operator. To show that A is dissipative (that is, $\operatorname{Re}\langle AU, U\rangle_H \le 0$, for each $U \in \mathcal{D}_i(A)$, $i = 1, 2$) we first find

$$\operatorname{Re}\langle AU, U\rangle_H = \langle AU, U\rangle_H = c^2[u_x z + v_x w]\,\Big|_0^1 \tag{14}$$

Applying (12) to (14), we obtain the dissipativeness of the unbounded linear operator A, i.e.

$$\langle AU, U\rangle_H = -\Big(\beta_1 z^2(1) + \beta_2 w^2(1)\Big) \le 0 \tag{15}$$

A similar result can be obtained from (13) and (14).

(ii) We have $A = \widehat{A} + \alpha\mathcal{B}$, where $\widehat{A}$ is the standard unbounded part, and $\mathcal{B}$ is a bounded perturbation to it. It is well known that with the boundary conditions (3) and (4), $(\lambda I - \widehat{A})^{-1}$ is compact for $\lambda > 0$. Since $\mathcal{B}$ is bounded, it follows that $(\lambda I - A)^{-1}$ exists and is compact for $\lambda > 0$ sufficiently large (Najafi *et al.*, 1997). This implies that A has compact resolvent, and as a consequence, the spectrum of A is discrete (Kato, 1966, Theorem 6.29 in Chapter 3).

(iii) This is a consequence of (i) and (ii) and Lumer-Phillips' Theorem (Pazy, 1983). ∎

The well-posedness of the problem (1), (2) is answered in terms of this semigroup $S(t)$ generated by A (Pazy, 1983). For any initial state $U_0 \in H$, the generalized solution of (1), (2) is given by $U = S(t)U_0$ and it becomes a classical solution for $U_0 \in \mathcal{D}_i(A)$, $i = 1, 2$.

3. Study of Uniform Stabilizability

We state the following theorem which is the essence of this section.

Theorem 1. *Consider the system (1) and (2) along with (3) and (4). Then the uniform exponential stability holds.*

To prove Theorem 1, the following lemmas are utilized.

Lemma 2. *For all sufficiently large T, there exists a constant C such that*

$$\int_0^T (u-v)^2(1,t)\,\mathrm{d}t \le C\left(\int_0^T (\beta_1 u_t^2 + \beta^2 v_t^2)(1,t)\,\mathrm{d}t + \int_0^T\int_0^1 (u-v)^2\,\mathrm{d}x\,\mathrm{d}t\right)$$

Lemma 3. *For all sufficiently large T, there exists a constant C such that*

$$\int_0^T\int_0^1 (u-v)^2\,\mathrm{d}x\,\mathrm{d}t \le C\int_0^T (\beta_1 u_t^2 + \beta^2 v_t^2)(1,t)\,\mathrm{d}t$$

The proofs of the above lemmas are provided in the Appendix.

Proof of Theorem 1. Without loss of generality we assume $c = 1$. Applying integration by parts and boundary conditions (3) and (4), we obtain

$$\dot{E} = -(\beta_1 u_t^2 + \beta_2 v_t^2)(1,t)$$

which leads to

$$E(T) - E(0) = -\int_0^T (\beta_1 u_t^2 + \beta_2 v_t^2)(1,t)\,\mathrm{d}t \tag{16}$$

We multiply the equations of (1) by xu_x and xv_x, respectively. Adding the products and integrating over $[0,1]\times[0,T]$, we obtain

$$\rho(T) - \rho(0) + \int_0^T E(t)\,\mathrm{d}t = \frac{1}{2}\int_0^T (u_t^2 + v_t^2)(1,t)\,\mathrm{d}t + \frac{1}{2}\int_0^T (u_x^2 + v_x^2)(1,t)\,\mathrm{d}t$$
$$+ \frac{\alpha}{2}\int_0^T (u-v)^2(1,t)\,\mathrm{d}t + \alpha\int_0^T\int_0^1 (u-v)^2\,\mathrm{d}x\,\mathrm{d}t \tag{17}$$

where $\rho(t) = \int_0^1 x(u_t u_x + v_t v_x)\,\mathrm{d}x$.

Since $|\rho(t)| \leq C_0 E(t)$, $C_0 > 0$, for a solution of (1)–(4) one gets the estimate

$$-C_0\Big(E(T)+E(0)\Big)+\int_0^T E(t)\,\mathrm{d}t \leq C_1\int_0^T (\beta_1 u_t^2+\beta_2 v_t^2)(1,t)\,\mathrm{d}t$$
$$+\frac{\alpha}{2}\int_0^T (u-v)^2(1,t)\,\mathrm{d}t+\alpha\int_0^T\int_0^1 (u-v)^2\,\mathrm{d}x\,\mathrm{d}t \qquad (18)$$

where C_1 is an appropriate positive constant. Using (16) in the last estimate, together with $\int_0^T E(t)\,\mathrm{d}t \geq TE(T)$, gives an estimate of the form

$$(T-2C_0)E(T) \leq C_1\int_0^T (\beta_1 u_t^2+\beta_2 v_t^2)(1,t)\,\mathrm{d}t$$
$$+\frac{\alpha}{2}\int_0^T (u-v)^2(1,t)\,\mathrm{d}t+\alpha\int_0^T\int_0^1 (u-v)^2\,\mathrm{d}x\,\mathrm{d}t \quad (19)$$

Now using Lemmas 2 and 3 in (19) results in the following estimation:

$$E(T) \leq C_T\int_0^T (\beta_1 u_t^2+\beta_2 v_t^2)(1,t)\,\mathrm{d}t, \qquad C_T > 0 \qquad (20)$$

Having considered (20) and (16) one can obtain

$$E(T) \leq \frac{C_T}{1+C_T}E(0)$$

Since the fixed time T is arbitrary, this shows that the strongly continuous semigroup $S(t)$ generated by A satisfies the following inequality:

$$\|S(t)\|_H < 1$$

Thus, the the proof of the theorem follows from (Balakrishnan, 1976). ■

Remark 2. Since the wave equation is reversible, with a similar approach as in (Chen, 1979), we can show that the system (1)–(4) is controllable using the "Controllability via Stability" argument employed in (Chen, 1979).

4. The Asymptotic Behavior of Eigenvalues of the System

The essence of this section can be summarized in the following theorem.

Theorem 2. *For the system (1)–(2) along with the boundary conditions (5)–(6) the following statements hold:*

1. *The system is strongly stable (i.e. (8)), provided that* $\alpha \notin \Lambda = \{(1/2)c^2(n^2 - m^2)\pi^2 \mid m,n \in \mathbb{N}\}$ *for the boundary conditions (5) and (6).*
2. *A sequence of eigenvalues can be constructed which approaches the imaginary axis. Hence, the system is not uniformly exponentially stable.*
3. *If* $\alpha \in \Lambda$, *then there exists* $U \in H$ *such that* $\|S(t)U\|_H = \text{constant}$.

Having compared the results obtained in Theorems 1 and 2, we observe in the parallel system (1)–(2) that the existence of two controllers does not guarantee the exponential decay of the system and hence the locations of theses controllers are crucial for uniform stabilizability.

To furnish the proof of the above theorem, we need to employ the following results in (Huang, 1985) and the Rouche theorem in (Ahlfors, 1979), respectively:

Theorem 3. (Strong stability) *Let $S(t)$ be a uniform bounded C_0-semigroup on a Banach space, and let* $\operatorname{Re}\lambda < 0$ *for all $\lambda \in \sigma(\mathcal{A})$. Then $S(t)$ is strongly asymptotically stable. Conversely, let $S(t)$ be strongly asymptotically stable. Then $S(t)$ is uniformly bounded,* $\operatorname{Re}\lambda \le 0$ *for all $\lambda \in \sigma(\mathcal{A})$, and there is on the imaginary axis neither point nor residual spectrum of $\mathcal{A}$.*

Theorem 4. (Rouche) *Let Γ be homologous to zero in Ω and such that $n(\Gamma, z)$ is either 0 or 1 for any point z not on Γ. Suppose that $\Psi(z)$ and $\Phi(z)$ are analytic in Ω and satisfy the inequality $|\Psi(z) - \Phi(z)| < |\Phi(z)|$ on Γ. Then $\Psi(z)$ and $\Phi(z)$ have the same number of zeros enclosed by Γ.*

Proof of Theorem 2.

1. From Theorem 1, we have concluded that the spectrum $\sigma(\mathcal{A})$ consists of only isolated eigenvalues. Now due to Theorem 3 it remains to claim $\rho(\mathcal{A}) \supset i\mathbb{R}$. To do this, it is sufficient to show that $(\mathcal{A} - \lambda I)^{-1}$ exists for any $\lambda \in i\mathbb{R}$. We assume the contrary, that $(\mathcal{A} - \lambda I)^{-1}$ does not exist; i.e. there exists $\lambda \in \sigma(\mathcal{A})$ such that $\lambda \in i\mathbb{R}$. Then $(\mathcal{A} - \lambda I)(u_0, z_0, v_0, w_0)^\top = 0$ has a nontrivial solution $(u_0, z_0, v_0, w_0)^\top \in \mathcal{D}(\mathcal{A})$, and explicitly satisfies:

$$
\begin{aligned}
&z_0 - \lambda u_0 = 0 \\
&c^2 u_0'' - \alpha u_0 - \lambda z_0 + \alpha v_0 = 0 \\
&w_0 - \lambda v_0 = 0 \\
&\alpha u_0 + c^2 v_0'' - \alpha v_0 - \lambda w_0 = 0
\end{aligned}
\tag{21}
$$

along with boundary conditions

$$
\begin{aligned}
&u_0(0) = 0, && u_0(1) = 0 \\
&c^2 v_0'(0) = -\beta_2 w_0(0), && c^2 v_0'(1) = -\beta_2 w_0(1), \qquad t \ge 0
\end{aligned}
\tag{22}
$$

Let

$$
(u, v)^\top = e^{\lambda t}\Big(u_0(x), v_0(x)\Big)^\top \tag{23}
$$

Then $(u, v)^\top$ satisfies the system of equations (1). The first observation is that $\|(u, z, v, w)^\top\| = \|e^{\lambda t}(u_0, z_0, v_0, w_0)^\top\| = \|(u_0, z_0, v_0, w_0)^\top\|$. This shows that the system is conservative, and therefore, $\mathrm{d}\,E(t)/\mathrm{d}t = \beta_2^2 v_t^2(0) = \beta_2^2 v_t^2(1) = 0$. Consequently, from (5) and (6), we have $v_x(0) = v_t(0) = 0$, $v_x(1) = v_t(1) = 0$ which imply from (22) and (23) that $v_0'(0) = v_0'(1) = 0$, $v_0(0) = v_0(1) = 0$. Thus

$(u_0, v_0)^\top$ is a solution to the boundary value problems:

$$\begin{aligned} c^2 u_0'' - \lambda^2 u_0 &= \alpha(u_0 - v_0) \\ c^2 v_0'' - \lambda^2 v_0 &= \alpha(v_0 - u_0) \end{aligned} \tag{24}$$

along with boundary conditions

$$\begin{aligned} &u_0(0) = 0, \quad u_0(1) = 0 \\ &v_0(0) = 0, \quad v_0(1) = 0, \quad v_0'(0) = 0, \quad v_0'(1) = 0 \end{aligned} \tag{25}$$

By defining $\xi = u_0 + v_0$, $\eta = u_0 - v_0$, (24) can be rewritten as:

$$\begin{aligned} &\xi'' - a^2 \xi = 0, \quad a = c^{-1}\lambda \\ &\eta'' - b^2 \eta = 0, \quad b = c^{-1}\sqrt{\lambda^2 + 2\alpha} \end{aligned} \tag{26}$$

Using the first four boundary conditions of (25) yields

$$\xi(0) = 0, \quad \xi(1) = 0, \quad \eta(0) = 0, \quad \eta(1) = 0 \tag{27}$$

The solutions to (26) and (27), which satisfy the boundary conditions at $x = 0$, are:

$$\xi(x) = C_1 \sinh ax, \qquad \eta(x) = C_2 \sinh bx \tag{28}$$

where C_1 and C_2 are arbitrary constants. To satisfy the boundary conditions at $x = 1$, provided $C_1 \neq 0$ and $C_2 \neq 0$, we should have

$$a = in\pi, \qquad b = im\pi, \qquad n, m \in \mathbb{N} \tag{29}$$

From (26) and (29) one can get $\alpha = (1/2)c^2(n^2 - m^2)\pi^2$ $(n > m)$ which violates the hypotheses, and consequently C_1 or C_2 must be zero. Without loss of generality consider $C_2 = 0$. Then from (28) and by reverse transformation, we have

$$v_0(x) = C_1 \sinh ax \tag{30}$$

Now, by applying the last boundary condition of (25) in (30), we obtain $C_1 = 0$. This shows that (24)–(25) has only the trivial solution, a contradiction. Hence, the system is strongly stable.

2. In order to show that the solution to the system (1) along with boundary conditions (5) and (6) is not uniformly exponential, we find a sequence of eigenvalues of the system which approaches the imaginary axis, and as a result the system is not uniformly stable. For this, we utilize Theorem 4. Since we are interested in finding this sequence close to the imaginary axis, we only consider the strip $S = \{\, z = x + iy \mid -1 < x < 1 \,\}$ in our discussion. Consequently, we obtain this fact that $\sinh z$ and $\cosh z$ are bounded in this strip. To find the eigenvalues, we solve the following equation:

$$(A - \omega I)(u, z, v, w)^\top = 0 \tag{31}$$

and we obtain

$$u = k_1(\cosh ax + \cosh bx) + k_2\left(\frac{\sinh ax}{a} + \frac{\sinh bx}{b}\right) + k_3(\cosh ax - \cosh bx) + k_4\left(\frac{\sinh ax}{a} - \frac{\sinh bx}{b}\right) \tag{32}$$

$$v = k_1(\cosh ax - \cosh bx) + k_2\left(\frac{\sinh ax}{a} - \frac{\sinh bx}{b}\right) + k_3(\cosh ax + \cosh bx) + k_4\left(\frac{\sinh ax}{a} + \frac{\sinh bx}{b}\right) \tag{33}$$

where

$$a = \frac{\omega}{c}, \qquad b = \frac{\sqrt{\omega^2 + 2\alpha}}{c} \tag{34}$$

Applying boundary conditions (5) and (6), one can get the following determinant for which we can compute the eigenvalues of the system

$$\det Q = a^2\Big[4h^{-1}\sinh a \sinh b + 4\beta c(\sinh a \cosh b + h^{-1}\cosh a \sinh b) - 2\beta^2c^2 + \beta^2c^2(h + h^{-1})\sinh a \sinh b + 2\beta^2c^2\cosh a \cosh b\Big] \tag{35}$$

where $h = a/b$. Now according to Theorem 4, we need to construct the following functions:

$$\Psi(a) = \frac{\det Q}{a^2} = \left(\beta^2c^2(h+h^{-1})\sinh a \sinh b + 2\beta^2c^2\cosh a \cosh b - 2\beta^2c^2\right) + 4\beta c(\sinh a \cosh b + h^{-1}\cosh a \sinh b) + 4h^{-1}\sinh a \sinh b \tag{36}$$

and

$$\begin{aligned}\Phi(a) &= 4\beta c(\sinh a \cosh b + \sinh b \cosh a) \\ &\quad + 2\beta^2c^2(\sinh a \sinh b + \cosh b \cosh a - 1) \\ &= 4\beta c \sinh(a + b) + 2\beta^2c^2\big(\cosh(a + 2b) - 1\big)\end{aligned} \tag{37}$$

Let $\Phi(a) = 0$. This implies $a + b = \pm in\pi$, from which we only consider a_n and b_n such that $a_n + b_n = i2n\pi$, where

$$a_n = i\Big(n\pi + \frac{\alpha}{2n\pi c^2}\Big), \quad b_n = i\Big(n\pi - \frac{\alpha}{2n\pi c^2}\Big), \quad n = 1, 2, 3, \ldots \tag{38}$$

Now, for large n one can construct the sequence of circles inside the strip S as follows:

$$\Gamma_n = \Big\{z \mid z = a_n + \frac{1}{n}e^{i\theta}, \ 0 \le \theta < 2\pi\Big\} \tag{39}$$

To apply Rouche's theorem, we should estimate $|\Psi - \Phi|$ and $|\Phi|$ on Γ_n for large n.

Estimation of $|\Psi - \Phi|$. For any $a \in \Gamma_n$,

$$|\Psi - \Phi| \le \beta^2 c^2 \,|(2 - h^{-1} + h)\sinh a \sinh b| + 4\beta c\,|(1 - h^{-1})\sinh b \cosh b| + 4h^{-1}\,|\sinh a|\,|\sinh b| \tag{40}$$

from which

$$\begin{aligned} |\sinh a| &= \left|\sinh\left(in\pi + i\frac{\alpha}{2n\pi c^2} + \frac{1}{n}e^{i\theta}\right)\right| \\ &= \left|\sinh(in\pi)\cosh\left[\frac{1}{n}\left(\frac{\alpha}{2\pi c^2} + e^{i\theta}\right)\right] + \cosh(in\pi)\sinh\left[\frac{1}{n}\left(i\frac{\alpha}{2\pi c^2} + e^{i\theta}\right)\right]\right| \\ &\le \left(\frac{\alpha}{2\pi c^2} + 1\right)\frac{1}{n} + O\left(\frac{1}{n^3}\right) \end{aligned} \tag{41}$$

Also

$$\begin{aligned} |\sinh b| &= \left|\sinh\left(a + (b - a)\right)\right| \\ &\le |\sinh a|\,|\cosh(b - a)| + |\cosh a|\,|\sinh(b - a)| \end{aligned}$$

The boundedness of $\cosh z$ in S and the fact that $|a - b| \le (2\alpha/c^2)(1/n) + O(1/n^3)$ leads to the following result:

$$\begin{aligned} |\sinh b| &\le c_1\left(\frac{\alpha}{2\pi c^2} + 1\right)\frac{1}{n} + O\left(\frac{1}{n^3}\right) + c_2|b - a| + O(|b - a|^3) \\ &\le c_1\left(\frac{\alpha}{2\pi c^2} + 1\right)\frac{1}{n} + c_2\left(\frac{2\alpha}{c^2}\right)\frac{1}{n} + O\left(\frac{1}{n^3}\right) \\ &= \left[c_1\left(\frac{\alpha}{2\pi c^2} + 1\right) + c_2\left(\frac{2\alpha}{c^2}\right)\right]\frac{1}{n} + O\left(\frac{1}{n^3}\right) \end{aligned} \tag{42}$$

where c_1 and c_2 are constants. Moreover,

$$|1 - h^{-1}| = \left|\frac{a - b}{a}\right| = \left|\frac{2\alpha}{a^2 c^2} + O\left(\frac{1}{|a|^4}\right)\right| = \frac{8\alpha}{n^2\pi^2 c^2} + O\left(\frac{1}{n^4}\right) \tag{43}$$

and similarly

$$|2 - h^{-1} + h| = \left|\frac{(a - b)^2}{ab}\right| \le O\left(\frac{1}{n^2}\right)$$

Now, from (40)–(43), one can get the following estimation for $|\Psi - \Phi|$:

$$|\Psi - \Phi| \le O\left(\frac{1}{n^2}\right) \tag{44}$$

for large n.

Estimation of $|\Phi|$. We need the following lemma whose proof is provided in the Appendix:

Lemma 4. $\frac{1}{n} \le |(a + b) - (a_n + b_n)| \le \frac{\text{constant}}{n}$ *for large* n.

Applying Lemma 4 we obtain

$$\begin{aligned} |\sinh(a+b)| &= \Big|\sinh\Big((a_n+b_n)+\big[(a+b)-(a_n+b_n)\big]\Big)\Big| \\ &= \Big|\sinh\big[(a+b)-(a_n+b_n)\big]\Big| = \frac{1}{n}+O\Big(\frac{1}{n^3}\Big) \geq \frac{1}{2n} \end{aligned} \tag{45}$$

and similarly

$$|\cosh(a+b)-1| \geq \frac{1}{2n} \tag{46}$$

From (37), (38), (45) and (46), the following estimation can be derived:

$$|\Phi(a)| \geq O\Big(\frac{1}{n}\Big) \tag{47}$$

Now, from (44) and (47), we finally obtain

$$|\Phi(a)| > |\Psi(a)-\Phi(a)|, \qquad a \in \Gamma_n, \; n \text{ large}$$

Hence, by Rouche's Theorem, $\Phi(z)$ and $\Psi(z)$ have the same number of zeros inside the disk Γ_n and the theorem follows.

3. This part is an immediate conclusion of Part 1. ■

5. Numerical Computation

Here, we exhibit numerical computations supporting our theoretical results (Theorems 1 and 2). Having considered this, the finite difference method (Richtmyer and Morton, 1967) was utilized for the system (1)–(2) along with boundary conditions (3)–(6). The sensitivity of this computational method due to its stability condition, i.e. $|\Delta t/\Delta x| < 1/c$ (Couront number), was carried out carefully in order to obtain reliable results. In Fig. 1 a confirmation of Theorem 2 is apparent, since the solutions $u(x,t)$ and $v(x,t)$ go to zero quickly as time T increases (see Figs. 1(a) and (b) for u and v, respectively). In Fig. 1(c), we see that the energy of the system decays uniformly exponentially due to energy absorbing boundaries at both right ends of upper and lower strings. In Fig. 2, we see that the solutions and the energy of the system (1) for u, v and E, are approaching zero as time increases. These observations agree strongly with Theorem 2. We have shown here, due to the rate of convergence of the solutions, that when the boundary conditions (5) and (6) are employed, the energy $E(t)$ converges strongly, i.e. $E(t) \to 0$ as $t \to \infty$, for all initial states $E(0) < \infty$. To solve the system numerically, we set the arbitrary quantities $c = 1$, $\alpha = 1$, $\beta_1 = 1$, $\beta_2 = 2$. We also consider the initial conditions as $u_0 = v_0 = \sin \pi x$, and $u_1 = v_1 = 0$, $x \in [0,1]$.

6. Conclusion

The study of the coupled wave equations in parallel shows that the behavior of the system is effected primarily by the location of the boundary controllers and not by the

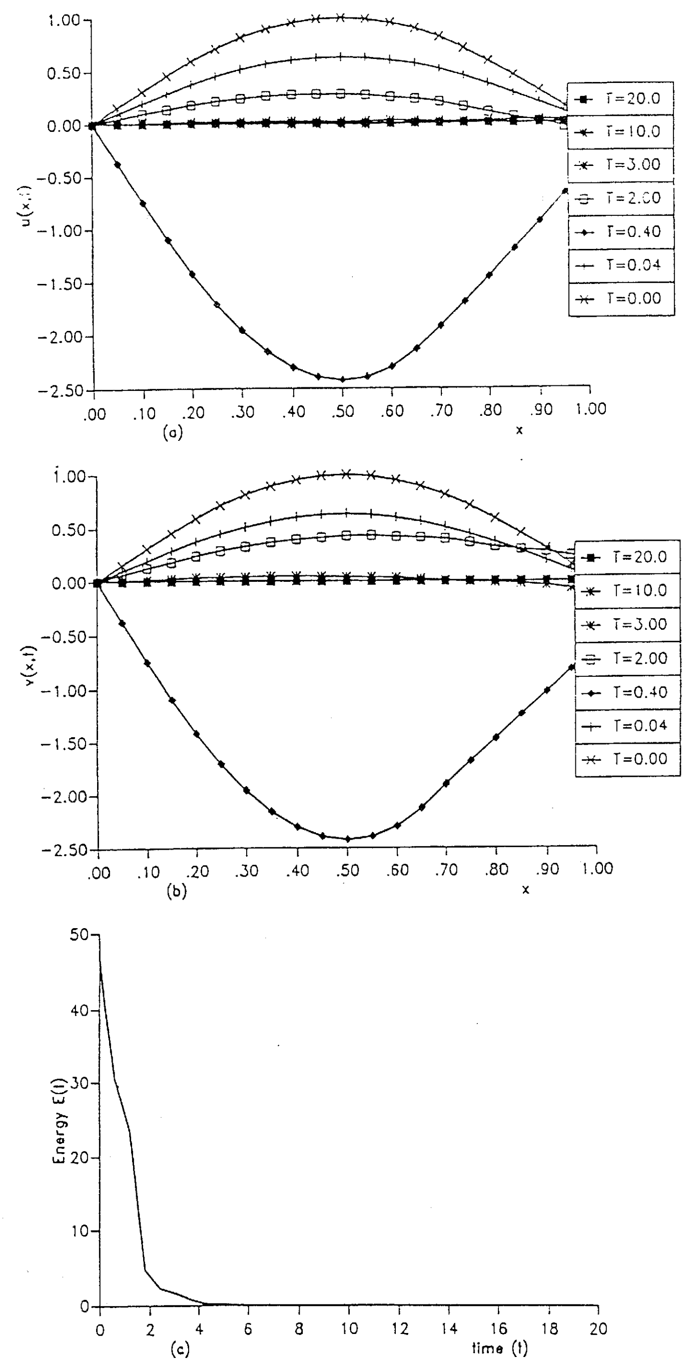

Fig. 1. (a) u-solution; (b) v-solution; and (c) energy of the system for boundary conditions (3) and (4).

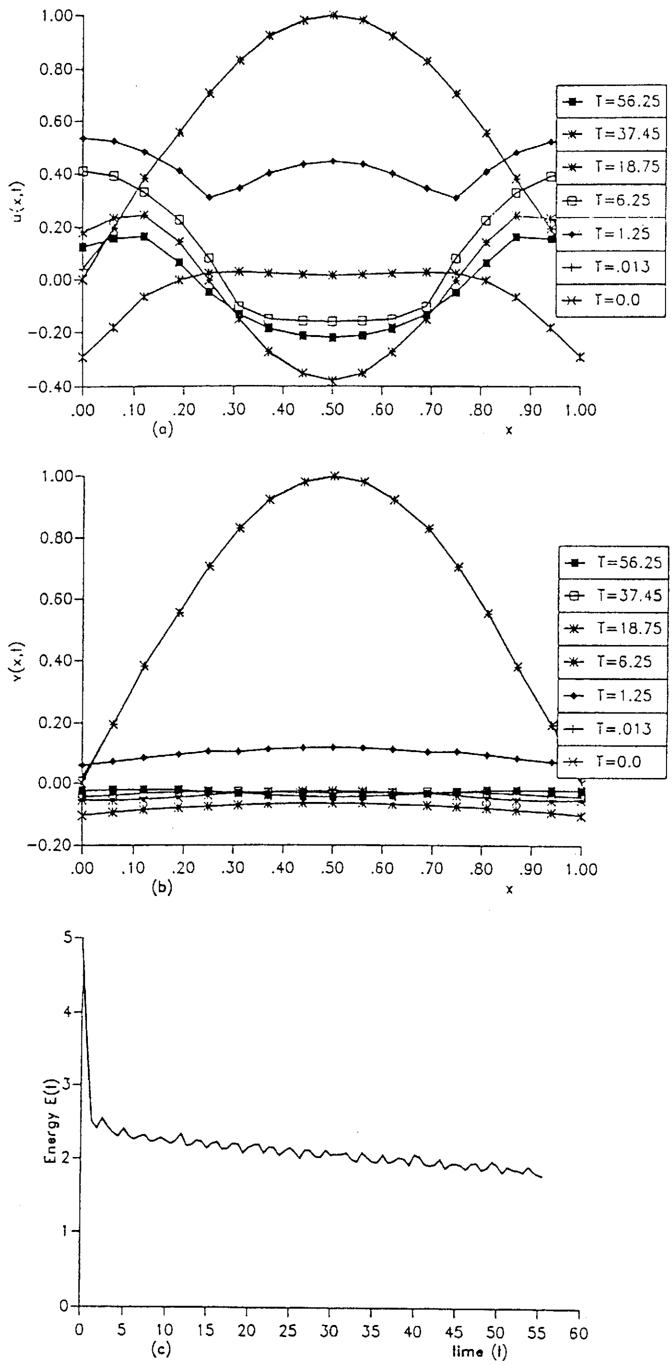

Fig. 2. (a) u-solution; (b) v-solution; and (c) energy of the system for boundary conditions (5) and (6).

minimum required controllers in the system. In Theorem 1, the uniform stabilization of coupled wave equations was achieved since the dissipative boundary conditions were imposed in both equations. However, in Theorem 2, we have found that if the system possesses both dissipative boundary controllers in only one equation, the most that can be expected is strong stability.

Appendix

Proof of Lemma 2. The lemma is proved by standard compactness-uniqueness arguments. For this we assume on the contrary the lemma is false. Then there is a sequence of solutions of (1)–(4) such that

$$\int_0^T (u_n - v_n)^2(1,t)\,\mathrm{d}t = 1 \tag{A1}$$

$$\int_0^T \Big(\beta_1(u_n)_t^2 + \beta^2(v_n)_t^2\Big)(1,t)\,\mathrm{d}t + \int_0^T\int_0^1 (u_n - v_n)^2\,\mathrm{d}x\mathrm{d}t \to 0 \tag{A2}$$

If T is large enough, it follows from (19) that $E(u_n, v_n)(T)$ is bounded and then, from (16), that $E(u_n, v_n)(0)$ is bounded. Therefore, since the energy is nonincreasing, $u_n - v_n$ is bounded in $L^\infty(0,T;H^1_{\{0\}}) \cap W^{1,\infty}(0,T;L^2)$. The injection

$$L^\infty(0,T;H^1_{\{0\}}) \cap W^{1,\infty}(0,T;L^2) \to L^\infty(0,T;[H^1_{\{0\}},L^2]_\gamma), \quad 0 < \gamma < 1$$

is compact, where $[H^1_{\{0\}},L^2]_\gamma$ is the interpolation space of order γ. Therefore, for a subsequence, $u_n - v_n$ converges strongly in $L^\infty(0,T;L^2)$ and $(u_n - v_n)(1,\cdot)$ converges strongly in $L^\infty(0,T)$. Calling the limit w, it follows from (A2) that $w = 0$. But (A1) implies that $\|w(1,\cdot)\|_{L^2(0,T)} = 1$, a contradiction. ■

Proof of Lemma 3. This lemma is also proved indirectly. If the lemma were false, for a sequence of solutions of (1)–(4) one would have

$$\int_0^T\int_0^1 (u - v)^2\,\mathrm{d}x\mathrm{d}t = 1 \tag{A3}$$

$$\int_0^T (\beta_1 u_t^2 + \beta^2 v_t^2)(1,t)\,\mathrm{d}t \to 0 \tag{A4}$$

As above, it follows from (19) and Lemma 1 that $u_n - v_n$ is bounded in $L^\infty(0,T;H^1_{\{0\}}) \cap W^{1,\infty}(0,T;L^2)$ and therefore, for a subsequence, $u_n - v_n$ converges strongly to w in $L^\infty(0,T;L^2)$ with $\|w\|_{L^2(0,T;L^2)} = 1$. The function $w_n = u_n - v_n$ satisfies the system

$$\ddot{w}_n - w_n'' + 2\alpha w_n = 0$$

$$w_n(0,t) = 0$$

$$w_n'(1,t) = -\beta_1 \dot{u}_n(1,t) + \beta_2 \dot{v}_n(1,t)$$

Passing to the limit in this system, using (A4), it follows that w satisfies

$$\ddot{w} - w'' + 2\alpha w = 0$$

$$w(0,t) = 0$$

$$w'(1,t) = 0, \qquad \dot{w}(1,t) = 0$$

Therefore $z = \dot{w}$ satisfies

$$\ddot{z} - z'' + 2\alpha z = 0$$

$$z(0,t) = 0$$

$$z'(1,t) = 0, \qquad z(1,t) = 0, \quad 0 < t < T$$

If T is large enough, it follows that $z = \dot{w} = 0$. Therefore w satisfies

$$w'' + 2\alpha w = 0$$

$$w(0) = 0$$

$$w'(1) = 0$$

which implies $w = 0$, a contradiction. ∎

Proof of Lemma 4. Let

$$\mathcal{A}(a) = a + b = a + \sqrt{a^2 + \frac{2\alpha}{2}} \tag{A5}$$

for which $\mathcal{A}'(a) = 1 + \dfrac{a}{\sqrt{a^2 + \frac{2\alpha}{c^2}}}$. Equation (A5) can be led to the following estimation:

$$\frac{\text{constant}}{n} \geq |\mathcal{A}(a) - \mathcal{A}(a_n)| = |(a+2) - (a_n + b_n)|$$

$$= \left| \left(1 + \frac{a_n}{\sqrt{a_n^2 + \frac{2\alpha}{c^2}}} \right) (a - a_n) + O\big(|a - a_n|^2\big) \right|$$

$$\geq \left| 1 + \frac{1}{\sqrt{1 - \dfrac{2\alpha}{c^2\left(n\pi + \frac{\alpha}{2n\pi c^2}\right)}}} \right| |a - a_n| - \frac{1}{n} \geq 2\frac{1}{n} - \frac{1}{n} = \frac{1}{n}, \quad \text{for large } n$$

∎

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