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nD CONTROL SYSTEMS IN A PRACTICAL SENSE

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In view of the features of practical nD signals and systems, this paper presents a comprehensive treatment on some theoretical and methodological results and possible applications for the control of nD systems in the practical sense that the input and output signals of the systems are unbounded in, at most, one dimension. Recent results on the basic properties and control problems for the case under consideration such as practical BIBO and internal stabilities, practical controllability and observability, feedback practical stabilization by both the algebraic and the state-space method, and the relation between the two methods will be summarized. Further contributions to the practical tracking problem and its applications will also be developed.

The results obtained make it clear that the nD control problems considered in the practical sense can be essentially reduced to the corresponding 1D problems, and thus can be solved, when compared with the conventional nD system theory, under less restrictive stability conditions and by much simpler 1D methods. In particular, it is shown that the proposed method for 2D practical tracking control in fact provides a general design approach for a class of iterative learning control systems and linear multipass processes. Therefore, the presented control theory for nD systems in the practical sense is of significance not only from the point of view of practical applications of nD system theory, but also from that of control of such iterative systems.

1. Introduction

In many practical situations of nD (multidimensional) signal processing, such as seismic and image processing, the independent variables $i_1, \ldots, i_n$ of an nD signal $x(i_1, \ldots, i_n)$ are usually spatial variables bounded in finite domains, except that perhaps one variable is the unbounded temporal variable. Based on this feature, Agathoklis and Bruton (1983) introduced the concept of practical BIBO (bounded-input bounded-output) stability for nD discrete systems, which proved to be less restrictive and more relevant for practical applications than the conventional BIBO stability. Some results on the design of nD digital filters based on practical BIBO stability have been documented in the literature (see e.g. Reddy *et al.*, 1990).

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Since practical BIBO stability conditions are much weaker than the conventional ones, there exist systems that are practical BIBO stable but not conventional BIBO stable (Agathoklis and Bruton, 1983). This fact means that the current design methods for nD ($n = 2$) systems, developed under the conventional BIBO stability concept, cannot be applied to design practical BIBO stable feedback systems (Xu *et al.*, 1994b). In fact, many new problems need to be investigated if we want to deal with the control problems of nD systems in such a practical sense.

It has been well-known that a large class of iterative learning control systems and linear multipass processes can be described by 2D system models (Boland and Owens, 1980; Geng *et al.*, 1990). As a common feature of these systems, it is observed that, though the iteration index is not subjected to any boundary condition, the dynamical processes on each trial or pass are always restricted in finite time intervals. It is natural, therefore, to consider that it would be possible to establish a simple and unified approach for the control of these systems under less restrictive stability conditions derived in some practical sense.

With such motivations, the authors have recently considered some fundamental control problems of nD systems, such as internal stability (Xu *et al.*, 1996a) and feedback stabilization (Xu *et al.*, 1994b; 1996b), in the practical sense of (Agathoklis and Bruton, 1983), i.e. under the assumption that the system input and output signals are unbounded in at most one dimension. The results obtained show that these nD control problems in the practical sense can be essentially reduced to the corresponding 1D problems and solved by the well-known 1D approaches. This property is of special significance as it implies that for some cases nD control problems may be unsolvable under the conditions of conventional stability, but can be solved under the less restrictive practical stability and meanwhile by much simpler 1D methods.

The purpose of this paper is to show the state of the art of the control theory for nD systems in the practical sense, by summarizing the above-mentioned previous results and showing some further results on the practical tracking problem and its applications.

The paper is organized as follows. Section 2 reviews the results on practical BIBO stability and practical internal stability. Sections 3 is devoted to treatments of the problem of practical stabilization by both the matrix fractional description (MFD) algebraic approach and the state-space approach. In Section 4, 2D practical tracking problem and its applications to iterative learning control systems and linear multipass processes will be investigated. Section 5 shows some numerical examples to support the theoretical results.

Troughhout the paper, $\mathbb{R}$ denotes the field of real numbers, $\mathbb{C}$ is the field of complex numbers, $\mathbb{Z}_+$ stands for the set of non-negative integers, $\mathbb{Z}_+^n = \{(i_1,\dots,i_n) \mid i_1,\dots,i_n \in \mathbb{Z}_+\}$, $\mathbb{Z}_+^{-n} = \{(i_1,\dots,i_n) \mid i_1,\dots,i_n \in \mathbb{Z}_+$, but if $i_j = +\infty$, then $i_k < +\infty$, $k = 1,2,\dots,n$, $k \neq j$, i.e. no more than one of $i_1,\dots,i_n$ can be infinite simultaneously$\}$. U, $\bar{U}$, T denote the open, closed unit disc and the unit circle in $\mathbb{C}$, respectively. U^n, $\bar{U}^n$, T^n stand for the open, closed unit polydisc and the unit torus in $\mathbb{C}^n$, respectively. A^T or x^T are the transposes of matrix A or vector x; I_m is the $m \times m$ identity matrix; $(i_1,\dots,0_j,\dots,i_n) \triangleq (i_1,\dots,i_{j-1},0,i_{j+1},\dots,i_n)$; $(0,\dots,i_j,\dots,0) \triangleq (0,\dots,0,i_j,0,\dots,0)$.

2. Practical Stabilities

In this section, we review the concepts of practical BIBO stability and practical internal (asymptotic) stability for nD discrete systems and the corresponding necessary and sufficient conditions shown by Agathoklis and Bruton (1983) and Xu *et al.* (1996a).

2.1. Practical BIBO Stability

Consider the class of SISO (single-input single-output) linear shift-invariant nD discrete systems for which the input $u(i_1, \ldots, i_n)$ and the output $y(i_1, \ldots, i_n)$ are related by the nD convolution sum:

$$y(i_1, \ldots, i_n) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} h(i_1 - k_1, \ldots, i_n - k_n) u(k_1, \ldots, k_n) \tag{1}$$

where $h(i_1, \ldots, i_n)$ is the impulse response. Using the nD z-transform

$$F(z_1, \ldots, z_n) = \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} f(i_1, \ldots, i_n) z_1^{i_1} \cdots z_n^{i_n} \tag{2}$$

we obtain the transfer function of (1)

$$H(z_1, \ldots, z_n) = \frac{n(z_1, \ldots, z_n)}{d(z_1, \ldots, z_n)} \tag{3}$$

For the nD system (1), the practical BIBO stability is defined by:

Definition 1. (Agathoklis and Bruton, 1983) An nD system is *practical BIBO stable* if and only if, for all input signals $u(i_1, \ldots, i_n)$ such that

$$|u(i_1, \ldots, i_n)| \le M < \infty \quad \forall (i_1, \ldots, i_n) \in \mathbb{Z}_+^{-n} \tag{4}$$

where M is a finite real number, there exists a finite real L such that, for the output of the system $y(i_1, \ldots, i_n)$, the relation

$$|y(i_1, \ldots, i_n)| \le L < \infty \tag{5}$$

is satisfied.

The difference to the conventional BIBO stability which is defined by replacing $\mathbb{Z}_+^{-n}$ by $\mathbb{Z}_+^n$ in (4) (Jury, 1978) is that, for the case of practical BIBO stability, the behaviour of the system at the points where more than one of the indeterminates take an infinite value are not considered (see Agathoklis and Bruton, 1983).

It is well-known that the nD system (1) is (conventional) BIBO stable if and only if its impulse response satisfies the following relation (Jury, 1978):

$$\sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} |h(i_1, i_2, \ldots, i_n)| < \infty \tag{6}$$

In contrast, Agathoklis and Bruton (1983) have proved the following theorem which reveals the relationship between the practical BIBO stability and the impulse response of an nD system.

Theorem 1. (Agathoklis and Bruton, 1983) *An nD discrete system is practical BIBO stable if and only if the n inequalities are satisfied:*

$$\sum_{i_1=0}^{N_1}\sum_{i_2=0}^{N_2}\cdots\sum_{i_k=0}^{N_k=\infty}\cdots\sum_{i_n=0}^{N_n}|h(i_1,i_2,\ldots,i_k,\ldots,i_n)|<\infty,\quad k=1,2,\ldots,n \tag{7}$$

where $N_1,\ N_2,\ldots,N_{k-1},\ N_{k+1},\ldots,N_n$ are any finite integers.

Further, the following theorem relates the practical BIBO stability to the singularities of an nD transfer function.

Theorem 2. (Agathoklis and Bruton, 1983) *The nD system (3) is practical BIBO stable if and only if*

$$d(0,\ldots,z_k,\ldots,0)\neq 0\quad \forall z_k\in\bar{U},\quad k=1,2,\ldots,n \tag{8}$$

It is well-known that, when an nD system given by (3) has no nonessential singularity of the second kind (Bose, 1982) on T^2 when $n=2$ (Goodman, 1977), or on $\bar{U}^n-U^n$ when $n>2$ (Swamy *et al.*, 1985), the system is conventional BIBO stable if and only if

$$d(z_1,\ldots,z_n)\neq 0\quad \forall(z_1,\ldots,z_n)\in\bar{U}^n \tag{9}$$

We therefore see that the condition (8) for practical BIBO stability is in fact equivalent to the stabilities of n 1D systems, and this is much weaker than the condition (9) for conventional BIBO stability.

2.2. Practical Internal Stability

Consider the nD Roesser state-space model (Kaczorek, 1985; Kurek, 1985) given by

$$\boldsymbol{x}'(i_1,\ldots,i_n)=A\boldsymbol{x}(i_1,\ldots,i_n)+B\boldsymbol{u}(i_1,\ldots,i_n) \tag{10a}$$

$$\boldsymbol{y}(i_1,\ldots,i_n)=C\boldsymbol{x}(i_1,\ldots,i_n)+D\boldsymbol{u}(i_1,\ldots,i_n) \tag{10b}$$

where $\boldsymbol{u}(i_1,\ldots,i_n)\in\mathbb{R}^m$ and $\boldsymbol{y}(i_1,\ldots,i_n)\in\mathbb{R}^l$ are the input and output vectors, respectively; $\boldsymbol{x}(i_1,\ldots,i_n)\in\mathbb{R}^{\tilde{n}}$ is the local state vector in the form

$$\boldsymbol{x}(i_1,\ldots,i_n)=\begin{bmatrix}x_1(i_1,\ldots,i_n)\\ x_2(i_1,\ldots,i_n)\\ \vdots\\ x_n(i_1,\ldots,i_n)\end{bmatrix},\quad \boldsymbol{x}'(i_1,i_2,\ldots,i_n)=\begin{bmatrix}x_1(i_1+1,i_2,\cdots,i_n)\\ x_2(i_1,i_2+1,\cdots,i_n)\\ \vdots\\ x_n(i_1,i_2,\cdots,i_n+1)\end{bmatrix}$$

with $x_i(i_1,\dots,i_n) \in \mathbb{R}^{n_i}$ $(i = 1,\dots,n,\ \tilde{n} = \sum_{i=1}^{n} n_i)$ being the i-th (sub-)vector of $x(i_1,\dots,i_n)$; and

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix}, \quad C = [C_1 \ \cdots \ C_n]$$

with A_{ij}, B_i, C_i and D being constant real matrices of suitable dimensions, in particular, $A_{ii} \in \mathbb{R}^{n_i \times n_i}$ $(i = 1,\dots,n)$.

When $f(i_1,\dots,i_n) = 0$ for $(i_1,\dots,i_n) \notin \mathbb{Z}_+^n$, the nD z-transform of (2) can also be written as (Bisiacco *et al.*, 1989)

$$F(z_1,\dots,z_n) = \sum_{i_1+\cdots+i_n \geq 0} f(i_1,\dots,i_n) z_1^{i_1} \cdots z_n^{i_n} \tag{11}$$

Applying this to the nD system (10) yields

$$x(z_1,\dots,z_n) = (I - ZA)^{-1}\Big(ZBu(z_1,\dots,z_n) + \mathcal{X}_0\Big) \tag{12}$$

$$y(z_1,\dots,z_n) = C\,x(z_1,\dots,z_n) + Du(z_1,\dots,z_n) \tag{13}$$

where

$$Z = \text{block diag}(z_1 I_{n_1}, z_2 I_{n_2}, \cdots, z_n I_{n_n}) \tag{14}$$

and

$$\mathcal{X}_0 = \sum_{i_1+\cdots+i_n = 0} x(i_1,\dots,i_n) z_1^{i_1} \cdots z_n^{i_n} \tag{15}$$

Let $\mathcal{X}_0 = 0$. Then we obtain the input/output relation

$$y(z_1,\dots,z_n) = G(z_1,\dots,z_n)\, u(z_1,\dots,z_n) \tag{16}$$

where $G(z_1,\dots,z_n)$ is the transfer matrix given by

$$G(z_1,\dots,z_n) = C(I - ZA)^{-1} ZB + D \tag{17}$$

The characteristic polynomial of the nD system (10) is defined as

$$\rho(z_1,\dots,z_n) = \det(I - ZA) \tag{18}$$

and we have the following relation (Xu *et al.*, 1996a):

$$\rho(0,\dots,z_k,\dots,0) = \det(I_{n_k} - z_k A_{kk}), \quad k = 1,\dots,n \tag{19}$$

Note that, in order to investigate the internal stability in this subsection, it will be enough to consider only the SISO case. Therefore we assume here $m = l = 1$ for the Roesser model (10). Introduce the following notation:

$$\tilde{\mathcal{X}}_r = \Big\{ x(i_1,\dots,i_n) \in \mathbb{R}^{\tilde{n}} \mid \sum_{j=1}^{n} i_j = r \Big\} \tag{20}$$

where it is assumed that $\boldsymbol{x}(i_1,\ldots,i_n)=0$ when $(i_1,\ldots,i_n)\notin\mathbb{Z}_+^n$, and $\boldsymbol{x}_j(i_1,\ldots,0_j,\ldots,i_n)=0$ $(j=1,\ldots,n)$ except $\boldsymbol{x}(0,\ldots,0)$. Note that this assumption implies that $\boldsymbol{x}(i_1,\ldots,i_n)\in\tilde{\mathcal{X}}_0$ may be of any given value, say $\boldsymbol{x}_0$, for $(i_1,\ldots,i_n)=(0,\ldots,0)$, but is zero for $(i_1,\ldots,i_n)\neq(0,\ldots,0)$. Further, denote by $\|\boldsymbol{x}(i_1,\ldots,i_n)\|$ the Euclidean norm of local state $\boldsymbol{x}(i_1,\ldots,i_n)$ in the state space $\mathbb{R}^{\bar{n}}$ and define, in the practical sense of (Agathoklis and Bruton, 1983), the following norm for $\tilde{\mathcal{X}}_r$:

$$\|\tilde{\mathcal{X}}_r\|_{h_0}=\max_{h\le h_0}\Big\{\|\boldsymbol{x}(i_1,\ldots,i_{j-1},r-h,i_{j+1},\ldots,i_n)\|\ \Big|\ h=\sum_{\substack{k=1\\k\neq j}}^{n}i_k,\ j=1,\ldots,n\Big\}\tag{21}$$

where h_0 is an arbitrary finite positive integer. For brevity, let $\|\tilde{\mathcal{X}}_r\|=\|\tilde{\mathcal{X}}_r\|_{h_0}$ in what follows.

Definition 2. (Xu *et al.*, 1996a) For the nD system (10), suppose that $u=0$ and $\|\tilde{\mathcal{X}}_0\|$ is finite. Then the system is said to be *practically internally* (or *asymptotically*) *stable* if $\|\tilde{\mathcal{X}}_r\|\to 0$ as $r\to\infty$ for any finite h_0.

It might be noted from (21) that, for a certain $j\in\{1,\ldots,n\}$, giving a finite h_0 in fact implies that only i_j may be infinite while $i_1,\ldots,i_{j-1},i_{j+1},\ldots,i_n$ are restricted to some finite values. Considering then the situations for any finite h_0 and for all $j=1,\ldots,n$, we see that the defined practical internal stability means that for any finite $\|\tilde{\mathcal{X}}_0\|$, $\|x(i_1,\ldots,i_n)\|$ approaches zero as (any) one of the variables $i_1,\ldots,i_n$ goes to infinity but all the others are finite, or equivalently, as $r=i_1+\cdots+i_n\to\infty$ but $(i_1,\ldots,i_n)\in\mathbb{Z}_+^{-n}$.

The following necessary and sufficient condition for practical internal stability has been shown in (Xu *et al.*, 1996a).

Theorem 3. *The nD system (10) is practically internally (or asymptotically) stable if and only if all the matrices A_{kk} $(k=1,\ldots,n)$ are stable in the* 1D *sense, i.e.*

$$\rho(0,\ldots,z_k,\ldots,0)=\det(I_{n_k}-z_kA_{kk})\neq 0\quad\forall z_k\in\bar{U},\quad k=1,\ldots,n\tag{22}$$

For the 2D case, another well-known state-space model is the Fornasini-Marchesini model (Fornasini and Machesini, 1978) described by the equation

$$\begin{aligned}\boldsymbol{x}(i_1+1,i_2+1)&=A_1\boldsymbol{x}(i_1,i_2+1)+A_2\boldsymbol{x}(i_1+1,i_2)\\&\quad+B_1\boldsymbol{u}(i_1,i_2+1)+B_2\boldsymbol{u}(i_1+1,i_2)\end{aligned}\tag{23a}$$

$$\boldsymbol{y}(i_1,i_2)=C\boldsymbol{x}(i_1,i_2)+D\boldsymbol{u}(i_1,i_2)\tag{23b}$$

The characteristic polynomial for the 2D system (23) is $\rho(z_1,z_2)=\det(I-z_1A_1-z_2A_2)$ (Fornasini and Marchesini, 1978). In the same way as for the Roesser model, one can also define the practical internal stability for a 2D Fornasini-Marchesini model, and show that a 2D system given by (23) is practically internally stable if and only if A_k $(k=1,2)$ are 1D stable, i.e. $\det(I-z_kA_k)\neq 0,\forall z_k\in\bar{U},\ k=1,2$.

The results obtained in Theorem 3 reveal that the practical internal stability of an nD system is also equivalent to the stabilities of n 1D systems, just like the case of practical BIBO stability. When $n = 2$, in particular, a system given by (10) or (23) is practically internally stable if and only if

$$\rho(z_1, 0) \neq 0, \quad \rho(0, z_2) \neq 0, \quad \forall z_1, z_2 \in \bar{U} \tag{24}$$

In contrast to this, however, a 2D system is internally stable in the conventional sense (Ahmed, 1980; Fornasini and Marchesini, 1979) if and only if $\rho(z_1, z_2) \neq 0$, $\forall (z_1, z_2) \in \bar{U}^2$, which is obviously much more restrictive.

It should also be noted that the conditions given in (24) are necessary and sufficient for 2D practical asymptotic stability, i.e. asymptotic stability in both cases where i_1 is unbounded (but i_2 bounded) and i_2 is unbounded (but i_1 bounded). If the case we have is known *a priori*, then only one of the conditions in (24) needs to be satisfied, as coincides with the result of (Kurek and Zaremba, 1993). A similar comment also applies to the nD case.

Let $G(z_1, \ldots, z_n) = C(I - ZA)^{-1}ZB + D = b(z_1, \ldots, z_n)/a(z_1, \ldots, z_n)$ be the transfer function for the nD system (10) when $m = l = 1$, and $G(0, \ldots, z_k, \ldots, 0) = b_k(0, \ldots, z_k, \ldots, 0)/a_k(0, \ldots, z_k, \ldots, 0)$, $k = 1, \ldots, n$. Since cancellations may occur between the numerators and the denominators, $a(z_1, \ldots, z_n)$ and $a_k(0, \ldots, z_k, \ldots, 0)$ are not in general equal to $\rho(z_1, \ldots, z_n)$ and $\rho(0, \ldots, z_k, \ldots, 0)$, respectively. Therefore, though practical internal stability implies practical BIBO stability, the converse is not necessarily true. It has been clarified in (Xu *et al.*, 1996a) that a state-space realization of a practical BIBO stable system is practically internally stable if and only if it is practically stabilizable and practically detectable, which will be defined in Section 3.2. Moreover, it has also been shown that, for the 2D transfer function $b(z_1, z_2)/a(z_1, z_2)$, a practically internally stable (Roesser or Fornasini-Marchesini model) realization can always be obtained if $b(z_1, z_2)/a(z_1, z_2)$ is practical BIBO stable (Xu *et al.*, 1996a).

3. Design of Practically Stable nD Feedback Control Systems

Detailed treatments concerning practical stabilization of nD systems by algebraic and state-space approaches and their relations will be stated in the following subsections. By practical stabilization, we mean to find a feedback compensator such that the resultant closed-loop system is practically stable.

3.1. Algebraic Approach

We describe here the MFD approach to the design of practically stable output feedback nD systems. Let $\boldsymbol{G}$ be the ring of nD causal rational functions, $\boldsymbol{H}$ be the ring of nD practically-stable rational functions, i.e.

$$\boldsymbol{G} = \left\{ \frac{n}{d} \mid n, d \in \mathbb{R}[z_1, \ldots, z_n],\ d(0, \ldots, 0) \neq 0 \right\}$$

$$\boldsymbol{H} = \left\{ \frac{n}{d} \in \boldsymbol{G} \mid d(0, \ldots, z_k, \ldots, 0) \neq 0 \ \ \forall z_k \in \bar{U},\ k = 1, 2, \ldots, n \right\}$$

and let $\boldsymbol{I} = \{h \in \boldsymbol{H} \mid h^{-1} \in \boldsymbol{G}\}$, $\boldsymbol{J} = \{h \in \boldsymbol{H} \mid h^{-1} \in \boldsymbol{H}\}$.

Denote by $\boldsymbol{M}(*)$ the set of the matrices with entries in the set $*$ (e.g. $\boldsymbol{G}, \boldsymbol{H}$). An element of $\boldsymbol{M}(\boldsymbol{H})$ is then said to be $\boldsymbol{G}$-unimodular (respectively $\boldsymbol{H}$-unimodular) if and only if it is square and its determinant belongs to $\boldsymbol{I}$ $(\boldsymbol{J})$. If $P \in \boldsymbol{M}(\boldsymbol{G})$ can be written as $P = N_p D_p^{-1}$, where D_p, $N_p \in \boldsymbol{M}(\boldsymbol{H})$ and D_p is $\boldsymbol{G}$-unimodular, we refer to such $N_p D_p^{-1}$ as a right MFD of P (on $\{\boldsymbol{G}, \boldsymbol{H}, \boldsymbol{I}, \boldsymbol{J}\}$).

Definition 3. For a right MFD $N_p D_p^{-1}$, we say that N_p and D_p are *right coprime* on $\boldsymbol{H}$ and that $N_p D_p^{-1}$ is a *right coprime MFD* on $\boldsymbol{H}$ if and only if there exist $W, V \in \boldsymbol{M}(\boldsymbol{H})$ such that the Bezout equation

$$WD_p + VN_p = I \tag{25}$$

is fulfilled.

The dual definitions of the left counterparts are given analogously. It is easy to see that, for any $P \in \boldsymbol{M}(\boldsymbol{G})$, we can always find N_p, $D_p \in \boldsymbol{M}(\mathbb{R}[z_1, \ldots, z_n]) \subset \boldsymbol{M}(\boldsymbol{H})$ with $\det D_p \in \boldsymbol{I}$ such that $P = N_p D_p^{-1}$, but N_p and D_p are not necessarily right coprime on $\boldsymbol{H}$. The following theorem gives a necessary and sufficient condition for the existence of a right coprime MFD of P on $\boldsymbol{H}$. Suppose, without loss of generality, N_p, $D_p \in \boldsymbol{M}(\mathbb{R}[z_1, \ldots, z_n])$ and $\det D_p \in \boldsymbol{I}$. Let $\mathcal{I}_k$ denote the ideal generated by all the maximal-order minors of the matrix

$$\begin{bmatrix} D_p(0, \ldots, z_k, \ldots, 0) \\ N_p(0, \ldots, z_k, \ldots, 0) \end{bmatrix} \tag{26}$$

and $\mathcal{V}(\mathcal{I}_k)$ be the algebraic variety of $\mathcal{I}_k$, i.e. the set of common zeros of the minors, where $k = 1, 2, \ldots, n$.

Theorem 4. (Xu *et al.*, 1994b) *For* $N_p D_p^{-1}$ *where* N_p, $D_p \in \boldsymbol{M}(\mathbb{R}[z_1, \ldots, z_n])$ *and* $\det D_p \in \boldsymbol{I}$, D_p *and* N_p *are right coprime on* $\boldsymbol{H}$ *if and only if*

$$\mathcal{V}(\mathcal{I}_k) \cap \bar{U} = \emptyset, \quad k = 1, 2, \ldots, n \tag{27}$$

The proof of this theorem in (Xu *et al.*, 1994b) gives a constructive solution algorithm for the Bezout eqn. (25):

Algorithm 1.

Step 1. When (27) holds, solve the following 1D polynomial matrix equations by the well-known 1D methods (Kučera, 1979):

$$\tilde{X}_k(z_k) D_p(0, \ldots, z_k, \ldots, 0) + \tilde{Y}_k(z_k) N_p(0, \ldots, z_k, \ldots, 0) = \tilde{\Phi}_k(z_k) \tag{28}$$

where $\tilde{X}_k(z_k)$, $\tilde{Y}_k(z_k) \in \boldsymbol{M}(\mathbb{R}[z_k])$, and

$$\det \tilde{\Phi}_k(z_k) \neq 0 \quad \forall z_k \in \bar{U}, \quad k = 1, 2, \ldots, n \tag{29}$$

Step 2. Construct the following solution to (28) for $k = 1, \ldots, n$:

$$\begin{cases} \bar{X}_k(z_k) = \tilde{X}_k(z_k) + R_k \tilde{N}_k(z_k) \\ \bar{Y}_k(z_k) = \tilde{Y}_k(z_k) - R_k \tilde{D}_k(z_k) \end{cases} \tag{30}$$

where $\tilde{D}_k(z_k),\ \tilde{N}_k(z_k) \in \boldsymbol{M}(\mathbb{R}[z_k])$ satisfy

$$\tilde{D}_k^{-1}(z_k)\tilde{N}_k(z_k) = N_p(0,\dots,z_k,\dots,0)D_p^{-1}(0,\dots,z_k,\dots,0) \tag{31}$$

and are left coprime, and

$$R_k = \tilde{Y}_k(0)\tilde{D}_k^{-1}(0) \in \boldsymbol{M}(\mathbb{R}) \tag{32}$$

Note that $\det \bar{X}_k(0) = \det\{\tilde{\Phi}_k(0)D_p^{-1}(0,\dots,0)\} \neq 0,\ \bar{Y}_k(0) = 0.$

Step 3. Write $\bar{X}_k(z_k)$ as

$$\bar{X}_k(z_k) \triangleq \hat{X}_k(z_k) + \bar{X}_k(0) \tag{33}$$

where $\bar{X}_k(0)$ corresponds to the constant terms of $\bar{X}(z_k)$ and $\hat{X}_k(z_k)$ denotes all the other terms which involve the variable z_k. Note that $\hat{X}_k(0) = 0$.

Substituting $\bar{X}_k(z_k),\ \bar{Y}_k(z_k)$ into (28) and premultiplying it by $\bar{X}_k^{-1}(0)$ give the result

$$X_k(z_k)D_p(0,\dots,z_k,\dots,0) + Y_k(z_k)N_p(0,\dots,z_k,\dots,0) = \Phi_k(z_k) \tag{34}$$

with

$$\begin{aligned} X_k(z_k) &= \bar{X}_k^{-1}(0)\bar{X}_k(z_k) = \bar{X}_k^{-1}(0)\hat{X}_k(z_k) + I \\ &\triangleq X_k'(z_k) + I && (35) \\ Y_k(z_k) &= \bar{X}_k^{-1}(0)\bar{Y}_k(z_k) && (36) \\ \Phi_k(z_k) &= \bar{X}_k^{-1}(0)\tilde{\Phi}_k(z_k) && (37) \end{aligned}$$

Note that $X_k'(0) = 0,\ Y_k(0) = 0$, and $\det \Phi_k(z_k) \neq 0,\ \forall z_k \in \bar{U}$.

Step 4. We now can construct a solution to the nD polynomial matrix equation

$$\begin{aligned} &X(z_1,\dots,z_n)D_p(z_1,\dots,z_n) + Y(z_1,\dots,z_n)N_p(z_1,\dots,z_n) \\ &\quad = \Phi(z_1,\dots,z_n) && (38) \end{aligned}$$

with

$$X(z_1,\dots,z_n) = \sum_{k=1}^{n} X_k'(z_k) + I \tag{39}$$

$$Y(z_1,\dots,z_n) = \sum_{k=1}^{n} Y_k(z_k) \tag{40}$$

and

$$\Phi(z_1,\dots,z_n) = \Big\{\sum_{k=1}^{n} X'_k(z_k)+I\Big\}D_p(z_1,\dots,z_n) + \Big\{\sum_{k=1}^{n} Y_k(z_k)\Big\}N_p(z_1,\dots,z_n) \tag{41}$$

Note that $\det\Phi(0,\dots,z_k,\dots,0)=\det\Phi_k(z_k)\neq 0,\ \forall z_k\in\bar{U},\ k=1,\dots,n$, i.e. $\det\Phi\in\boldsymbol{J}$. Therefore the solution to (25) is of the form

$$W=\Phi^{-1}X, \quad V=\Phi^{-1}Y \tag{42}$$

Using the right and left coprime MFD's on $\boldsymbol{H}$, a doubly coprime MFD relation on $\boldsymbol{H}$ can also be given.

Theorem 5. (Xu *et al.*, 1994b) *Suppose $P\in\boldsymbol{M}(\boldsymbol{G})$, and let $N_pD_p^{-1}$, $\tilde{D}_p^{-1}\tilde{N}_p$ be any right and left coprime MFD of P on $\boldsymbol{H}$, respectively. Then there exist $W,V,\tilde{W},\tilde{V}\in\boldsymbol{M}(\boldsymbol{H})$ such that*

$$\begin{bmatrix} W & V \\ -\tilde{N}_p & \tilde{D}_p \end{bmatrix}\begin{bmatrix} D_p & -\tilde{V} \\ N_p & \tilde{W} \end{bmatrix}=\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{43}$$

Based on the above results, the output feedback practical stabilization problem of nD systems can easily be solved (Xu *et al.*, 1994b).

Consider the MIMO (multi-input multi-output) nD feedback system shown in Fig. 1, where $N_pD_p^{-1}$, with $N_p,\ D_p\in\boldsymbol{M}(\mathbb{R}[z_1,\ \dots,z_n])$, is a right MFD on $\boldsymbol{H}$ for the plant $P\in\boldsymbol{M}(\boldsymbol{G})$, and $D_c^{-1}[N_{c1}\ N_{c2}]$, with $D_c,N_{c1},N_{c2}\in\boldsymbol{M}(\boldsymbol{H})$, is a left MFD on $\boldsymbol{H}$ for the controller $C\in\boldsymbol{M}(\boldsymbol{G})$. Let $y=[y_1\ y_2]^T$, $u=[u_1\ u_2\ u_3]^T$. It is then easy to see (Vidyasagar, 1985)

$$y=H_{yu}u \tag{44}$$

where

$$H_{yu}=\begin{bmatrix} D_p\Delta^{-1}N_{c1} & -I+D_p\Delta^{-1}D_c & -D_p\Delta^{-1}N_{c2} \\ N_p\Delta^{-1}N_{c1} & N_p\Delta^{-1}D_c & -N_p\Delta^{-1}N_{c2} \end{bmatrix} \tag{45}$$

and

$$\Delta=N_{c2}N_p+D_cD_p \tag{46}$$

If $\det\Delta\neq 0$ and $H_{yu}\in\boldsymbol{M}(\boldsymbol{H})$, we say that the nD feedback system of Fig. 1 is practically stable, P is practically stabilizable, and further C is a practically stabilizing compensator for P. It has been shown in (Xu *et al.*, 1994b) that the nD feedback system of Fig. 1 is practically stable if and only if Δ is $\boldsymbol{H}$-unimodular. Based on this result, we obtain the following theorem.

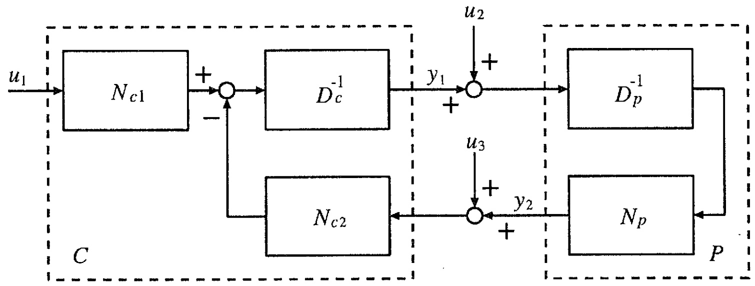

Fig. 1. An nD feedback control system.

Theorem 6. (Xu *et al.*, 1994b) *An nD plant $P = N_pD_p^{-1} \in \boldsymbol{M}(\boldsymbol{G})$ is practically stabilizable if and only if D_p and N_p are right coprime on $\boldsymbol{H}$.*

Further, the parametrization of all nD (ouput feedback) practically stabilizing compensators can be given as follows.

Theorem 7. (Xu *et al.*, 1994b) *Suppose that $N_pD_p^{-1}$ and $\tilde{D}_p^{-1}\tilde{N}_p$ are respectively any right and left coprime MFD on $\boldsymbol{H}$ for a given plant $P \in \boldsymbol{M}(\boldsymbol{G})$, and that W, $V \in \boldsymbol{M}(\boldsymbol{H})$ satisfy $WD_p + VN_p = I$. Then the set of all practically stabilizing compensators of P is given by*

$$C \in \Big\{(W + S\tilde{N}_p)^{-1}[Q \ \ V - S\tilde{D}_p] \;\Big|\; Q, S \in \boldsymbol{M}(\boldsymbol{H}),\ \det(W + S\tilde{N}_p) \in \boldsymbol{I}\Big\} \tag{47}$$

and the set of all possible practically-stable transfer matrices is in the form

$$\begin{bmatrix} D_pQ & D_p(W + S\tilde{N}_p) - I & -D_p(V - S\tilde{D}_p) \\ N_pQ & N_p(W + S\tilde{N}_p) & -N_p(V - S\tilde{D}_p) \end{bmatrix} \tag{48}$$

3.2. State-Space Approach

First, we see some basic properties in the practical sense of (Agathoklis and Bruton, 1983) for an nD system described by the Roesser state-space model. Consider the linear nD system (10) with the boundary conditions

$$\mathcal{X}_0 = \Big\{\boldsymbol{x}_j(i_1, \ldots, 0_j, \ldots, i_n) \;\Big|\; i_k \in \mathbb{Z}_+, k, j = 1, \ldots, n\Big\} \tag{49}$$

The practical controllability (reachability) and the practical observability are respectively defined as follows (Xu *et al.*, 1996b).

Definition 4. The nD system (10) is said to be *practically controllable* (equivalently, *practically reachable*) if and only if, for all $p = 1, \ldots, n$, there exists $t_p \geq 0$ such that for any finite $i_k \geq 0$, $k = 1, \ldots, n$, $k \neq p$, a local state $\boldsymbol{x}_p(i_1, \ldots, t_p, \ldots, i_n)$

can be reached from $\bar{\mathcal{X}}_0 = 0$ by using the input signal sequence $\{u(i_1, \ldots, i_p, \ldots, i_n) \mid 0 \le i_p < t_p\}$.

Definition 5. The nD system (10) is said to be *practically observable* if and only if, for all $p = 1, \ldots, n$, there exists $s_p > 0$ such that whenever $u = 0$ and $\bar{\mathcal{X}}_0 = 0$ except $x_p(i_1, \ldots, 0_p, \ldots, i_n) \neq 0$ for any finite $i_k \geq 0$, $k = 1, \ldots, n$, $k \neq p$, $y(i_1, \ldots, s_p, \ldots, i_n)$ is not the same as when $x_p(i_1, \ldots, 0_p, \ldots, i_n) = 0$.

The following theorems show the corresponding necessary and sufficient conditions for practical controllability and practical observability (Xu *et al.*, 1996b).

Theorem 8. *The nD system (10) is practically controllable if and only if the pairs* (A_{ii}, B_i) *are controllable in the 1D sense for all* $i = 1, \ldots, n$.

Theorem 9. *The nD system (10) is practically observable if and only if the pair* (A_{ii}, C_i) *is observable in the 1D sense for all* $i = 1, \ldots, n$.

By comparing the results of Theorems 8 and 9 with those of (Ciftcibasi and Yüksel, 1983; Eising, 1979; Kurek, 1987), we see that practical controllability is equivalent to r(real)-controllability for any n and implies modal controllability for $n = 2$, while practical observability implies modal observability for $n = 2$.

From 1D system theory and the results of Theorems 8 and 9, it is easy to see the duality between the practical controllability and the practical observability.

We call the nD system (10) practically internally (or asymptotically) stabilizable, or simply practically stabilizable, if there exists a local state feedback such that the resultant feedback system is practically internally stable. Further, we call the system practically detectable if there exists an asymptotic observer for the local state $x(i_1, \ldots, i_n)$ whose estimate error vanishes as $i_1 + \cdots + i_n \to \infty$ but $(i_1, \ldots, i_n) \in \mathbb{Z}_+^{-n}$. Consider now the practical stabilizability and the practical detectability of (10).

Define the local state feedback:

$$u(i_1, \ldots, i_n) = v(i_1, \ldots, i_n) - Kx(i_1, \ldots, i_n) \tag{50}$$

where $v \in \mathbb{R}^m$ is a new input vector and $K = [K_1\ K_2 \cdots K_n]$, $K_i, i = 1, \ldots, n$, are real feedback gain matrices of appropriate dimensions. Substituting (50) into (10) yields the closed-loop system:

$$x'(i_1, \cdots, i_n) = A_c\, x(i_1, \cdots, i_n) + Bv(i_1, \ldots, i_n) \tag{51}$$

where $A_c \triangleq A - BK$. Then the closed-loop characteristic polynomial has the form

$$\rho_c(z_1, \ldots, z_n) = \det(I - ZA_c) \tag{52}$$

The following theorem gives necessary and sufficient conditions for (10) to be practically stabilizable, or equivalently, for the closed-loop nD system (51) to be practically stable.

Theorem 10. (Xu *et al.*, 1996b) *The following conditions are equivalent:*

(i) *the nD system (10) is practically stabilizable;*

(ii) *all the pairs* $(A_{kk},\ B_k)$, $k = 1, \dots, n$, *are stabilizable in the* 1D *sense, i.e.*

$$\operatorname{rank}[I_{n_k} - z_k A_{kk} \;\; z_k B_k] = n_k \quad \forall z_k \in \bar{U},\ k = 1, \dots, n \tag{53}$$

(iii) $(I - ZA)$ *and* ZB *are left coprime on* $\boldsymbol{H}$, *i.e. there exist* $X(z_1, \dots, z_n)$, $Y(z_1, \dots, z_n) \in \boldsymbol{M}(\boldsymbol{H})$ *such that*

$$(I - ZA)X(z_1, \dots, z_n) + (ZB)Y(z_1, \dots, z_n) = I \tag{54}$$

Theorem 10 reveals that the practical stabilization problem of the nD system (10) by the state feedback (50) is equivalent to the stabilization problems of n 1D systems described by $(A_{ii},\ B_i)$, $i = 1, \dots, n$. Therefore, the feedback gain matrices K_i, $i = 1, \dots, n$, can be determined by well-known 1D methods.

For the characteristic polynomial $\rho(z_1, \dots, z_n)$ of (10), if we define the zeros of $\rho(0, \dots, z_k, \dots, 0)$, $k = 1, \dots, n$, as the practical zeros of $\rho(z_1, \dots, z_n)$, or equivalently the practical poles of (10), and the problem of the practical pole assignment of (10) as to locate its closed-loop practical poles $\{z_k \mid \rho_c(0, \dots, z_k, \dots, 0) = 0, k = 1, \dots, n\}$, then it is clear that these practical poles are arbitrarily assignable if and only if the system, or simply the pair (A, B), is practically controllable.

To realize the state feedback we may need to construct an observer to estimate the states if they cannot be completely measured. Consider an nD observer described by

$$\begin{aligned} \hat{\boldsymbol{x}}'(i_1, \dots, i_n) = {} & A\hat{\boldsymbol{x}}(i_1, \dots, i_n) + B\boldsymbol{u}(i_1, \dots, i_n) \\ & + F\{\boldsymbol{y}(i_1, \dots, i_n) - \hat{\boldsymbol{y}}(i_1, \dots, i_n)\} \end{aligned} \tag{55a}$$

$$\hat{\boldsymbol{y}}(i_1, \dots, i_n) = C\hat{\boldsymbol{x}}(i_1, \dots, i_n) \tag{55b}$$

where $\hat{\boldsymbol{x}}(i_1, \dots, i_n) \in \mathbb{R}^{\bar{n}}$ is the local state vector, $\hat{\boldsymbol{y}}(i_1, \dots, i_n) \in \mathbb{R}^l$ is the output vector of the observer, and $F = [F_1^T \cdots F_n^T]^T$, $F_i, i = 1, \dots, n$, are real matrices with suitable dimensions. To estimate the local state $\boldsymbol{x}(i_1, \dots, i_n)$, we should choose F such that the error

$$\boldsymbol{e}(i_1, \dots, i_n) = \boldsymbol{x}(i_1, \dots, i_n) - \hat{\boldsymbol{x}}(i_1, \dots, i_n) \tag{56}$$

can be properly controlled. From (10) and (55), the error $\boldsymbol{e}(i_1, \dots, i_n)$ obeys the equation

$$\boldsymbol{e}'(i_1, \dots, i_n) = (A - FC)\boldsymbol{e}(i_1, \dots, i_n) \tag{57}$$

According to the results of Theorem 3, if there exists some F such that (57) is practically asymptotically stable, the estimate error $\boldsymbol{e}(i_1, \dots, i_n) \to 0$ as $i_1 + \cdots + i_n \to \infty$ but $(i_1, \dots, i_n) \in \mathbb{Z}_+^{-n}$, i.e. the nD system (10) will be practically detectable.

Theorem 11. (Xu *et al.*, 1996b) *The following conditions are equivalent:*

(i) *the* nD *system (10) is practically detectable;*

(ii) *all the pairs* (A_{kk}, C_k), $k = 1, \dots, n$, *are detectable in the* 1D *sense, i.e.*

$$\operatorname{rank}\begin{bmatrix} C_k \\ I_{n_k} - z_k A_{kk} \end{bmatrix} = n_k \quad \forall z_k \in \bar{U},\ k = 1, \dots, n \tag{58}$$

(iii) $(I - ZA)$ *and* C *are right coprime on* $\boldsymbol{H}$, *i.e. there exist* $W(z_1, \dots, z_n)$, $V(z_1, \dots, z_n) \in \boldsymbol{M}(\boldsymbol{H})$ *such that*

$$W(z_1, \dots, z_n)(I - ZA) + V(z_1, \dots, z_n)C = I \tag{59}$$

The results of Theorem 11 show that an observer for (10) in the practical sense can be constructed by using 1D approaches. Moreover, it is also obvious that the practical zeros of $\det(I - Z(A - FC))$ can be arbitrarily assigned if and only if the nD system (10), or simply the pair (A, C), is practically observable.

The observer of (55) can be combined with the feedback controller of (50). Then, from (56), (57), (55) and (10), the state equation for the overall system is obtained as

$$\begin{bmatrix} \boldsymbol{x}' \\ \boldsymbol{e}' \end{bmatrix} = \begin{bmatrix} (A - BK) & BK \\ 0 & (A - FC) \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{e} \end{bmatrix} \triangleq \bar{A} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{e} \end{bmatrix} \tag{60}$$

The characteristic polynomial of this system is easily seen to be

$$\begin{aligned} \det(I - \bar{Z}\bar{A}) &= \det\left(I - \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix} \begin{bmatrix} A - BK & BK \\ 0 & A - FC \end{bmatrix}\right) \\ &= \det(I - Z(A - BK)) \det(I - Z(A - FC)) \end{aligned} \tag{61}$$

Therefore the practical stability of the combined system is guaranteed and the controller and the observer can be designed independently.

3.3. A Connection Between the State-Space Representation and the Doubly Coprime MFD on $\boldsymbol{H}$

A matrix $A \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$ is said to be practically stable if $\det(I - ZA) \in \boldsymbol{J}$. By Theorem 10, there exists $K \in \mathbb{R}^{m \times \tilde{n}}$ such that $A - BK$ is practically stable whenever $(A, B) \in \mathbb{R}^{\tilde{n} \times \tilde{n}} \times \mathbb{R}^{\tilde{n} \times m}$ is practically stabilizable. Moreover, by Theorem 11, there exists $F \in \mathbb{R}^{\tilde{n} \times l}$ such that $A - FC$ is practically stable whenever $(A, C) \in \mathbb{R}^{\tilde{n} \times \tilde{n}} \times \mathbb{R}^{l \times \tilde{n}}$ is practically detectable. A connection between the state-space representation and the doubly coprime MFD on $\boldsymbol{H}$ is characterized by the following theorem (Xu *et al.*, 1996b).

Theorem 12. *Suppose that the transfer matrix of the nD system (10) is*

$$G(z_1,\dots,z_n) = C(I - ZA)^{-1}ZB \in \boldsymbol{M}(\boldsymbol{G}) \tag{62}$$

where (A, B) is practically stabilizable and $(A,\ C)$ is practically detectable. Choose K and F such that $A - BK$ and $A - FC$ are practically stable. Define

$$N_p(z_1,\dots,z_n) = C\Big[I - Z(A - BK)\Big]^{-1} ZB \tag{63}$$

$$D_p(z_1,\dots,z_n) = I - K\Big[I - Z(A - BK)\Big]^{-1} ZB \tag{64}$$

$$V(z_1,\dots,z_n) = K\Big[I - Z(A - FC)\Big]^{-1} ZF \tag{65}$$

$$W(z_1,\dots,z_n) = I + K\Big[I - Z(A - FC)\Big]^{-1} ZB \tag{66}$$

$$\tilde{D}_p(z_1,\dots,z_n) = I - C\Big[I - Z(A - FC)\Big]^{-1} ZF \tag{67}$$

$$\tilde{N}_p(z_1,\dots,z_n) = C\Big[I - Z(A - FC)\Big]^{-1} ZB \tag{68}$$

$$\tilde{W}(z_1,\dots,z_n) = I + C\Big[I - Z(A - BK)\Big]^{-1} ZF \tag{69}$$

$$\tilde{V}(z_1,\dots,z_n) = K\Big[I - Z(A - BK)\Big]^{-1} ZF \tag{70}$$

Then

1) *all matrices defined in (63)–(70) are practically stable;*

2) $\tilde{D}(z_1,\dots,z_n)$ *and* $D(z_1,\dots,z_n)$ *are nonsingular;*

3) *we have*

$$\begin{aligned} G(z_1,\dots,z_n) &= N_p(z_1,\dots,z_n)D_p^{-1}(z_1,\dots,z_n) \\ &= \tilde{D}_p^{-1}(z_1,\dots,z_n)\tilde{N}_p(z_1,\dots,z_n) \end{aligned} \tag{71}$$

4)
$$\begin{bmatrix} W & V \\ -\tilde{N}_p & \tilde{D}_p \end{bmatrix} \begin{bmatrix} D_p & -\tilde{V} \\ N_p & \tilde{W} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{72}$$

Similarly as in (Nett *et al.*, 1984), the results of Theorem 12 can be easily generalized to the case of $G(z_1,\dots,z_n) = C(I - ZA)^{-1}ZB + D \in \boldsymbol{M}(\boldsymbol{G})$.

4. Practical Tracking Control and Its Applications

In this section, we investigate the nD practical tracking control problem, i.e. the tracking control problem for nD systems in the practical sense of (Agathoklis and Bruton, 1983). The proposed solution method is based on the concept of skew primeness on $\boldsymbol{H}$ and a solution algorithm for the skew prime equation, which can be regarded as a kind of generalization of the results of the 1D case (Wolovich, 1978). It will also be shown that the results can be applied, as a unified and general design approach, to iterative learning control systems and linear multipass processes under much less restrictive convergence or stability conditions and without requiring any *a priori* restriction on the structure of controllers.

4.1. Skew Primeness of Matrices over $\boldsymbol{H}$

Definition 6. Consider matrices $D, N \in \boldsymbol{M}(\boldsymbol{H})$. D and N are said to be *(externally) skew prime* on $\boldsymbol{H}$ if and only if there exist $X, Y \in \boldsymbol{M}(\boldsymbol{H})$ such that

$$DX + YN = I \tag{73}$$

Let a and b be common denominators for the entries of D and N, respectively. It is obvious that $a, b \in \boldsymbol{J}$ for any $D, N \in \boldsymbol{M}(\boldsymbol{H})$. It is also evident that (73) can always be rewritten in the form of $D'X' + Y'N' = I$, where $D' = Da$, $N' = bN$, $X' = a^{-1}X$ and $Y' = Yb^{-1}$, and clearly D', $N' \in \boldsymbol{M}(\mathbb{R}[z_1, \ldots, z_n]) \subset \boldsymbol{M}(\boldsymbol{H})$ and X', $Y' \in \boldsymbol{M}(\boldsymbol{H})$. This fact shows that, when studying the skew primeness of D, N on $\boldsymbol{H}$, one can limit D and N to be polynomial matrices without loss of generality. In what follows, therefore, we assume that $D, N \in \boldsymbol{M}(\mathbb{R}[z_1, \ldots, z_n])$.

Theorem 13. *Consider D, $N \in \boldsymbol{M}(\mathbb{R}[z_1, \ldots, z_n])$ and let D be nonsingular. Then D and N are skew prime on $\boldsymbol{H}$ if there exist $\bar{D}, \bar{N} \in \boldsymbol{M}(\boldsymbol{H})$ such that*

$$ND = \bar{D}\bar{N} \tag{74}$$

where D, $\bar{N}$ are right coprime and N, $\bar{D}$ are left coprime on $\boldsymbol{H}$. When $n \leq 2$, the condition is also necessary.

Proof. First show the sufficiency. If D, $\bar{N}$ are right coprime and N, $\bar{D}$ are left coprime on $\boldsymbol{H}$, and (74) holds, then according to Theorem 5 there exist X_1, X_2, X_3, $X_4 \in \boldsymbol{M}(\boldsymbol{H})$ such that the doubly coprime MFD relation

$$\begin{bmatrix} X_1 & X_2 \\ -\bar{D} & N \end{bmatrix} \begin{bmatrix} \bar{N} & -X_3 \\ D & X_4 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{75}$$

takes place. Obviously, (75) is equivalent to

$$\begin{bmatrix} \bar{N} & -X_3 \\ D & X_4 \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ -\bar{D} & N \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{76}$$

which gives a solution to the skew prime equation (73) as

$$DX_2 + X_4N = I \tag{77}$$

Now consider the necessity for the case $n \leq 2$. If D and N are skew prime on $\boldsymbol{H}$, D and Y must be left coprime on $\boldsymbol{H}$ (and X and N must be right coprime on $\boldsymbol{H}$) in view of (73). For the left MFD $D^{-1}Y$, we can always find the representation

$$D^{-1}Y = D''^{-1}Y'' = \bar{Y}\bar{D}^{-1} \tag{78}$$

where D'', Y'', $\bar{Y}$ and $\bar{D} \in \boldsymbol{M}(\mathbb{R}[z_1, z_2])$, and D'', Y'' are left factor coprime and $\bar{Y}$, $\bar{D}$ are right factor coprime (Xu *et al.*, 1990).

By using the Cauchy-Binet Theorem, it is easy to see that, for $[D' \;\; Y'] = R[D'' \;\; Y'']$, D', Y' are left coprime on $\boldsymbol{H}$ if and only if $\det R \in \boldsymbol{J}$ and D'', Y'' are left coprime on $\boldsymbol{H}$.

Denote by $\mathcal{I}_1$ and $\bar{\mathcal{I}}_1$, respectively, the ideal generated by the all maximal order minors of $[D''(z_1, 0) \; Y''(z_1, 0)]$ and $[\bar{D}^T(z_1, 0) \; \bar{Y}^T(z_1, 0)]^T$. In the same way, define $\mathcal{I}_2$ and $\bar{\mathcal{I}}_2$ for the case when $z_1 = 0$ but z_2 is free. Then by the results of (Bisiacco *et al.*, 1989), the algebraic variety $\mathcal{V}(\mathcal{I}_k)$ is identical with $\mathcal{V}(\bar{\mathcal{I}}_k)$, $k = 1, 2$. Therefore, if D'' and Y'' are left coprime on $\boldsymbol{H}$, i.e. $\mathcal{V}(\mathcal{I}_k) \cap \bar{U} = \emptyset$ $(k = 1, 2)$, $\bar{D}$ and $\bar{Y}$ must be right coprime on $\boldsymbol{H}$.

In view of Theorem 4, we can find W, $V \in \boldsymbol{M}(\boldsymbol{H})$ such that

$$W\bar{Y} + V\bar{D} = I \tag{79}$$

From (73), (78) and (79), we have

$$\begin{bmatrix} D & Y \\ -W & V \end{bmatrix} \begin{bmatrix} X & -\bar{Y} \\ N & \bar{D} \end{bmatrix} = \begin{bmatrix} I & 0 \\ -WX + VN & I \end{bmatrix} \tag{80}$$

Premultiplying (80) by

$$\begin{bmatrix} I & 0 \\ WX - VN & I \end{bmatrix} \tag{81}$$

yields

$$\begin{bmatrix} D & Y \\ -\bar{N} & \bar{X} \end{bmatrix} \begin{bmatrix} X & -\bar{Y} \\ N & \bar{D} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{82}$$

where $\bar{N} = W - (WX - VN)D$, $\bar{X} = V + (WX - VN)Y$.

The identity (82) obviously implies

$$\begin{bmatrix} X & -\bar{Y} \\ N & \bar{D} \end{bmatrix} \begin{bmatrix} D & Y \\ -\bar{N} & \bar{X} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{83}$$

Therefore we have

$$ND = \bar{D}\bar{N}$$

with D and $\bar{N}$ right coprime and N and $\bar{D}$ left coprime on $\boldsymbol{H}$. This completes the proof. ■

In view of Theorem 13, a solution to the skew prime equation (73) will be directly obtained if $\bar{D}$ and $\bar{N}$ satisfying (74) can be found. For brevity, we only consider the 2D case here and show a constructive procedure for the determination of $\bar{D}$ and $\bar{N}$ under the assumptions that $D,\ N \in \boldsymbol{M}(\mathbb{R}[z_1, z_2])$ and D is square and nonsingular.

As shown in (Wolovich, 1978; Xu *et al.*, 1990), our attention can be restricted, without loss of generality, to the case where D and N are square matrices of equal order. In what follows, therefore, let $D,\ N \in \mathbb{R}^{p\times\times p}[z_1, z_2]$.

Under these assumptions, we can always find $\hat{D}$ and $\hat{N}$ such that

$$ND = \hat{D}\hat{N} \tag{84}$$

with $\det \hat{D} = \det D$ in the following way (Xu *et al.*, 1990). Denote by D^+ the adjoint of D and let $\Delta = \det D$. Then we have $ND/\Delta = N[D^+]^{-1}$ which can be represented as

$$ND/\Delta = N[D^+]^{-1} = N_r D_r^{-1} = D_l^{-1} N_l \tag{85}$$

where N_r, D_r are right factor coprime and D_l, N_l left factor coprime. Let E be any greatest common right factor of N and D^+. Since D_l and N_l are left factor coprime, we have (Morf *et al.*, 1977)

$$\det D_l \det E = \det D^+ = \Delta^{p-1} \tag{86}$$

Multiplying both the sides of (85) by Δ gives

$$ND = \Delta D_l^{-1} N_l = \bar{D}\bar{N} \tag{87}$$

where $\bar{D} = \Delta D_l^{-1}$, $\bar{N} = N_l$.

Since D_l and N_l are left factor coprime and ND is a 2D polynomial matrix, $\bar{D}$ must also be a 2D polynomial matrix (Morf *et al.*, 1977). Moreover, since $\det \bar{D} = \det E \cdot \Delta$, it is possible, by the general factorization theorem (Morf *et al.*, 1977), to factorize $\bar{D}$ as $\bar{D} = \hat{D}\hat{E}$ such that $\det \hat{D} = \Delta$, $\det \hat{E} = \det E$. If $\hat{N} = \hat{E}\bar{N}$, then we have the result of (84).

Now, the following theorem can be given in the same way as in (Xu *et al.*, 1990).

Theorem 14. *Consider a pair of square matrices* $D, N \in \boldsymbol{M}(\mathbb{R}[z_1, z_2])$ *of equal order with* D *nonsingular. Let* $ND = \hat{D}\hat{N}$ *with* $\det \hat{D} = \det D$*. Then* D *and* N *are skew prime on* $\boldsymbol{H}$ *if and only if* D *and* $\hat{N}$ *are right coprime on* $\boldsymbol{H}$*, or* N *and* $\hat{D}$ *are left coprime on* $\boldsymbol{H}$*.*

4.2. Practical Tracking Control

The objective here is to investigate the tracking problem of 2D systems in the practical sense of (Agathoklis and Bruton, 1983).

Consider a plant characterized by the input/output relation

$$\boldsymbol{y}(z_1, z_2) = A_p^{-1} B_p\, \boldsymbol{u}(z_1, z_2) + A_p^{-1} C_p \boldsymbol{y}_0(z_1, z_2) \tag{88}$$

where $A_p, B_p, C_p \in \boldsymbol{M}(\mathbb{R}[z_1, z_2])$; A_p is $\boldsymbol{G}$-unimodular; $\boldsymbol{u}$, $\boldsymbol{y}$ are vectors over $\mathbb{R}[z_1, z_2]$ corresponding to the input and output sequences, respectively; $\boldsymbol{y}_0$ is a vector over $\mathbb{R}[z_1, z_2]$ denoting arbitrary initial or boundary conditions of the considered plant. Let

$$A_p^{-1} B_p = \tilde{B}_p \tilde{A}_p^{-1} \tag{89}$$

with $\tilde{B}_p$ and $\tilde{A}_p$ right factor coprime.

Similarly, let the class of 2D reference signals $\boldsymbol{r}(z_1, z_2)$ be given by the equation

$$\boldsymbol{r}(z_1, z_2) = A_s^{-1} B_s \boldsymbol{r}_0(z_1, z_2) \tag{90}$$

where $A_s, B_s \in \boldsymbol{M}(\mathbb{R}[z_1, z_2])$, and let $\boldsymbol{r}_0$ correspond to the initial or boundary conditions of the reference generator.

A general linear 2D controller can be described by

$$\boldsymbol{u}(z_1, z_2) = A_c^{-1} B_{c1}\, \boldsymbol{r}(z_1, z_2) - A_c^{-1} B_{c2} \boldsymbol{y}(z_1, z_2) + A_c^{-1} C_c \boldsymbol{u}_0(z_1, z_2) \tag{91}$$

where A_c, B_{c2}, $B_{c1} \in \boldsymbol{M}(\boldsymbol{H})$, and $\boldsymbol{u}_0$ depends on the initial or boundary conditions of the controller.

The practical tracking problem to be considered can be stated as follows: Given A_p, B_p, A_s, find A_c, B_{c1} and B_{c2} for arbitrary $C_p \boldsymbol{y}_0$, $B_s \boldsymbol{r}_0$ and $C_c \boldsymbol{u}_0$ such that

- the resultant closed-loop feedback system

$$\begin{aligned} \boldsymbol{y}(z_1, z_2) &= \tilde{B}_p (A_c \tilde{A}_p + B_{c2} \tilde{B}_p)^{-1} B_{c1} \boldsymbol{r}(z_1, z_2) \\ &+ \Big[I - \tilde{B}_p (A_c \tilde{A}_p + B_{c2} \tilde{B}_p)^{-1} B_{c2}\Big] A_p^{-1} C_p \boldsymbol{y}_0(z_1, z_2) \\ &+ \tilde{B}_p (A_c \tilde{A}_p + B_{c2} \tilde{B}_p)^{-1} C_c \boldsymbol{u}_0(z_1, z_2) \end{aligned} \tag{92}$$

  is stable, and

- the output $\boldsymbol{y}$ asymptotically tracks the reference signal $\boldsymbol{r}$

in the practical sense of (Agathoklis and Bruton, 1983).

In view of the results of practical internal stability and the remarks on its relation to practical BIBO stability in Section 2.2, it is easy to see that the practical tracking problem is equivalent to the following one: Under the given conditions, find A_c, B_{c1}, B_{c2} such that the output $\boldsymbol{y}(z_1, z_2)$ of the closed-loop system and the tracking error $\boldsymbol{e}(z_1, z_2) = \boldsymbol{r}(z_1, z_2) - \boldsymbol{y}(z_1, z_2)$ are vectors with entries in $\boldsymbol{H}$, for any $C_p \boldsymbol{y}_0$, $B_s \boldsymbol{r}_0$ and $C_c \boldsymbol{u}_0$.

It follows from (90) and (92) that

$$\begin{aligned} \boldsymbol{e}(z_1, z_2) &= \Big[I - \tilde{B}_p (A_c \tilde{A}_p + B_{c2} \tilde{B}_p)^{-1} B_{c1}\Big] A_s^{-1} B_s\, \boldsymbol{r}_0(z_1, z_2) \\ &- \Big[I - \tilde{B}_p (A_c \tilde{A}_p + B_{c2} \tilde{B}_p)^{-1} B_{c2}\Big] A_p^{-1} C_p \boldsymbol{y}_0(z_1, z_2) \\ &- \tilde{B}_p (A_c \tilde{A}_p + B_{c2} \tilde{B}_p)^{-1} C_c \boldsymbol{u}_0(z_1, z_2) \end{aligned} \tag{93}$$

The following theorem gives a necessary and sufficient condition for the solution of the practical tracking problem.

Theorem 15. *Suppose that A_s and B_s are left coprime on $\boldsymbol{H}$. Then the necessary and sufficient condition for the practical tracking problem to be solvable is as follows:*

(a) *the plant (88) is practical stabilizable, i.e. A_p and B_p are left coprime on $\boldsymbol{H}$, and*

(b) *$\tilde{B}_p$ and A_s are skew prime on $\boldsymbol{H}$.*

Proof. First show the necessity. The necessity of (a) is clear from Theorem 6. When (a) holds, as in the proof of Theorem 13 we can show that $\tilde{A}_p$ and $\tilde{B}_p$ are right coprime on $\boldsymbol{H}$. Therefore, there exist some $A_c, B_{c2} \in \boldsymbol{M}(\boldsymbol{H})$ such that

$$A_c\tilde{A}_p + B_{c2}\tilde{B}_p = I \tag{94}$$

Letting $\boldsymbol{y}_0 = 0$, $\boldsymbol{u}_0 = 0$, it follows from (93) that

$$\boldsymbol{e}(z_1, z_2) = (I - \tilde{B}_pB_{c1})A_s^{-1}B_s\boldsymbol{r}_0(z_1, z_2) \tag{95}$$

In view of Theorem 3, for any bounded $\boldsymbol{r}_0$, $\boldsymbol{e}(z_1, z_2) \in \boldsymbol{M}(\boldsymbol{H})$ only if $(I - \tilde{B}_pB_{c1})A_s^{-1}B_s \in \boldsymbol{M}(\boldsymbol{H})$. Based on the condition that A_s and B_s are left coprime on $\boldsymbol{H}$, we can show the following relation in the same way as for the 1D case (Vidyasagar, 1985):

$$(I - \tilde{B}_pB_{c1})A_s^{-1}B_s \in \boldsymbol{M}(\boldsymbol{H}) \Leftrightarrow (I - \tilde{B}_pB_{c1})A_s^{-1} \in \boldsymbol{M}(\boldsymbol{H}) \tag{96}$$

Therefore, letting $(I - \tilde{B}_pB_{c1})A_s^{-1} \triangleq Q \in \boldsymbol{M}(\boldsymbol{H})$, we obtain

$$\tilde{B}_pB_{c1} + QA_s = I \tag{97}$$

which shows that $\tilde{B}_p$ and A_s are skew prime on $\boldsymbol{H}$.

The sufficiency can be shown as follows. When the condition of (a) is satisfied, $\tilde{A}_p$ and $\tilde{B}_p$ are also right coprime on $\boldsymbol{H}$ as mentioned above. According to Theorem 7, the practically stabilizing controller can be constructed by the solution A_c, B_{c2} of (94). In fact, by Theorems 5, we can get $V, W \in \boldsymbol{M}(\boldsymbol{H})$ such that the double MFD relation

$$\begin{bmatrix} A_c & B_{c2} \\ -B_p & A_p \end{bmatrix} \begin{bmatrix} \tilde{A}_p & -V \\ \tilde{B}_p & W \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{98}$$

holds. Using this result, it is easy to see that, for $\boldsymbol{r} = 0$, the closed-loop system (92) can be rewritten as

$$\boldsymbol{y}(z_1, z_2) = WC_p\boldsymbol{y}_0(z_1, z_2) + \tilde{B}_pC_c\boldsymbol{u}_0(z_1, z_2) \tag{99}$$

with WC_p, $\tilde{B}_pC_c \in \boldsymbol{M}(\boldsymbol{H})$. Due to Theorem 3, it is clear that the closed-loop system is practically stable for any bounded $\boldsymbol{y}_0$ and $\boldsymbol{u}_0$.

On the other hand, by the results of Theorems 13 and 14, if (b) holds true, B_{c1}, $Q \in \boldsymbol{M}(\boldsymbol{H})$ satisfying (97) can be obtained. Substituting this result and that of (98) into (93) yields

$$\boldsymbol{e}(z_1, z_2) = QB_s\boldsymbol{r}_0(z_1, z_2) - WC_p\boldsymbol{y}_0(z_1, z_2) - \tilde{B}_pC_c\,\boldsymbol{u}_0(z_1, z_2) \tag{100}$$

where QB_s, WC_p, $\tilde{B}_pC_c \in \boldsymbol{M}(\boldsymbol{H})$. Obviously, $\boldsymbol{e}(z_1, z_2)$ is practically stable.

The following design procedure for practical tracking control systems can now be stated:

Algorithm 2.

Step 1. For given A_p, B_p, verify their practical stabilizability according to Theorem 4. Find $\tilde{A}_p$ and $\tilde{B}_p \in \boldsymbol{M}(\mathbb{R}[z_1, z_2])$ which satisfy $A_p^{-1}D_p = \tilde{B}_p\tilde{A}_p^{-1}$ and are right factor coprime by the methods of, e.g. (Morf *et al.*, 1977; Guiver and Bose, 1982).

Step 2. Solve the Bezout equation (94) by Algorithm 1. Here, first find a particular solution X, $Y \in \boldsymbol{M}(\boldsymbol{H})$ such that $X\tilde{A}_p + Y\tilde{B}_p = I$, then construct the following general solution to (94):

$$A_c = X + SB_p \tag{101}$$

$$B_{c2} = Y - SA_p \tag{102}$$

where $S \in \boldsymbol{M}(\boldsymbol{H})$ is an arbitrary matrix satisfying $\det A_c(0,0) \neq 0$.

Step 3. Verify the solvability of the skew prime equation (97) and, if it is solvable, find the solution B_{c1}, $Q \in \boldsymbol{M}(\boldsymbol{H})$, by the methods given in Section 4.1.

4.3. Application to Discrete Linear Multipass Processes

Multipass processes constitute a class of dynamic systems characterized by a series of sweeps, or passes, through a set of dynamics defined over a finite time interval, with the output of previous passes contributing to the output of the new current pass. It has been observed that the effect from the outputs of previous passes may generate oscillations of the output increasing in amplitude from pass to pass (Rogers and Owens, 1992). This property makes attempts of direct utilization of techniques from conventional system theory almost always end in failure (see Rogers and Owens, 1992 and the references given therein).

Connections between linear multipass processes and 2D systems have been considered by several researchers (Boland and Owens, 1980; Gałkowski *et al.*, 1997; Rocha *et al.*, 1996; Rogers and Owens, 1992). In particular, it has been shown that 2D stability is equivalent to the so-called stability along the pass for multipass processes (Rocha *et al.*, 1996; Rogers and Owens, 1992). It should be noted, however, that the stability along the pass would in general be too restrictive for practical applications, since it is defined for the case where the fact that the pass length is finite is ignored. Moreover, it seems that no satisfactory design method is currently available for the control of multipass processes.

In what follows, we give some further results on the connection between multipass processes and 2D systems, and show how we can apply the results for practical tracking control to design a control system for multipass processes.

Consider the so-called non-unit memory discrete linear multipass processes described by the state-space model (Rogers and Owens, 1992)

$$\boldsymbol{x}_{k+1}(i+1) = A\boldsymbol{x}_{k+1}(i) + B\boldsymbol{u}_{k+1}(i) + \sum_{j=1}^{M} B_{j-1}\boldsymbol{y}_{k+1-j}(i) \tag{103a}$$

$$\boldsymbol{y}_{k+1}(i) = C\boldsymbol{x}_{k+1}(i) + D_0\boldsymbol{u}_{k+1}(i) + \sum_{j=1}^{M} D_j\boldsymbol{y}_{k-j}(i) \tag{103b}$$

$$0 \le i \le N, \quad \boldsymbol{x}_k(0) = d_k, \quad k \ge 0$$

where $\boldsymbol{u}_k(i) \in \mathbb{R}^l$, $\boldsymbol{y}_k(i) \in \mathbb{R}^m$, and $\boldsymbol{x}_k(i) \in \mathbb{R}^p$ are the input, output and state vectors on the k-th pass, respectively; A, B, C, B_j, D_j are constant matrices with suitable dimensions, and the pass length N is a fixed finite constant.

By defining $\boldsymbol{x}_k(i) = \boldsymbol{x}(i,k)$, $\boldsymbol{y}_k(i) = \boldsymbol{y}(i,k)$ and $\boldsymbol{u}_k(i) = \boldsymbol{u}(i,k)$ in (103) and applying the 2D z-transform, we can have the input/output relation (Rogers and Owens, 1992)

$$Y(z_1,z_2) = G(z_1,z_2)U(z_1,z_2) \tag{104}$$

where $G(z_1,z_2)$ is the $m \times l$ transfer matrix given by

$$G(z_1,z_2) = \Big(I_m - \sum_{j=1}^{M} G_j(z_1)z_2^j\Big)^{-1} G_0(z_1) \tag{105}$$

with

$$G_0(z_1) = C(I_p - z_1A)^{-1}z_1B + D_0 \tag{106}$$

$$G_j(z_1) = C(I_p - z_1A)^{-1}z_1B_{j-1} + D_j, \quad 1 \le j \le M \tag{107}$$

and z_1, z_2 can be regarded as backward shift operators.

The multipass process (103) is said to be asymptotically stable if and only if, for any given bounded initial output $\boldsymbol{y}_0(i), 0 \le i \le N$ and initial states d_k, $k \ge 0$, the sequence of outputs $\{\boldsymbol{y}_k\}$ converges to an equilibrium output $\boldsymbol{y}_\infty$ (Rogers and Owens, 1992). It has been shown in (Rogers and Owens, 1989), by using an abstract function analysis approach, that (103) is asymptotically stable if and only if

$$\rho(z_2) \triangleq \det\Big(I_m - \sum_{j=1}^{M} D_j z_2^j\Big) \ne 0 \quad \forall z_2 \in \bar{U} \tag{108}$$

It has been known (Xu *et al.*, 1994) that the non-unit memory multipass processes given by (103) can be simply represented as a 2D Fornasini-Marchesini model. In fact, let

$$\boldsymbol{\xi}(i,k) = \Big[\boldsymbol{x}_k^T(i),\ \boldsymbol{y}_{k-1}^T(i),\ \ldots,\ \boldsymbol{y}_{k-M}^T(i)\Big]^T \in \mathbb{R}^{p+mM} \tag{109}$$

It then follows immediately that

$$\begin{aligned}\boldsymbol{\xi}(i+1,k+1) = \tilde{A}_1\boldsymbol{\xi}(i,k+1) + \tilde{A}_2\boldsymbol{\xi}(i+1,k)\\ + \tilde{B}_1\boldsymbol{u}(i,k+1) + \tilde{B}_2\boldsymbol{u}(i+1,k)\end{aligned} \tag{110a}$$

$$\boldsymbol{y}(i,k) = \tilde{C}\boldsymbol{\xi}(i,k) + D_0\boldsymbol{u}(i,k) \tag{110b}$$

where

$$\tilde{A}_1 = \begin{bmatrix} A & B_0 & B_1 & \cdots & B_{M-1} \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ C & D_1 & \cdots & D_{M-1} & D_M \\ 0 & I & \cdots & 0 & 0 \\ \vdots & & \ddots & 0 & 0 \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}$$

$$\tilde{B}_1 = [B^T,\ 0,\ \ldots,\ 0]^T, \quad \tilde{B}_2 = [0,\ D_0,\ 0,\ \ldots, 0]^T, \quad \tilde{C} = [C,\ D_1,\ \ldots, D_M]$$

Though the manipulation is rather tedious, it is trivial to show that the transfer matrix $\tilde{G}(z_1, z_2)$ of (110) satisfies the relation

$$\begin{aligned}\tilde{G}(z_1,z_2) &= \tilde{C}(I_{p+mM} - z_1\tilde{A}_1 - z_2\tilde{A}_2)^{-1}(z_1\tilde{B}_1 + z_2\tilde{B}_2) + D_0 \\ &= \Big(I_m - \sum_{j=1}^{M} G_j(z_1)z_2^j\Big)^{-1} G_0(z_1) = G(z_1,z_2)\end{aligned} \tag{111}$$

In view of the structures of $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{B}_1$, $\tilde{B}_2$ and the results of (Fornasini and Marchesini, 1978; Kaczorek, 1985), it follows immediately that the 2D Fornasini-Marchesini model of (110) directly corresponds to a Roesser model which is in fact simply equivalent to the ones shown by Rocha *et al.* (1996) and Gałkowski *et al.* (1997). It should now be clear that the class of multipass processes is just a special case of standard 2D systems, and all the well-developed results of 2D system theory can be applied directly.

For $G(z_1, z_2)$ given in (105) or (111), it is trivial to see that

$$G(z_1, 0) = G_0(z_1) \tag{112}$$

$$G(0, z_2) = \operatorname{adj}\Big(I_m - \sum_{j=1}^{M} D_j z_2^j\Big)\frac{D_0}{\rho(z_2)} \tag{113}$$

We then have that $G(z_1, z_2)$ is practically stable if and only if $G(z_1, 0)$ and $G(0, z_2)$ are both 1D stable. In particular, the latter is equivalent to the condition (108). Therefore, the asymptotic stability for linear multipass processes may be viewed a special case of the practical (internal) stability. Further, by practical stability theory, if we also need to consider the stability as the pass length $N \to \infty$ but the iteration index k is finite, then the 1D stability condition of $G(z_1, 0)$ has to be satisfied. It is reasonable to consider such a situation since it has been observed (also see Kurek and Zaremba, 1993 for a similar situation) that, if the pass length N is large, there may exist the cases where the system output value is excessively large for large i and small iteration number k. The practical stability theory, therefore, not only shows some significant insights but also supplies a rather reasonable design requirement for linear multipass processes. In the following, we consider the control problem of linear multipass processes under the condition of practical stability. That is, consider the problem how to find an (output) feedback controller for a given multipass process such that the resultant closed-loop system is (asymptotically) stable and the output converges to a sequence of specified reference signals with finite pass length.

In fact, it is easy to formulate the above problem as the practical tracking problem stated in Section 4.2. A reference signal $r_k(i)$ for multipass processes with finite pass length N can be regarded as 2D signal $r(i,k)$ in the form

$$r(i,k) = \begin{cases} r_k(i) & (0 \le i \le N) \\ 0 & (i > N) \end{cases} \tag{114}$$

Applying the 2D z-transform to $r(i,k)$ yields

$$r(z_1,z_2) = \sum_{i=0}^{N}\sum_{k=0}^{\infty} r(i,k) z_2^k z_1^i \in \boldsymbol{M}\Big(\mathbb{R}(z_2)[z_1]\Big) \tag{115}$$

i.e. the entries of $r(z_1,z_2)$ are polynomials in z_1 having polynomial fractions in z_2 as coefficients. Therefore, we can always have

$$r(z_1,z_2) = A_s^{-1}(z_2) B_s(z_1,z_2) \tag{116}$$

where $A_s(z_2) \in \boldsymbol{M}(\mathbb{R}[z_2])$ and $B_s(z_1,z_2) \in \boldsymbol{M}(\mathbb{R}[z_1,z_2])$. Suppose that $A_s(z_2)$ and $B_s(z_1,z_2)$ are left coprime on $\boldsymbol{H}$, which means that $[A_s(z_2) \;\; B_s(0,z_2)]$ and $[A_s(0) \;\; B_s(z_1,0)]$ are full rank for any $z_1, z_2 \in \bar{U}$.

Letting $G(z_1,z_2) = A_p^{-1}(z_1,z_2)B_p(z_1,z_2)$, we see that the problem under consideration is equivalent to the 2D practical tracking problem with the reference signal specified by (116), so Algorithm 2 can be applied directly.

4.4. Application to Iterative Learning Control Systems

A discrete iterative learning control system (ILCS) for a given (1D) plant is demanded to repetitively track a specified output trajectory $y_d(i)$ on a finite discrete-time interval $i \in [1,N]$ with the system performance improved based on the evaluation of the performance in previous iterations. Some design approaches to ILCS have been proposed based on either artificial intelligence or conventional control system design principles. However, it has been indicated that the techniques involved for convergence analysis and learning system design are of an *ad hoc* type, and few of them show insights into the fundamental properties of ILCS, such as stability, learning convergence and so on (Geng and Jamshidi, 1990; Geng *et al.*, 1990). This is mainly due to the lack of mathematical model for the entire ILCS which could explicitly describe both the dynamics of the control system itself on the time interval and the behaviour of the learning process along the direction of iterations (Geng *et al.*, 1990).

Geng *et al.* (1990) and Kurek and Zaremba (1993) have recently shown that 2D system theory may offer a highly promising approach to ILCS by revealing the following facts. Namely, the two kinds of dynamics involved in ILCS can be easily characterized by a 2D model, and 2D stability theory provides a useful method for analysing the learning convergence and stability of ILCS. However, although Geng *et al.* (1990) proposed a general type of learning controller structure based on the 2D model and showed a convergence condition for the learning controller in terms of 2D stability, no constructive procedures for the test of the condition and the design of such a general learning controller were provided except for a very restricted particular case. In fact, it would be extremely difficult, if not impossible, to find such general procedures, since the convergence condition itself contains the undetermined parameters of the controller under design. Further, the feature that the length of the time interval for i is finite was not taken into account in (Geng *et al.*, 1990) and the conventional 2D stability was directly applied to the analysis of learning convergence, which is in fact too restrictive (Kurek and Zaremba, 1993).

The objective here is to show the possibility to design an ILCS, by the method proposed previously for 2D practical tracking systems, under less restrictive conditions of practical stability and without requiring *a priori* restriction on the controller structure.

Consider the 1D system given by

$$\boldsymbol{x}(i+1) = A\boldsymbol{x}(i) + B\boldsymbol{u}(i) \tag{117a}$$

$$\boldsymbol{y}(i) = C\boldsymbol{x}(i) + D\boldsymbol{u}(i) \tag{117b}$$

where $\boldsymbol{u}$, $\boldsymbol{y}$, $\boldsymbol{x}$ are the input, output and state vectors, respectively; A, B, C, D are real matrices with suitable dimensions.

The problem may now be formulated as follows: Given the system (117) with boundary condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$, and a reference output trajectory $\boldsymbol{y}_d(i)$ $(i = 1,\dots,N)$, find a learning control scheme such that the tracking error $\boldsymbol{y}(i) - \boldsymbol{y}_d(i)$ for all $i \in [1,N]$ converges asymptotically to zero along the direction of learning iterations.

Denote by j the iteration number, and by $\tilde{\boldsymbol{u}}(i,j)$, $\tilde{\boldsymbol{y}}(i,j)$, $\tilde{\boldsymbol{x}}(i,j)$ the values of the input, output and state vectors at the time point i of the j-th iteration, respectively. Then the plant (117) can be viewed as a 2D discrete system in the form

$$\tilde{\boldsymbol{x}}(i+1,j) = A\,\tilde{\boldsymbol{x}}(i,j) + B\tilde{\boldsymbol{u}}(i,j) \tag{118a}$$

$$\tilde{\boldsymbol{y}}(i,j) = C\,\tilde{\boldsymbol{x}}(i,j) + D\tilde{\boldsymbol{u}}(i,j) \tag{118b}$$

where the boundary conditions can be considered as $\tilde{\boldsymbol{x}}(i,0), 0 \le i \le N$ and $\tilde{\boldsymbol{x}}(0,j), j \ge 0$. Similarly, denote by $\tilde{\boldsymbol{y}}_d(i,j)$ the reference trajectory. Since the identical $\boldsymbol{y}_d(i)$ is used for all iterations, $\tilde{\boldsymbol{y}}_d(i,j)$ can be represented in the form

$$\tilde{\boldsymbol{y}}_d(i,k) = \begin{cases} \boldsymbol{y}_d(i) & (0 \le i \le N) \\ 0 & (i > N) \end{cases} \tag{119}$$

Further, define the tracking error $\tilde{e}(i,j)$ as

$$\tilde{e}(i,j)=\tilde{y}_d(i,j)-\tilde{y}(i,j) \tag{120}$$

Applying the 2D z-transform to (118) gives

$$\tilde{Y}(z_1,z_2)=\Big[C(I-z_1A)^{-1}z_1B+D\Big]\tilde{U}(z_1,z_2)+C(I-z_1A)^{-1}\sum_{j=0}^{\infty}\tilde{x}(0,j)z_2^j \tag{121}$$

Find $D_p(z_1), C_p(z_1)\in \boldsymbol{M}(\mathbb{R}[z_1])$ which satisfy $C(I-z_1A)^{-1}=D_p^{-1}(z_1)C_p(z_1)$, $\det D_p(0)\neq 0$ and are left coprime. Then (121) can be rewritten as

$$\tilde{Y}(z_1,z_2)=D_p^{-1}(z_1)N_p(z_1)\tilde{U}(z_1,z_2)+D_p^{-1}(z_1)C_p(z_1)\tilde{X}_0(z_2) \tag{122}$$

where $N_p(z_1)=C_p(z_1)z_1B+D_p(z_1)D$ and $\tilde{X}_0(z_2)=\sum_{j=0}^{\infty}\tilde{x}(0,j)z_2^j$.

The 2D z-transform of (119) is given by

$$\begin{aligned}\tilde{Y}_d(z_1,z_2) &= (1-z_2)^{-1}\sum_{i=0}^{N}\tilde{y}_d(i,0)z_1^i=(1-z_2)^{-1}\sum_{i=0}^{N}y_d(i)z_1^i\\ &\triangleq (1-z_2)^{-1}Y_d(z_1)\end{aligned} \tag{123}$$

Obviously, the entries of $Y_d(z_1)$ are 1D polynomials of N-th degree in z_1.

Considering that a general learning controller may depend on the information of the input and the tracking error in both the present iteration and a finite number of previous iterations, we use a general linear controller structure as follows:

$$\begin{aligned}\tilde{U}(z_1,z_2) &= D_c^{-1}(z_1,z_2)N_{c1}(z_1,z_2)\tilde{Y}_d(z_1,z_2)\\ &\quad - D_c^{-1}(z_1,z_2)N_{c2}(z_1,z_2)\tilde{Y}(z_1,z_2)\end{aligned} \tag{124}$$

where $D_c, N_{c1}, N_{c2}\in \boldsymbol{M}(\boldsymbol{H})$.

Similarly as in previous discussions for multipass processes, besides the convergence of ILCS in the direction of learning iterations, it is also reasonable to consider the stability of the system in the time direction $(i\to\infty)$ at a fixed iteration. It should be clear that, in this case, the problem will be equivalent to the 2D practical tracking problem.

In view of Algorithm 2, D_c and N_{c2} can be determined by solving the equation

$$D_c(z_1,z_2)\tilde{D}_p(z_1)+N_{c2}(z_1,z_2)\tilde{N}_p(z_1)=I \tag{125}$$

where $\tilde{N}_p(z_1)\tilde{D}_p^{-1}(z_1)=D_p^{-1}(z_1)N_p(z_1)$, $\tilde{D}_p(z_1), \tilde{N}_p(z_1)\in \boldsymbol{M}(\mathbb{R}[z_1])$, $\det\tilde{D}_p(0)\neq 0$. If the 1D plant (117) is stabilizable, there exist $X_1, Y_1, \Phi_1\in \boldsymbol{M}(\mathbb{R}[z_1])$ such that

$$X_1(z_1)\tilde{D}_p(z_1)+Y_1(z_1)\tilde{N}_p(z_1)=\Phi_1(z_1) \tag{126}$$

with $X_1(0)=I, Y_1(0)=0, \det\Phi_1(z_1)\neq 0, \forall z_1\in\bar{U}$.

It is clear that $\Phi_1(0) = \tilde{D}_p(0)$. On the other hand, since $\tilde{D}_p(0)$ and $\tilde{N}_p(0)$ are real matrices, the equation

$$X_2(z_2)\tilde{D}_p(0) + Y_2(z_2)\tilde{N}_p(0) = \Phi_2(z_2) \tag{127}$$

will always be solvable for the solution $X_2(z_2) = I, Y_2(z_2) = 0, \Phi_2(z_2) = \tilde{D}_p(0)$ and $\det \Phi_2(z_2) = \det D_p(0) \neq 0$. Therefore, the following general solution to (125) can be obtained.

$$D_c(z_1, z_2) = \Phi_1^{-1}(z_1)X_1(z_1) + S(z_1, z_2)N_p(z_1) \tag{128a}$$

$$N_{c2}(z_1, z_2) = \Phi_1^{-1}(z_1)Y_1(z_1) - S(z_1, z_2)D_p(z_1) \tag{128b}$$

where $S(z_1, z_2) \in \boldsymbol{M}(\boldsymbol{H})$ is an arbitrary matrix satisfying $\det D_c(0,0) \neq 0$.

Further, due to Theorem 15 and (123), N_{c1} can be obtained by solving

$$\tilde{N}_p(z_1)N_{c1}(z_1, z_2) + T(z_1, z_2)(1 - z_2) = I \tag{129}$$

Several features for the design method proposed above can be observed. Since $D_c(z_1, z_2)$, $N_{c1}(z_1, z_2), N_{c2}(z_1, z_2) \in \boldsymbol{M}(\boldsymbol{H})$ and $\det D_c(0,0) \neq 0$, we can have

$$\begin{aligned}\tilde{U}(z_1, z_2) &= D_c'(z_1, z_2)\tilde{U}(z_1, z_2) + N_{c1}'(z_1, z_2)Y_d(z_1) \\ &\quad - N_{c2}'(z_1, z_2)\tilde{Y}(z_1, z_2)\end{aligned} \tag{130}$$

for some $D_c', N_{c1}', N_{c2}' \in \boldsymbol{M}(\mathbb{R}[z_1, z_2])$ and $D_c'(0,0) = 0$. It then follows that

$$\begin{aligned}\tilde{u}(i,j) &= \sum_{q=0}^{d_{z_1}(D_c')} \sum_{\substack{r=0 \\ (q,r)\neq(0,0)}}^{d_{z_2}(D_c')} Q_{qr}\tilde{u}(i-q, j-r) + \sum_{k=0}^{d_{z_1}(N_{c1}')} \sum_{l=0}^{d_{z_2}(N_{c1}')} L_{kl}y_d(i-k) \\ &\quad - \sum_{m=0}^{d_{z_1}(N_{c2}')} \sum_{p=0}^{d_{z_2}(N_{c2}')} M_{mp}\tilde{y}(i-m, j-p)\end{aligned} \tag{131}$$

where $d_{z_i}(*)$ denotes the maximum degree of polynomial entries of the matrix $*$ in z_i, and Q_{qr}, L_{kl}, M_{mp} correspond to the coefficient matrices of D_c', N_{c1}' and N_{c2}', respectively. In most existing design methods for ILCS, the updating control input

$$\tilde{u}(i,j) = \tilde{u}(i, j-1) + \Delta\tilde{u}(i, j-1) \tag{132}$$

is used where the input modification $\Delta\tilde{u}(i, j-1)$ is a (linear) function of the tracking errors in previous iterations (see e.g. Geng *et al.*, 1990). Comparing (132) with (131), we see that (132) may be considered as a special case of (131). In consequence, with the use of the proposed method, it is possible to obtain a more general learning controller without requiring any *a priori* restriction on its structure, and the convergence of the learning process and the stability in time at a fixed learning iteration are guaranteed according to the practical internal stability.

Another interesting and important feature is that the proposed design procedure does not require any information about the value of the reference trajectory. As a matter of fact, besides the model of the plant, what we need is just to know that the reference signal is restricted in a finite time interval and it does not vary with the iterations. This means that, for a given plant, it is possible to design a learning scheme such that the resultant ILCS tracks repetitively an arbitrary trajectory as long as it is restricted in a finite time interval.

5. Numerical Examples

Examples on practical stabilities and practical stabilization can be found in (Agathoklis and Bruton, 1983; Xu *et al.*, 1994b, 1996a). Here we only show a few regarding practical tracking control and its application.

Example 1. Construct a learning controller with the structure of (124) for the plant:

$$y(i) - 1.96y(i-1) + 1.06y(i-2) = 0.06u(i) \tag{133}$$

Suppose the boundary condition is zero. The transfer function $p(z_1)$ of (133) is as follows:

$$p(z_1) = \frac{\tilde{n}_p(z_1)}{\tilde{d}_p(z_1)} = \frac{3}{53z_1^2 - 98z_1 + 50} \tag{134}$$

Solving (125) and (129) in scalar form for the above plant, we can obtain the controller:

$$d_c(z_1, z_2) = \tilde{w}(z_1) + s(z_1, z_2)\tilde{n}_p(z_1) \tag{135a}$$

$$n_{c2}(z_1, z_2) = \tilde{v}(z_1) - s(z_1, z_2)\tilde{d}_p(z_1) \tag{135b}$$

$$n_{c1}(z_1, z_2) = \frac{z_2}{3} \tag{135c}$$

where $\tilde{w}(z_1) = 1/50$ and $\tilde{v}(z_1) = z_1(-53z_1 + 98)/150$ is a particular solution to (125).

Let $s = z_2$. For the desired trajectory given by

$$y_d(i) = \begin{cases} \dfrac{4i}{N} & 1 \le i \le \dfrac{N}{4} \\ 1 - \left(i - \dfrac{N}{4}\right)\dfrac{4}{N} & \dfrac{N}{4} < i \le \dfrac{3N}{4} \\ -1 + \left(i - \dfrac{3N}{4}\right)\dfrac{4}{N} & \dfrac{3N}{4} < i \le N \end{cases} \tag{136}$$

where i $(1 \le i \le N = 50)$, the output and the tracking error of the designed ILCS is shown in Fig. 2. Further, it has been verified by simulation that, using the very same controller given by (135), the system can also track various kinds of different trajectories in a finite time interval. In particular, Fig. 3 shows the simulation results for a reference signal similar to the one used in the example of (Geng and Jamshidi, 1990). ♦

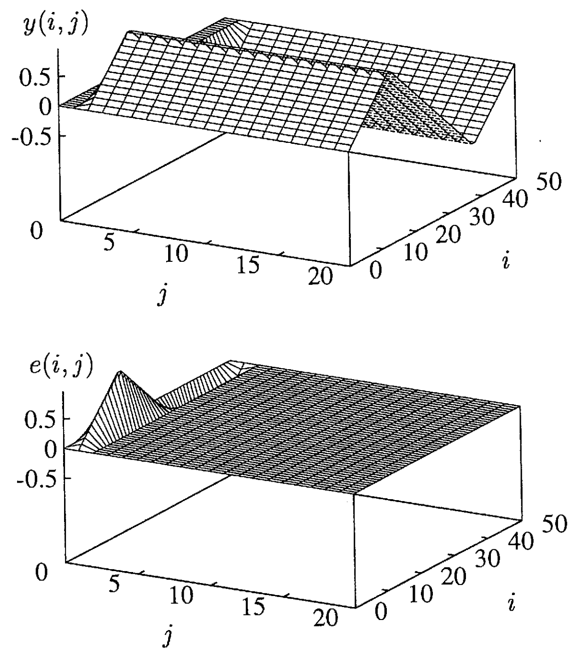

Fig. 2. Simulation results of Example 1.

Next, we would like to show a numerical example for practical tracking control of a 2D plant. It should be noted that the reference signal used here varies not only in the direction of time, but also along the direction of iterations.

Example 2. Consider the 2D plant given by

$$p(z_1, z_2) = \frac{\tilde{b}_p(z_1, z_2)}{\tilde{a}_p(z_1, z_2)} \tag{137}$$

where

$$\tilde{a}_p(z_1, z_2) = 2(-27z_1z_2 + 25z_1 + 25z_2 + 75)/25$$

$$\tilde{b}_p(z_1, z_2) = (25z_1^2z_2 - 25z_1^2 - 4z_1z_2 + 100z_2 + 100)/25$$

Design a controller such that the system output tracks the following 2D reference signal:

$$r(i, j) = \sin\frac{\pi j}{8} y_d(i) \tag{138}$$

where $y_d(i)$ is just the one defined by (136).

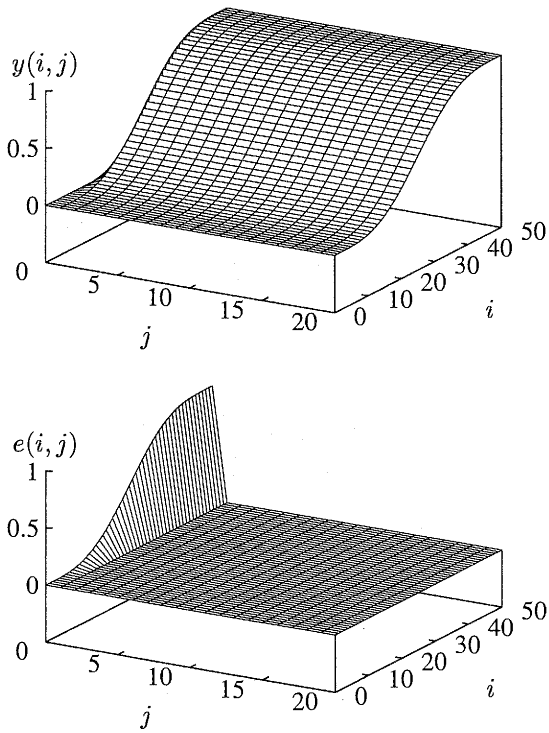

Fig. 3. Simulation results of Example 1 for another trajectory.

Since $\tilde{a}_p(z_1, z_2)$ is practical stable, we simply have the following solution for (125).

$$a_c(z_1, z_2) = 1/\tilde{a}_p(z_1, z_2) + s(z_1, z_2)\tilde{b}_p(z_1, z_2) \tag{139a}$$

$$b_{c2}(z_1, z_2) = -s(z_1, z_2)\tilde{a}_p(z_1, z_2) \tag{139b}$$

On the other hand, the z-transform of $r(i, j)$ is as follows:

$$\begin{aligned} r(z_1, z_2) &= \frac{z_2 \sin(\pi/8) Y_d(z_1)}{z_2^2 - 2z_2 \cos(\pi/8) + 1} \\ &\triangleq \frac{b_s(z_1, z_2)}{a_s(z_1, z_2)} \end{aligned} \tag{140}$$

Therefore, the skew equation (97) becomes

$$\tilde{b}_p(z_1, z_2) n_{c1}(z_1, z_2) + t(z_1, z_2) a_s(z_1, z_2) = 1 \tag{141}$$

and its solution can be obtained as follows:

$$b_{c1}(z_1, z_2) = \frac{w(z_1, z_2)}{\phi(z_1, z_2)} \tag{142}$$

$$t(z_1, z_2) = \frac{v(z_1, z_2)}{\phi(z_1, z_2)} \tag{143}$$

where

$$w(z_1, z_2) = z_2^2 - 2.847759z_2 + 3.847759 \tag{144}$$

$$v(z_1, z_2) = 3.847759z_1^2 - 4.0z_2 \tag{145}$$

$$\begin{aligned} \phi(z_1, z_2) = {} & z_1^2 z_2^3 - 0.414214 z_1^2 z_2 - 0.16 z_1 z_2^3 \\ & + 0.455641 z_1 z_2^2 - 0.615641 z_1 z_2 + 15.391036 \end{aligned} \tag{146}$$

Let $s = z_2/2$. Then we have the simulation results shown in Fig. 4, from which excellent tracking performance for the desired 2D trajectory can be confirmed. ♦

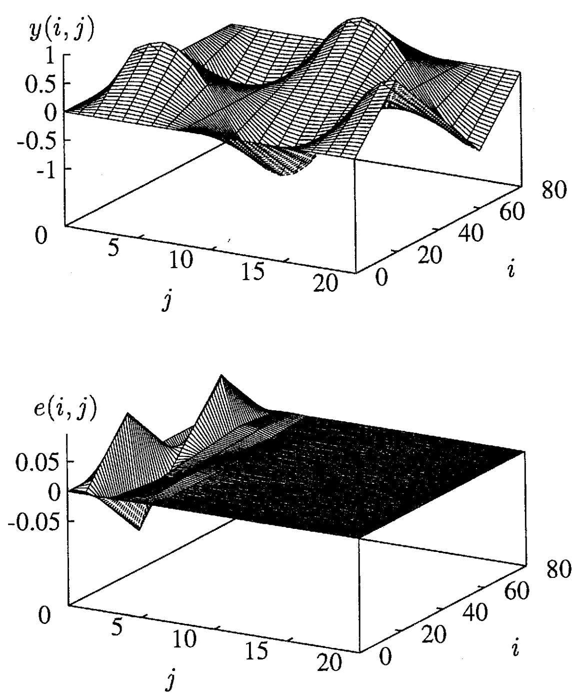

Fig. 4. Simulation results of Example 2.

6. Concluding Remarks

A comprehensive treatment has been presented of some theoretical and methodological results and their possible applications for the control of nD systems in the practical sense that the system inputs and outputs are unbounded in, at most, one dimension. Some recent results have been reviewed on the basic properties and control problems such as practical BIBO and internal stabilities, practical controllability and observability, practical stabilization by both the algebraic and the state-space method, and the relation between the two methods. Moreover, further contributions to the practical tracking problem and its applications have been shown.

The results obtained reveal that the nD control problems considered in the practical sense can be essentially reduced to the corresponding 1D problems, and thus can be solved, when compared with the conventional nD system theory, under less restrictive stability conditions and by much simpler methods. In particular, it is shown that the proposed method for 2D practical tracking control provides in fact a general design approach for a class of iterative learning control systems and linear multipass processes. Therefore, the presented control theory for nD systems in the practical sense is of significance not only from the point of view of practical applications of nD system theory, but also from that of control of such iterative systems.

It should be remarked here that the problems considered in this paper are the most basic ones for the control of nD systems in the practical sense, and there remain many problems to be solved for actual applications.

First of all, the results are mainly based on considering the steady-state performance of the resultant control systems, therefore no guarantee is provided for the transient performance. Optimal control would be naturally considered as an effective way to improve the system transient performance. As a special case of the 2D LQR (linear quadratic regulator) problem, the 2D minimum-energy problem in the finite-horizon case (both variables bounded) and its application to multipass processes have been considered in e.g. (Kaczorek and Klamka, 1986; Li and Fadali, 1991). The obtained optimal control for this case, however, is an open-loop one. The 2D LQR problem in the infinite-horizon case (both variables unbounded) have also been considered in (Bisiacco, 1995; Bisiacco and Fornasini, 1990) where, however, a substantial difficulty is encountered; namely: an infinite-dimensional global state feedback is in general required. Therefore, it is found again that it would be reasonable to consider 2D optimal control in some more practical sense, such as for the case where one variable is bounded while the other is unbounded. Such research is now in progress and some preliminary results can be found e.g. in (Yamada *et al.*, 1996).

Also, for actual situations, the ability to obtain, in an appropriate way, the model of the underlying system is of great importance and an adaptive scheme would be desirable. To this end, some efforts and interesting results have recently been reported (see e.g. Geng *et al.*, 1990; Heath, 1994).

Finally, we would like to mention that, in the case where the conventional 2D stability is required e.g. to design a controller for a multipass process under the stability along the pass (Rogers and Owens, 1992), the well-developed results for 2D stabilization (Bisiacco *et al.*, 1986; Guiver and Bose, 1985; Lin, 1988; Xu *et al.*, 1994a) and

2D tracking control (Xu *et al.*, 1990) can be directly applied in the way shown in this paper.

References

Agathoklis P. and Bruton L. (1983): *Practical-BIBO stability of n-dimensional discrete systems.* — IEE Proc. G. Electron. Circuits Syst., Vol.130, No.6, pp.236–242.

Ahmed A.R.E. (1980): *On the stability of two-dimensional discrete systems.* — IEEE Trans. Automat. Contr., Vol.25, No.3, pp.551–552.

Bisiacco M. and Fornasini E. (1990): *Optimal control of two-dimensional systems.* — SIAM J. Contr. Optim., Vol.28, No.3, pp.582–601.

Bisiacco M., Fornasini E. and Marchesini G. (1986): *Controller design for 2D systems*, In: Frequency Domain and State Space Methods for Linear Systems (C.I. Byrnes and A. Lindquist, Eds.). — North Holland: Elservier, pp.99–113.

Bisiacco M., Fornasini E. and Marchesini G. (1989): *Dynamic regulation of 2D systems: A state-space approach.* — Lin. Alg. Appl., Vol.122/123/124, pp.195–218.

Bisiacco M. (1995): *New results in 2D optimal control theory.* — Multidim. Sys. Signal Process., Vol.6, No.3, pp.189–222.

Boland F.M. and Owens D.H. (1980): *Linear multipass processes: a two-dimensional interpretation.* — IEE Proc., Vol.127, Pt.D, No.5, pp.189–193.

Bose N.K. (1982): *Applied Multidimensional Systems Theory.* — New York: Van Nostrand Reinhold.

Ciftcibasi T. and Yüksel Ö. (1983): *Sufficient or necessary conditions for modal controllability and observability of Roesser's 2-D system model.* — IEEE Trans. Automat. Contr., Vol.28, No.4, pp.527–529.

Eising R. (1979): *Controllability and observability of 2-D systems.* — IEEE Trans. Automat. Contr., Vol.24, No.1, pp.132–133.

Fornasini E. and Marchesini G. (1978) *Doubly-indexed dynamical systems: State-space models and structural properties.* — Math. Syst. Theory, Vol.12, pp.59–72.

Fornasini E. and Marchesini G. (1979): *On the internal stability of two-dimensional filters.* — IEEE Trans. Automat. Contr., Vol.24, No.1, pp.129–130.

Guiver J.P. and Bose N.K. (1982): *Polynomial matrix primitive factorization over arbitrary coefficient field and related results.* — IEEE Trans. Circuits Syst., Vol.29, No.10, pp.649–657.

Guiver J.P. and Bose N.K. (1985): *Causal and weakly causal 2-D filters with applications in stabilization*, In: Multidimensional Systems Theory (N.K. Bose, Ed.). — Dordrecht: Reidel, p.52.

Geng Z., Carroll R. and Xie J. (1990): *Two-dimensional model and algorithm analysis for a class of iterative learning control systems.* — Int. J. Contr., Vol.52, No.4, pp.833–862.

Geng Z. and Jamshidi M. (1990): *Learning control system analysis and design based on 2-d system theory.* — J. Intell. Robot. Syst., Vol.3, pp.17–26.

Goodman D. (1977): *Some stability properties of two-dimensional linear shift-invariant digital filters.* — IEEE Trans. Circuits Syst., Vol.24, pp.201–208.

Gałkowski K., Rogers E. and Owens D. (1997): *Basic systems theory for discrete linear repetitive processes using 2D Roesser model interpretations.* — Appl. Math. Comp. Sci., Vol.7, No.1, pp.101–116.

Heath W.P. (1994): *Self-Tuning Control for Two-Dimensional Processes.* — England: Research Studies Press Ltd.

Jury E.T. (1978): *Stability of multidimensional scalar and matrix polynomials.*— Proc. IEEE, Vol.66, No.9, pp.1018–1047.

Kaczorek T. (1985): *Two-Dimensional Linear Systems.* — Berlin: Springer-Verlag.

Kaczorek T. and Klamka J. (1986): *Minimum energy control of 2-D linear systems with variable coefficients.* — Int. J. Contr., Vol.44, No.3, pp.645–650.

Kučera V. (1979): *Discrete Linear Control: The Polynomial Equation Approach.* — Chichester: Wiley.

Kurek J.E. (1985): *Basic properties of q-dimensional linear digital systems.* — Int. J. Control, Vol.42, No.1, pp.119–128.

Kurek J.E. (1987): *Reachability of a system described by the multidimensional Roesser model.* — Int. J. Contr., Vol.45, No.2, pp.1559–1563.

Kurek J.E. and Zaremba M.B. (1993): *Iterative learning control synthesis based on 2-D system theory.* — IEEE Trans. Automat. Contr., Vol.38, No.1, pp.121–125.

Li C. and Fadali M.S. (1991): *Optimal control of 2-D systems.* — IEEE Trans. Automat. Contr., Vol.36, No.2, pp.223–228.

Lin Z. (1988): *Feedback stabilization of multivariable two-dimensional linear systems.* — Int. J. Contr., Vol.48, No.3, pp.1301–1317.

Morf M., Levy B. and Kung S.Y. (1977): *New result in 2-D systems theory, Part I: 2-D polynomial matrices, factorization and coprimeness.* — Proc. IEEE, Vol.65, No.6, pp.861–872.

Nett C.N., Jacobson C.A. and Balas M.J. (1984): *A connection between state-space and doubly coprime fractional representations.* — IEEE Trans. Automat. Contr., Vol.29, No.9, pp.831–832.

Rogers E. and Owens D.H. (1989): *2D-transfer functions and stability tests for differential non-unit memory linear multipass processes.* — Int. J. Contr., Vol.50, No.2, pp.651–666.

Rogers E. and Owens D.H. (1992): *Stability Analysis for Linear Repetitive Processes.* — Berlin: Springer Verlag.

Reedy P.S., Palacherla S.R. and Swamy M.N.S. (1990): *Least squares inverse polynomial and a proof of practical-BIBO stability of n-D digital filters.* — IEEE Trans. Circuits Syst., Vol.37, No.12, pp.1565–1567.

Rocha P., Rogers E. and Owens D.H. (1996): *Stability of discrete non-unit memory linear repetitive processes — a two-dimensional systems interpretation.* — Int. J. Contr., Vol.63, No.3, pp.457–482.

Swamy M.N., Roytman L.M. and Plotkin E.I. (1985): *On stability properties of three- and higher dimensional linear shift-invariant digital filters.* — IEEE Trans. Circuits Syst., Vol.32, No.9, pp.888–891.

Vidyasagar M. (1985): *Control System Synthesis: A Factorization Approach.* — Boston: MIT Press.

Wolovich W.A. (1978): *Skew prime polynomial matrices.* — IEEE Trans. Automat. Contr., Vol.23, No.5, pp.880–887.

Xu L., Saito O. and Abe K. (1990): *Bilateral polynomial matrix equations in two indeterminates.* — Multidim. Syst. Signal Process., Vol.1, No.4, pp.363–379.

Xu L., Saito O. and Abe K. (1994a): *Output feedback stabilizability and stabilization algorithms for 2D systems.* — Multidim. Syst. Signal Process., Vol.5, No.1, pp.41–60.

Xu L., Saito O. and Abe K. (1994b): *The design of practically stable nD feedback systems.* — Automatica, Vol.30, No.9, pp.1389–1397.

Xu L., Saito O. and Abe K. (1996a): *Practical internal stability of nD discrete systems.* — IEEE Trans. Automat. Contr., Vol.41, No.5, pp.756–761.

Xu L., Saito O. and Abe K. (1996b): *Design of practically stable nD feedback systems: A state-space approach.* — Int. J. Contr., Vol.64, No.1, pp.29–39.

Xu L., Yamada M., Saito O. and Abe K. (1994): *Practical tracking problem of nd systems and its applications to iterative control systems.* — Proc. 1st *Asian Contr. Conf.*, Tokyo, Japan, Vol.III, pp.447–450.

Yamada M., Xu L. and Saito O. (1996): *2D model-following servo systems.* — Proc. 13th *IFAC World Congress*, San Francisco, Vol.D, pp.291–296.