Appl. Math. and Comp. Sci., 1997, Vol.7, No.3, 513–520

UNREACHABILITY AND UNCONTROLLABILITY OF 2-D LINEAR SYSTEMS WITH BOUNDED INPUTS

TADEUSZ KACZOREK*

It is shown that the general 2-D model, with constant coefficients, of linear systems with bounded inputs and bounded norms of input matrices is not locally reachable and locally controllable if the norms of its system matrices are less than or equal to one. The classisal 2-D Roesser model with bounded inputs and a bounded norm of input matrix is not locally reachable and locally controllable if the norm of its system matrix is less than or equal to one. Similar results have been proved for the models with variable coefficients.

1. Introduction

The most popular models of two-dimensional (2-D) linear systems are the discrete models proposed by Roesser (1975), Fornasini and Marchesini (1976; 1978), and Kurek (1985). A survey of the current state of the theory of 2-D linear systems is given in (Kaczorek, 1993; Lewis, 1992; 1995).

It is well-known (Mohler, 1991) that the linear continuous-time system $\dot{x} = Ax + Bu$ with u bounded is not completely controllable if the eigenvalues of the system matrix A have negative real parts. Similarly, the linear discrete-time system $x_{i+1} = Ax_i + Bu_i$ with u_i bounded is not completely reachable (controllable) if the eigenvalues of A have magnitudes less than one.

In this short paper, a counterpart of 2-D linear systems described by the general model and the 2-D Roesser model will be established. It will be shown that the general 2-D model with bounded inputs and bounded norms of input matrices is not locally reachable and locally controllable if the norms of its system matrices are less than or equal to one.

2. Preliminaries

Consider 2-D linear systems described by the state equations (Kurek, 1985)

$$x_{i+1,j+1} = A_0 x_{ij} + A_1 x_{i+1,j} + A_2 x_{i,j+1} + B_0 u_{ij} + B_1 u_{i+1,j} + B_2 u_{i,j+1} \qquad i,j \in \mathbb{Z}_+ \tag{1}$$

* Warsaw University of Technology, Institute of Control and Industrial Electronics, 00–662 Warszawa, ul. Koszykowa 75, Poland, e-mail: kaczorek@nov.isep.pw.edu.pl.

where $x_{ij} \in \mathbb{R}^n$ is the local state vector at the point (i,j), $u_{ij} \in \mathbb{R}^m$ is the input, $A_t \in \mathbb{R}^{n \times n}$, $B_t \in \mathbb{R}^{n \times m}$, $t = 0, 1, 2$, $\mathbb{Z}_+$ is the set of nonnegative integers and $\mathbb{R}^{n \times m}$ ($\mathbb{R}^n$) is the set of $n \times m$ (n-dimensional) real matrices (vectors). The boundary conditions for (1) are given by

$$x_{i0} \quad \text{for} \quad i \geq 0 \qquad \text{and} \qquad x_{0j} \quad \text{for} \quad j \geq 0 \tag{2}$$

The solution x_{ij} of eqn. (1) with (2) is given by (Kaczorek, 1993; Kurek, 1985)

$$\begin{aligned} x_{ij} = & \sum_{p=1}^{i} T_{i-p,j-1}\left(A_1 x_{p0} + B_1 u_{p0}\right) + \sum_{q=1}^{j} T_{i-1,j-q}\left(A_2 x_{0q} + B_2 u_{0q}\right) \\ & + \sum_{p=1}^{i-1} T_{i-p-1,j-1} A_0 x_{p0} + \sum_{q=1}^{j-1} T_{i-1,j-q-1} A_0 x_{0q} \\ & + T_{i-1,j-1} A_0 x_{00} + \sum_{p=0}^{i-1} \sum_{q=0}^{j-1} T_{i-p-1,j-q-1} B_0 u_{pq} \\ & + \sum_{p=0}^{i} \sum_{q=0}^{j} \left(T_{i-p-1,j-q} B_1 T_{i-p,j-q-1} B_2\right) u_{pq} \quad i,j \in \mathbb{Z}_+ \end{aligned} \tag{3}$$

where T_{pq} is the transition matrix of (1) defined as follows:

$$\begin{aligned} & T_{00} := I \quad \text{(the identity matrix)} \\ & T_{pq} := A_0 T_{p-1,q-1} + A_1 T_{p,q-1} + A_2 T_{p-1,q} \ \text{for} \ i,j \in \mathbb{Z}_+ \ \text{and} \ i+j > 0 \\ & T_{pq} := 0 \quad \text{(the zero matrix) for } p < 0 \text{ and/or } q < 0 \end{aligned} \tag{4}$$

Definition 1. The system (1) is called *locally reachable* at (h,k) if for any boundary condition (2) and every vector $x_f \in \mathbb{R}^n$, there exists a sequence of inputs u_{ij} for $(i,j) \in D_{hk} := \{(i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : 0 \leq i \leq h,\ 0 \leq j \leq k,\ i+j \neq h+k\}$ such that $x_{hk} = x_f$.

Definition 2. The system (1) is called *locally controllable* at (h,k) if for any boundary condition (2) there exists a sequence of inputs u_{ij} for $(i,j) \in D_{hk}$ such that $x_{hk} = 0$.

3. General 2-D Model with Constant Coefficients

Let us denote by $\|A\|$ the norm of a matrix A and by $\|x\|$ the associated norm of a vector x ($\|Ax\| \leq \|A\|\|x\|$).

Lemma 1. *Let* $\|A_t\| \leq m \leq 1$ *for* $t = 0, 1, 2$. *Then* $\|T_{pq}\|$ *satisfies the inequality*

$$\|T_{pq}\| \leq 3^p 2^q m \quad \text{for} \quad p,q \in \mathbb{Z}_+ \ (p+q > 0) \tag{5}$$

Proof. Using (4) and (5) it is easy to check that the hypothesis is true for the pairs $(1,0)$, $(0,1)$ and $(1,1)$, since

$$\|T_{10}\| = \|A_2\| \leq 3m, \qquad \|T_{01}\| = \|A_1\| \leq 2m$$

$$\|T_{11}\| = \|A_0 + A_1A_2 + A_2A_1\| \leq \|A_0\| + 2\|A_1\|\|A_2\| \leq 6m$$

Assuming that the hypothesis is true for the pairs (i,j), $(i+1,j)$ and $(i,j+1)$, $i+j>0$, we shall show that it is also valid for the pair $(i+1,j+1)$. Using (4) for $p=i+1$, $q=j+1$ and the well-known properties of the matrix norm we obtain

$$\|T_{i+1,j+1}\| = \|A_0T_{ij} + A_1T_{i+1,j} + A_2T_{i,j+1}\| \leq \|A_0\|\|T_{ij}\| + \|A_1\|\|T_{i+1,j}\| + \|A_2\|\|T_{i,j+1}\| \leq 3^i2^jm + 3^{i+1}2^jm + 3^i2^{j+1}m \leq 3^{i+1}2^{j+1}m$$

Therefore the hypothesis (5) is true. ■

For simplicity it is assumed in (1) that $B_0 = B$, $B_1 = B_2 = 0$, i.e.

$$x_{i+1,j+1} = A_0x_{ij} + A_1x_{i+1,j} + A_2x_{i,j+1} + Bu_{ij} \quad i,j \in \mathbb{Z}_+ \tag{6}$$

It is also assumed that

$$\|B\| = b < \infty \tag{7}$$

Without loss of generality it may be assumed that the boundary conditions (2) are zero. In this case, from (3) for $B_0 = B$, $B_1 = B_2 = 0$ we get

$$x_{ij} = \sum_{p=0}^{i-1}\sum_{q=0}^{j-1} T_{pq}Bu_{i-p-1,j-q-1} \quad i,j \in \mathbb{Z}_+ \ (i+j>0) \tag{8}$$

Taking into account (5) and (7) from (8) we conclude that

$$\|x_{ij}\| = \|\sum_{p=0}^{i-1}\sum_{q=0}^{j-1} T_{pq}Bu_{i-p-1,j-q-1}\| \leq \sum_{p=0}^{i-1}\sum_{q=0}^{j-1} \|T_{pq}\|\|B\|\|u_{i-p-1,j-q-1}\|$$

$$\leq \sum_{p=0}^{i-1}\sum_{q=0}^{j-1} 3^p2^qmbu \leq \sum_{p=0}^{i-1}\sum_{q=0}^{j-1} 3^p2^qbu \tag{9}$$

where

$$\|u_{ij}\| \leq u \leq \infty \quad \text{for all } i,j \in \mathbb{Z}_+ \tag{10}$$

From (9) it is easy to show that if (10) holds and

$$\|A_t\| \leq 1 \quad \text{for} \quad t = 1,2,3 \tag{11}$$

then for a given arbitrary point (h,k) there is no bounded sequence $u_{ij}(i,j)$ for $(i,j) \in D'_{hk} = \{(i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : \ 0 \leq i < h, \ 0 \leq j < k\}$ satisfying (10) for every x_f such that $x_{hk} = x_f$. For example, if $\|x_f\| > \sum_{p=0}^{h-1}\sum_{q=0}^{k-1} 3^p2^qbu$, then there is no sequence u_{ij} satisfying (10) for $(i,j) \in D'_{hk}$ such that $x_{hk} = x_f$. Therefore the following theorem has been proved.

Theorem 1. *The system (6) is not locally reachable at any point (h,k) if (7), (10) and (11) hold.*

Remark 1. It can be shown that if h and k are finite, then the system (6) is not locally reachable at (h,k) provided that the sequence u_{ij} is bounded and the norms $\|B\|$ and $\|A_t\|$, $t = 1,2,3$ are also bounded.

Example 1. Consider the system (6) with

$$A_0 = 0, \quad A_1 = \frac{1}{6}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \quad A_2 = \frac{1}{4}\begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{12}$$

and zero boundary conditions for $h = 2$ and $k = 1$. It is easy to check that the system (6) with (12) is locally reachable at $(2,1)$ for unbounded inputs u_{ij}, since (Lewis, 1992; Kaczorek, 1993)

$$\operatorname{rank}[B,\ A_2B] = \operatorname{rank}\begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix} = 2$$

Let $|u_{ij}| \le u = 1$ for $(i,j) \in D'_{21}$ and $x_f = [3 \quad 2]^T$. In this case, from (8) we have

$$x_{21} = A_2Bu_{00} + Bu_{10} \tag{13}$$

Taking $\|x\| = \max_{l \le i \le n} |x_i|$ for the vector x and $\|A\| = \max_{l \le i \le n} \sum_{j=1}^{n} |a_{ij}|$ for the matrix $A = [a_{ij}]$ we obtain $\|A_2\| = 3/4$, $\|B\| = 1$ and from (13) $\|x_{21}\| \le \|A_2\|\|B\|\|u_{00}\| + \|B\|\|u_{10}\| \le (7/4)u$.

Note that in this case there is no sequence satisfying $|u_{ij}| \le 1$ for $(i,j) \in D'_{21}$ such that $x_{21} = x_f = [3 \quad 2]^T$, since $\|x_f\| = 3$. Therefore the system is not locally reachable at $(2,1)$ if $|u_{ij}| \le 1$ for $(i,j) \in D'_{21}$. $\diamondsuit$

Now let us consider the local controllability of the system (6) with nonzero boundary conditions (2). Define

$$x_{bc}(i,j) = \sum_{p=1}^{i} T_{i-p,j-1}A_1x_{p0} + \sum_{q=1}^{j} T_{i-1,j-q}A_2x_{0q} + \sum_{p=1}^{i-1} T_{i-p-1,j-1}A_0x_{p0} + \sum_{q=1}^{j-1} T_{i-1,j-q-1}A_0x_{0q} + T_{i-1,j-1}A_0x_{00} \tag{14}$$

For arbitrary conditions (2), (14) is an arbitrary vector in $\mathbb{R}^n$.

Using (3) and (14) we may write the solution x_{ij} of (6) with nonzero boundary conditions in the form

$$x_{ij} = x_{bc}(i,j) + \sum_{p=0}^{i-1}\sum_{q=0}^{j-1} T_{pq}Bu_{i-p-1,j-q-1} \tag{15}$$

From comparison of (8) with (15) it follows that for $i = h$, $j = k$ and $x_{hk} = 0$ we obtain the same relations as in the reachability case if we take $x_f = -x_{bc}(h,k)$. Therefore we have the following result.

Theorem 2. *The system (6) is not locally controllable at an arbitrary point (h,k) if (7), (10) and (11) hold.*

Similarly, we may prove for the system (1) the following.

Theorem 3. *The system (1) is not locally reachable and locally controllable at an arbitrary point (h,k) if (7) and (10) hold and*

$$\|B_t\| < \infty \quad \textit{for} \quad t = 1,2,3 \tag{16}$$

4. 2-D Roesser Model with Constant Coefficients

Consider the 2-D Roesser model (Roesser, 1975).

$$\begin{bmatrix} x^h_{i+1,j} \\ x^v_{i,j+1} \end{bmatrix} = A \begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix} + Bu_{ij}, \quad A := \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B := \begin{bmatrix} B_{11} \\ B_{22} \end{bmatrix} \tag{17}$$

where $x^h_{ij} \in \mathbb{R}^{n_1}$ is the horizontal state vector, $x^v_{ij} \in \mathbb{R}^{n_2}$ is the vertical state vector, $u_{ij} \in \mathbb{R}^m$ is the input.

It is well-known (Kaczorek, 1993) that defining

$$x_{ij} := \begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix}$$

we may write (17) in the form

$$x_{i+1,j+1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} x_{i,j+1} + \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} x_{i+1,j}$$

$$+ \begin{bmatrix} B_{11} \\ 0 \end{bmatrix} u_{i,j+1} + \begin{bmatrix} 0 \\ B_{22} \end{bmatrix} u_{i+1,j} \tag{18}$$

It follows from a comparison of (18) with (1) that the Roesser model (17) is a special case of (1) with

$$A_0 = 0, \quad A_1 = \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad A_2 = \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}$$

$$B_0 = 0, \quad B_1 := \begin{bmatrix} 0 \\ B_{22} \end{bmatrix}, \quad B_2 := \begin{bmatrix} B_{11} \\ 0 \end{bmatrix}$$

Therefore from Theorem 3 we have

Theorem 4. *The Roesser model (17) with bounded inputs u_{ij} is not locally reachable and locally controllable at an arbitrary point (h, k) if*

$$\|A\| \leq 1 \quad \textit{and} \quad \|B\| < \infty \tag{19}$$

5. Extensions for 2-D Models with Variable Coefficients

Consider the 2-D Roesser model with variable coefficients

$$\begin{bmatrix} x^h_{i+1,j} \\ x^v_{i,j+1} \end{bmatrix} = A_{ij} \begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix} + B_{ij} u_{ij} \tag{20}$$

where x^h_{ij}, x^v_{ij} and u_{ij} are defined in the same way as for the model (17) and the entries of

$$A_{ij} := \begin{bmatrix} A^{11}_{ij} & A^{12}_{ij} \\ A^{21}_{ij} & A^{22}_{ij} \end{bmatrix} \quad \text{and} \quad B_{ij} := \begin{bmatrix} B^1_{ij} \\ B^2_{ij} \end{bmatrix}$$

depend on i and j.

The transition matrix T^{ij}_{pq} of (16) is defined as follows (Kaczorek, 1993):

$$\begin{aligned} & T^{00}_{pq} := I \quad \text{(the identity matrix)} \quad for \ p, q \in \mathbb{Z}_+ \\ & T^{ij}_{pq} := A^{10}_{p-1,q} T^{i-1,j}_{p-1,q} + A^{01}_{p,q-1} T^{i,j-1}_{p,q-1} \\ & T^{ij}_{pq} := 0 \quad \text{(the zero matrix) for } i < 0 \text{ and/or} \\ & \qquad j < 0 \text{ and/or } p < 0 \text{ and/or } q < 0 \end{aligned} \tag{21}$$

where

$$A^{01}_{p-1,q} := \begin{bmatrix} A^{11}_{p-1,q} & A^{12}_{p-1,q} \\ 0 & 0 \end{bmatrix}, \quad A^{10}_{p,q-1} := \begin{bmatrix} 0 & 0 \\ A^{21}_{p,q-1} & A^{22}_{p,q-1} \end{bmatrix}$$

The solution to (20) with boundary conditions x_{0j}^h, $j \in \mathbb{Z}_+$, x_{i0}^v, $i \in \mathbb{Z}_+$ is given by (Kaczorek, 1993)

$$\begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} = \sum_{k=0}^{i} T_{ij}^{i-k,j} \begin{bmatrix} 0 \\ x_{k0}^v \end{bmatrix} + \sum_{l=0}^{j} T_{ij}^{i,j-1} \begin{bmatrix} x_{0l}^h \\ 0 \end{bmatrix}$$

$$+ \sum_{k=0}^{i-1} \sum_{l=0}^{j} T_{ij}^{i-k-1,j-l} \begin{bmatrix} B_{kl}^1 \\ 0 \end{bmatrix} + \sum_{k=0}^{i} \sum_{l=0}^{j-1} T_{ij}^{i-k,j-l-1} \begin{bmatrix} 0 \\ B_{kl}^2 \end{bmatrix} u_{kl} \qquad (22)$$

Likewise, using (21) and (22) the following theorem can be proved:

Theorem 5. *The 2-D Roesser model (20) with variable coefficients is not locally reachable and locally controllable at an arbitrary point (p, q) if the input sequence u_{ij} is bounded,*

$$\|A_{ij}\| < 1 \quad \textit{and} \quad \|B_{ij}\| < \infty \quad \textit{for all} \quad i, j \in \mathbb{Z}_+ \qquad (23)$$

Similar results can be obtained for the 2-D general model with variable coefficients.

6. Concluding Remarks

It has been shown that the general 2-D model, with constant coefficients, of linear systems with bounded inputs is not locally reachable and locally controllable if (7), (10) and (16) hold. The 2-D Roesser model (17) with bounded inputs is not locally reachable and locally controllable if the condition (19) is satisfied. If the coefficients of 2-D Roesser model are variable, then the conditions (19) should be replaced with the conditions (23). With slight modifications the considerations can be extended to n-D linear systems, $n > 2$. An open problem is an extension of the considerations to singular (implicit) linear systems.

References

Fornasini E. and Marchesini G. (1976): *State space realization of two-dimensional filters.* — IEEE Trans. Automat. Contr., Vol.AC-21, No.4, pp.484–491.

Fornasini E. and Marchesini G. (1978): *Doubly indexed dynamical systems: State-space models and structural properties.* — Math. Syst. Theory, Vol.12, pp.59–72.

Kaczorek T. (1993): *Linear Control Sytems, Vol.II: Synthesis of Multivariable Systems and Multidimensional Systems.* — New York: Research Studies Press and J. Wiley.

Klamka J. (1991): *Controllability of Dynamical Systems.* — Warsaw-London: PWN and Kluwer Academic Publisher.

Kurek J. (1985): *The general state-space model for a two-dimensional linear digital systems.* — IEEE Trans. Automat. Contr., Vol.AC-30, No.5, pp.600–602.

Lewis F.L. (1992): *A review of 2-D implicit systems.* — Automatica, Vol.28, No.2, pp.345–354.

Lewis F.L. (1995): *Techniques in 2-D implicit systems.* — Contr. Dynam. Syst., Vol.69, pp.89–131.

Mohler R.R. (1991): *Nonlinear Systems.* — Englewood Cliffs, New York: Prentice Hall.

Roesser R.P. (1975): *A discrete state-space model for linear image processing.* — IEEE Trans. Automat. Contr. AC-21, No.1, pp.1–10.

Received: 15 March 1997