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THE DIAMETER OF THE REACHABILITY SET FOR A 2-D CONTINUOUS-DISCRETE LINEAR SYSTEM WITH DISTURBANCES

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The methods are presented for computing the support functions and diameters of reachability sets for a certain type of 2-D continous-discrete linear system with disturbances limited to a rectangle and an ellipsoid. These methods are based on the idea of the reachability set for 1-D systems with disturbances and the 2-D continuous-discrete linear system theory.

1. Introduction

2-D continuous-discrete models of linear systems have been investigated in (Kaczorek, 1994; 1995; Kaczorek and Stajniak, 1994; Stajniak, 1995), where the respective solutions, local reachability and controllability, and minimum-energy control problems for various models have been considered. In this paper, the countepart of the reachability set for a 2-D continuous-discrete linear system with disturbances, known for continuous and discrete 1-D systems (Kurzhanski, 1977; Schweppe, 1973), is introduced. The formulae for the support function of the reachability set X^r_{tk} and X^e_{tk} for a certain type of 2-D continuous-discrete linear system with disturbances limited to a rectangle and an ellipsoid in $\mathbb{R}^q$ are established (Theorems 3 and 5).

Computation of the diameters of the sets X^r_{tk} and X^e_{tk} according to Definition 2 is illustrated with examples. The formulae for the support functions of reachability sets $X^r_{tk}(u)$ and $X^e_{tk}(u)$ for 2-D continuous-discrete linear control systems are also given.

2. Reachability Set for 2-D Continuous-Discrete Linear Systems with Disturbances from a Rectangle in $\mathbb{R}^q$

Consider the following 2-D continuous-discrete linear system:

$$\dot{x}(t, k+1) = Ax(t,k) + Cw(t,k), \quad t \in [0,\ T], \quad k \in [0,\ N] \tag{1}$$

where $\dot{x}(t,k) = \partial x(t,k)/\partial t$, $x(t,k) \in \mathbb{R}^n$ is the state vector, $w(t,k) \in \mathbb{R}^q$ stands for the disturbance vector, $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{n\times q}$ are real matrices.

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The corresponding boundary conditions are given by

$$x(t,0) = x_1(t), \quad t \in [0,T] \quad \text{and} \quad x(0,k) = x_2(k), \quad k \in [0,N] \tag{2}$$

where $x_1(t)$ and $x_2(k)$ are known, and $x_1(0) = x_2(0)$.

Assume that the disturbance vectors belong to a rectangle in $\mathbb{R}^q$.

$$\begin{cases} w(t,k) \in W_r \\ W_r = \left\{ w : w^T = [w^1, w^2, \ldots, w^q] \wedge \underline{w}^j \le w^j \le \overline{w}^j,\ j = 1, 2, \cdots, q \right\} \end{cases} \tag{3}$$

where $\underline{w}^j$ and $\overline{w}^j$ are given, $\overline{w}^j - \underline{w}^j = \Delta w^j > 0$, and let

$$w_c^T = \left[\frac{\underline{w}^1 + \overline{w}^1}{2}, \frac{\underline{w}^2 + \overline{w}^2}{2}, \ldots, \frac{\underline{w}^q + \overline{w}^q}{2} \right]$$

Definition 1. The *reachability set* X_{tk}^r is the set of all possible states of the system (1) at the moment (t,k) with boundary conditions (2) for all possible disturbances from the set W_r.

Theorem 1. *The set* X_{tk}^r *is convex.*

Proof. Let $x_1,\ x_2 \in X_{tk}^r$. From (Kaczorek, 1994; Kaczorek and Stajniak, 1994) we know that x_1 and x_2 have the form:

$$\begin{aligned} x_{1,2}(t,k) = {} & A^k \int_0^t \frac{(t-\tau)^{k-1}}{(k-1)!} x_1(\tau)\,\mathrm{d}\tau + \sum_{i=0}^{k-1} \frac{t^i}{i!} A^i x_2(k-i) \\ & + \sum_{i=0}^{k-1} A^{k-i-1} C \int_0^t \frac{(t-\tau)^{k-i-1}}{(k-i-1)!} w_{1,2}(\tau,i)\,\mathrm{d}\tau, \quad k \in [1,N] \end{aligned} \tag{4}$$

where $w_{1,2}(\tau,i) \in W_r$ for $\tau \in [0,t]$, $i \in [0,k-1]$. For $0 \le \alpha \le 1$, it is easy to show that

$$\begin{aligned} x &= \alpha x_1 + (1-\alpha) x_2 \\ &= A^k \int_0^t \frac{(t-\tau)^{k-1}}{(k-1)!} x_1(\tau)\,\mathrm{d}\tau + \sum_{i=0}^{k-1} \frac{t^i}{i!} A^i x_2(k-i) \\ &\quad + \sum_{i=0}^{k-1} A^{k-i-1} C \int_0^t \frac{(t-\tau)^{k-i-1}}{(k-i-1)!} \Big[\alpha w_1(\tau,i) + (1-\alpha) w_2(\tau,i) \Big] \mathrm{d}\tau \end{aligned} \tag{5}$$

and $x \in X_{tk}^r$. It is the solution to (1) and (2) with the disturbance

$$w(\tau,i) = \alpha w_1(\tau,i) + (1-\alpha) w_2(\tau,i) \tag{6}$$

Since the rectangle W_r is convex, $w(\tau,i)$ of the form (6) belongs to W_r, too. ■

The convex set X^r_{tk} in $\mathbb{R}^n$ can be described by the support function (Rockafellar, 1972)

$$h(z \mid X^r_{tk}) = \max_{x \in X^r_{tk}} z^T x, \quad z \in \mathbb{R}^n \tag{7}$$

where T denotes transposition. Let a^{ki}_{pj} denote elements of the matrix $A^{k-i-1}C$.

Theorem 2. *If $x \in X^r_{tk}$, then for every $z^T = [z^1, z^2, \dots, z^n]$ the following condition is satisfied:*

$$z^T x \le z^T \left[A^k \int_0^t \frac{(t-\tau)^{k-1}}{(k-1)!} x_1(\tau)\,\mathrm{d}\tau + \sum_{i=0}^{k-1} \frac{t^i}{i!} A^i x_2(k-i) \right] + \sum_{i=0}^{k-1} \frac{t^{k-i}}{(k-i)!} \left[z^T A^{k-i-1} C w_c + \frac{1}{2} \sum_{p=1}^{n} |z^p| \left| \sum_{j=1}^{q} a^{ki}_{pj} \Delta w^j \right| \right] \tag{8}$$

Proof. Let $x \in X^r_{tk}$. According to (4), for $z \in \mathbb{R}^n$ we can write

$$z^T x = z^T \left[A^k \int_0^t \frac{(t-\tau)^{k-1}}{(k-1)!} x_1(\tau)\,\mathrm{d}\tau + \sum_{i=0}^{k-1} \frac{t^i}{i!} A^i x_2(k-i) \right] + z^T \left[\sum_{i=0}^{k-1} A^{k-i-1} C \int_0^t \frac{(t-\tau)^{k-i-1}}{(k-i-1)!} w(\tau, i)\,\mathrm{d}\tau \right] \tag{9}$$

Denote by $f_z(z)$ and $f_2(z)$ the first and the second term of this sum, respectively. We have

$$f_2(z) \le \sum_{i=0}^{k-1} \max_{s \in [0,t]} \left[z^T A^{k-i-1} C w(s,i) \right] \int_0^t \frac{(t-\tau)^{k-i-1}}{(k-i-1)!}\,\mathrm{d}\tau \tag{10}$$

It is easy to verify that

$$z^T A^{k-i-1} C w = \sum_{p=1}^{n} z^p \sum_{j=1}^{q} a^{ki}_{pj} w^j \tag{11}$$

From (3) it follows that

$$f_2(z) \le \sum_{i=0}^{k-1} \frac{t^{k-i}}{(k-i)!} \left[z^T A^{k-i-1} C w_c + \frac{1}{2} \sum_{p=1}^{n} |z^p| \left| \sum_{j=1}^{q} a^{ki}_{pj} \Delta w^j \right| \right] \tag{12}$$

which completes the proof. ■

Theorem 3. *The support function of the reachability set X^r_{tk} of the system (1) with boundary conditions (2) and disturbances (3) has the form:*

$$h(z \mid X^r_{tk}) = z^T \Bigg[A^k \int_0^t \frac{(t-\tau)^{k-1}}{(k-1)!} x_1(\tau)\,\mathrm{d}\tau + \sum_{i=0}^{k-1} \frac{t^i}{i!} A^i x_2(k-i)$$

$$+ \sum_{i=0}^{k-1} \frac{t^{k-i}}{(k-i)!} A^{k-i-1} C w_c \Bigg]$$

$$+ \frac{1}{2} \sum_{i=0}^{k-1} \frac{t^{k-i}}{k-i)!} \sum_{p=1}^{n} |z^p| \Big| \sum_{j=1}^{q} a^{ki}_{pj} \Delta w^j \Big| \tag{13}$$

Proof. Let us denote by $f(z)$ the right-hand side of (8). Since $f(z)$ is positive-homogeneous, convex and continuous function, and X^r_{tk} is a convex and closed set, from Theorem 2 and the properties of the support function, we conclude that

$$f(z) = h(z \mid X^r_{tk}) \tag{14}$$

■

Remark 1. In case $k = 0$, according to the boundary conditions (2), we have $x(t,0) = x_1(t)$. Then the reachability set X^r_{t0}, for a fixed t, is a one-point set in $\mathbb{R}^n$ and

$$h(z \mid X^r_{t0}) = z^T x_1(t), \quad z \in \mathbb{R}^n$$

Corollary 1. *If $k = 1$ and $x_1(t) = 0$ for $t \in [0, T]$, then the support function of the reachability set X^r_{t1} has the form*

$$h(z \mid X^r_{t1}) = z^T \Big[x_2(1) + tCw_c \Big] + \frac{1}{2} t \sum_{p=1}^{n} |z^p| \Big| \sum_{j=1}^{q} c_{pj} \Delta w^j \Big|, \quad z \in \mathbb{R}^n$$

It is equal to the support function of the reachability set for the continuous linear system $\dot{x}(t) = Cw(t)$ with $x(0) = x_2(1)$, where $w(t) \in W_r$, calculated according to (Barmish et al., 1978).

Definition 2. The *diameter* of the set X is defined by

$$d(X) = \max_{\|z\| \le 1} \Big[h(z \mid X) + h(-z \mid X) \Big] \tag{15}$$

It is a maximum length of the projection of the set X onto the straight line αz, $\alpha \in \mathbb{R}$, for $\|z\| \le 1$.

Theorem 4. *The diameter of the reachability set X^r_{tk} for the system (1) with boundary conditions (2) and disturbances (3) is given by*

$$d(X^r_{tk}) = \Bigg[\sum_{p=1}^{n} \Big(\sum_{i=0}^{k-1} \frac{t^{k-i}}{(k-i)!} \Big| \sum_{j=1}^{q} a^{ki}_{pj} \Delta w^j \Big| \Big)^2 \Bigg]^{1/2} \tag{16}$$

Proof. Using (13), we obtain

$$h(z \mid X_{tk}^r) + h(-z \mid X_{tk}^r) = \sum_{i=0}^{k-1} \frac{t^{k-i}}{(k-i)!} \sum_{p=1}^{n} |z^p| \Big| \sum_{j=1}^{q} a_{pj}^{ki} \Delta w^j \Big| \tag{17}$$

Let us maximize this function subject to $\sum_{p=1}^{n} (z^p)^2 - 1 \leq 0$. We first form the Lagrangian

$$L_1(z_1, z_2, \dots, z_n, \lambda) = -\sum_{i=0}^{k-1} \frac{t^{k-i}}{(k-i)!} \sum_{p=1}^{n} |z^p| \Big| \sum_{j=1}^{q} a_{pj}^{ki} \Delta w^j \Big| + \lambda \Big[\sum_{p=1}^{n} (z^p)^2 - 1 \Big] \tag{18}$$

and then use the differential Kuhn-Tucker conditions (Dubnicki and Zorychta, 1972). Hence for $z^p > 0$ we have

$$\frac{\partial L_1}{\partial z^p} = -\sum_{i=0}^{k-1} \frac{t^{k-i}}{(k-i)!} \Big| \sum_{j=1}^{q} a_{pj}^{ki} \Delta w^j \Big| + 2\lambda z^p = 0 \tag{19}$$

and therefore

$$z^p = \frac{1}{2\lambda} \sum_{i=0}^{k-1} \frac{t^{k-i}}{(k-i)!} \Big| \sum_{j=1}^{q} a_{pj}^{ki} \Delta w^j \Big|, \quad p = 1, 2, \dots, n \tag{20}$$

Moreover,

$$\frac{\partial L_1}{\partial \lambda} \lambda = \Big[\sum_{p=1}^{n} (z^p)^2 - 1 \Big] \lambda = 0 \tag{21}$$

When $\lambda \neq 0$, we have

$$\sum_{p=1}^{n} (z^p)^2 = 1 \tag{22}$$

In order to find λ, we apply (20) and (22):

$$\lambda = \frac{1}{2} \Big[\sum_{p=1}^{n} \Big(\sum_{i=0}^{k-1} \frac{t^{k-i}}{(k-i)!} \Big| \sum_{j=1}^{q} a_{pj}^{ki} \Delta w^j \Big| \Big)^2 \Big]^{1/2} \tag{23}$$

Substituting (23) into (20) yields

$$z^p = \Big[\sum_{p=1}^{n} \Big(\sum_{i=0}^{k-1} \frac{t^{k-i}}{(k-i)!} \Big| \sum_{j=1}^{q} a_{pj}^{ki} \Delta w^j \Big| \Big)^2 \Big]^{-1/2} \sum_{i=0}^{k-1} \frac{t^{k-i}}{(k-i)!} \Big| \sum_{j=1}^{q} a_{pj}^{ki} \Delta w^j \Big| \tag{24}$$

Finally,

$$\begin{aligned} d(X^{r}_{tk}) &= \sum_{i=0}^{k-1}\frac{t^{k-i}}{(k-i)!}\sum_{p=1}^{n}\Big(\sum_{i=0}^{k-1}\frac{t^{k-i}}{(k-i)!}\Big|\sum_{j=1}^{q}a^{ki}_{pj}\Delta w^{j}\Big|\Big)\Big|\sum_{j=1}^{q}a^{ki}_{pj}\Delta w^{j}\Big| \\ &\quad\times\Big[\sum_{p=1}^{n}\Big(\sum_{i=0}^{k-1}\frac{t^{k-i}}{(k-i)!}\Big|\sum_{j=1}^{q}a^{ki}_{pj}\Delta w^{j}\Big|\Big)^{2}\Big]^{-1/2} \\ &= \Big[\sum_{p=1}^{n}\Big(\sum_{i=0}^{k-1}\frac{t^{k-i}}{(k-i)!}\Big|\sum_{j=1}^{q}a^{ki}_{pj}\Delta w^{j}\Big|\Big)^{2}\Big]^{1/2} \qquad (25)\end{aligned}$$

For $z^p < 0$ we obtain the same result. ■

Remark 2. In case $k = 0$, based on Corollary 1 and Definition 2, we can write $d(X^r_{t0}) = 0$.

Example 1. Consider the system (1) with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$x_1(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad t \in [0, T]; \qquad x_2(k) = \begin{bmatrix} k \\ 1 \end{bmatrix}, \quad k \in [0, N]$$

Assume $-0.1 \le w^j(\tau, i) \le 0.1$ for $j = 1, 2$. We calculate the support function of the reachability set X^r_{12} using (13):

$$\begin{aligned} h(z \mid X^r_{12}) &= z^T\left(\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\int_0^1 (1-\tau)\,\mathrm{d}\tau + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right. \\ &\quad \left. + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + \frac{1}{2}\left[\frac{1}{2}\Big(|z^1|0.2 + |z^2|2\cdot 0.2\Big)\right. \\ &\quad \left. + \Big(|z^1|0.2 + |z^2|0.2\Big)\right] = 3z^1 + 5z^2 + 0.15|z^1| + 0.2|z^2| \qquad (26)\end{aligned}$$

The diameter of the set X^r_{12}, according to (16), has the value

$$d(X^r_{12}) = \left[\Big(\frac{1}{2}\cdot 0.2 + 0.2\Big)^2 + \Big(\frac{1}{2}\cdot 2\cdot 0.2 + 0.2\Big)^2\right]^{1/2} = 0.5 \qquad (27)$$

The same result can be obtained directly from Definition 2:

$$d(X^r_{12}) = \max_{(z^1)^2+(z^2)^2\le 1}\Big[0.3|z^1| + 0.4|z^2|\Big] \qquad (28)$$

This maximum is attained for $|z^1| = 0.6$ and $|z^2| = 0.8$. ♦

3. Reachability Set for a 2-D Continuous-Discrete Linear System with Disturbances from an Ellipsoid in $\mathbb{R}^q$

Consider the system (1) with boundary conditions (2). Assume that the disturbance vectors $w(t,k)$ belong to the ellipsoid

$$W_e = \left\{ w : (w-m)^T Q^{-1} (w-m) \le 1 \right\} \tag{29}$$

where $m \in \mathbb{R}^q$ denotes its centre and Q is a symmetric positive-definite matrix in $\mathbb{R}^{q\times q}$.

Definition 3. The *reachability set* X^e_{tk} is the set of all possible states of the system (1) at the moment (t,k) with boundary conditions (2) for all possible disturbances from the set W_e. It easy to verify that X^e_{tk} is a convex set as the ellipsoid W_e is convex.

Theorem 5. *The support function of the reachability set* X^e_{tk} *has the form:*

$$\begin{aligned} h(z \mid X^e_{tk}) &= z^T \Bigg[A^k \int_0^t \frac{(t-\tau)^{k-1}}{(k-1)!} x_1(\tau)\,\mathrm{d}\tau \\ &\quad + \sum_{i=0}^{k-1} \frac{t^i}{i!} A^i x_2(k-i) + \sum_{i=0}^{k-1} \frac{t^{k-i}}{(k-i)!} A^{k-i-1} C m \Bigg] \\ &\quad + \sum_{i=0}^{k-1} \sqrt{2k-2i-1}\, \frac{t^{k-i}}{(k-i)!} \\ &\quad \times \left[z^T (A^{k-i-1}C) Q (A^{k-i-1}C)^T z \right]^{1/2}, \quad z \in \mathbb{R}^n \end{aligned} \tag{30}$$

Proof. From (4) and the definition of the support function of X^e_{tk}, it follows that

$$\begin{aligned} h(z \mid X^e_{tk}) &= z^T \Bigg[A^k \int_0^t \frac{(t-\tau)^{k-1}}{(k-1)!} x_1(\tau)\,\mathrm{d}\tau + \sum_{i=0}^{k-1} \frac{t^i}{i!} A^i x_2(k-i) \Bigg] \\ &\quad + \max_{w\in W_e} \sum_{i=0}^{k-1} z^T (A^{k-i-1}C) \int_0^t \frac{(t-\tau)^{k-i-1}}{(k-i-1)!} w(\tau,i)\,\mathrm{d}\tau \end{aligned} \tag{31}$$

Let us find

$$\min_w \left[-\sum_{i=0}^{k-1} z^T (A^{k-i-1}C) \int_0^t \frac{(t-\tau)^{k-i-1}}{(k-i-1)!} w(\tau,i)\,\mathrm{d}\tau \right] \tag{32}$$

subject to the condition $w(\tau,i) \in W_e$ for $0 \le \tau \le t$, $0 \le i \le k-1$, and

$$\left[w(\tau,i) - m \right]^T Q^{-1} \left[w(\tau,i) - m \right] \le 1 \tag{33}$$

otherwise, i.e.

$$\int_0^t \Big[w(\tau,i)-m\Big]^T Q^{-1}\Big[w(\tau,i)-m\Big]\,\mathrm{d}\tau - t \le 0 \tag{34}$$

First, we form the Lagrangian

$$\begin{aligned} L_2(w,\lambda_0,\lambda_1,\dots,\lambda_{k-1}) &= -\sum_{i=0}^{k-1} z^T(A^{k-i-1}C)\int_0^t \frac{(t-\tau)^{k-i-1}}{(k-i-1)!}w(\tau,i)\,\mathrm{d}\tau \\ &\quad + \sum_{i=0}^{k-1}\lambda_i\Big\{\int_0^t\Big[w(\tau,i)-m\Big]^T Q^{-1}\Big[w(\tau,i)-m\Big]\,\mathrm{d}\tau - t\Big\} \end{aligned} \tag{35}$$

Assume $w(0,0)=0$. Using the differential Kuhn-Tucker conditions, we obtain

$$2\lambda_i Q^{-1}\int_0^t \Big[w(\tau,i)-m\Big]\,\mathrm{d}\tau = \frac{t^{k-i}}{(k-i)!}(A^{k-i-1}C)^T z, \quad i\in[0,k-1] \tag{36}$$

Hence

$$w(t,i) = m + \frac{1}{2\lambda_i}\frac{t^{k-i-1}}{(k-i-1)!}Q(A^{k-i-1}C)^T z, \quad \lambda_i>0 \tag{37}$$

Moreover

$$\lambda_i\int_0^t \Big[w(\tau,i)-m\Big]^T Q^{-1}\Big[w(\tau,i)-m\Big]\,\mathrm{d}\tau = t\lambda_i \tag{38}$$

Substituting (37) into (38) gives

$$\lambda_i = \frac{1}{2}\frac{t^{k-i-1}}{(k-i-1)!\sqrt{2k-2i-1}}\Big[z^T(A^{k-i-1}C)Q(A^{k-i-1}C)^T z\Big]^{1/2} \tag{39}$$

and therefore

$$w(t,i) = m + \sqrt{2k-2i-1}\,Q(A^{k-i-1}C)^T z\Big[z^T(A^{k-i-1}C)Q(A^{k-i-1}C)^T z\Big]^{-1/2} \tag{40}$$

The minimum value in (32) equals

$$\begin{aligned} &-\sum_{i=0}^{k-1}\frac{t^{k-i}}{(k-i)!}\Big\{z^T(A^{k-i-1}C)m \\ &\qquad + \sqrt{2k-2i-1}\Big[z^T(A^{k-i-1}C)Q(A^{k-i-1}C)^T z\Big]^{1/2}\Big\} \end{aligned} \tag{41}$$

which yields (30). ■

Remark 3. Remark 1 is still valid.

Corollary 2. *If* $k=1$ *and* $x_1(t)=0$ *for* $t\in[0,T]$, *then*

$$h(z \mid X^e_{t1}) = z^T x_2(1) + tz^T Cm + t\big[z^T CQC^T z\big]^{1/2}, \quad z\in\mathbb{R}^n$$

It is equal to the support function of the reachability set for the continuous linear system $\dot{x}(t) = Cw(t)$ *with* $x(0) = x_2(1)$, *where* $w(t) \in W_e$, *calculated according to (Barmish et al., 1978).*

Corollary 3. *If* $m = 0$, *then the support function of the reachability set* X_{tk}^e *has the form*

$$h(z \mid X_{tk}^e) = z^T \left[A^k \int_0^t \frac{(t-\tau)^{k-1}}{(k-1)!} x_1(\tau)\,\mathrm{d}\tau + \sum_{i=0}^{k-1} \frac{t^i}{i!} A^i x_2(k-i) \right]$$

$$+ \sum_{i=0}^{k-1} \sqrt{2k-2i-1} \frac{t^{k-i}}{(k-i)!} \Big[z^T (A^{k-i-1}C) Q (A^{k-i-1}C)^T z \Big]^{1/2}, \quad z \in \mathbb{R}^n$$

Remark 4. The same result can be obtained when computing this support function directly from (7) (just as in the proof of Theorem 5). The multipliers λ_i, $i = 0, 1, \dots, k-1$ in the Lagrangian, for which the maximum is attained, have the same values independently of m. The values of disturbances w at the maximum in both the cases differ by m.

In order to determine the diameter of the reachability set X_{tk}^e based on Definition 2 and Theorem 5, it is necessary to find

$$\max_{\|z\| \le 1} 2 \sum_{i=1}^{k-1} \sqrt{2k-2i-1} \frac{t^{k-i}}{(k-i)!} \Big[z^T (A^{k-i-1}C) Q (A^{k-i-1}C)^T z \Big]^{1/2} \tag{42}$$

Then, for $z^T = [z^1, z^2, \dots, z^n]$,

$$d(X_{tk}^e) = \max_{\sum_{j=1}^{n} (z^j)^2 \le 1} 2 \sum_{i=0}^{k-1} \sqrt{2k-2i-1} \frac{t^{k-i}}{(k-i)!} \Big[\sum_{j=1}^{n} z^j \sum_{m=1}^{n} q_{jm}^{ki} z^m \Big]^{1/2} \tag{43}$$

where q_{jm}^{ki} are elements of the matrix $(A^{k-i-1}C)Q(A^{k-i-1}C)^T$.

The Kuhn-Tucker conditions lead to

$$z^p = \frac{\displaystyle\sum_{i=1}^{k-1} \sqrt{2k-2i-1} \frac{t^{k-i}}{(k-i)!} \sum_{m=1}^{n} z^m (q_{mp}^{ki} + q_{pm}^{ki})}{\displaystyle\Big\{ \sum_{j=1}^{n} \Big[\sum_{i=1}^{k-1} \sqrt{2k-2i-1} \frac{t^{k-i}}{(k-i)!} \sum_{m=1}^{n} z^m (q_{mj}^{ki} + q_{jm}^{ki}) \Big]^2 \Big\}^{1/2}} \tag{44}$$

for $p = 1, 2, \dots, n$.

The matrix $(A^{k-i-1}C)Q(A^{k-i-1}C)^T$ is symmetric, which implies

$$z^p = \frac{\displaystyle\sum_{i=1}^{k-1} \sqrt{2k-2i-1} \frac{t^{k-i}}{(k-i)!} \sum_{m=1}^{n} z^m q_{pm}^{ki}}{\displaystyle\Big\{ \sum_{j=1}^{n} \Big[\sum_{i=1}^{k-1} \sqrt{2k-2i-1} \frac{t^{k-i}}{(k-i)!} \sum_{m=1}^{n} z^m q_{pm}^{ki} \Big]^2 \Big\}^{1/2}} \tag{45}$$

In contrast to the previous section, solution to (45) in a general case is cumbersome. In particular cases, however, for a small n, the presented method of computing $d(X^e_{tk})$ may give explicit results.

Example 2. We consider, as previously, the system (1) with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \qquad C \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$x_1(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad t \in [0,T]; \qquad x_2(k) = \begin{bmatrix} k \\ 1 \end{bmatrix}, \quad k \in [0,N]$$

$$W_e = \left\{ w : w^T w \le 0.01 \right\}, \qquad m = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}$$

For $t = 1$ and $k = 2$, according to (30), the support function of the reachability set X^e_{tk} has the form

$$h(z \mid X^e_{12}) = z^T \begin{bmatrix} 3 \\ 5 \end{bmatrix} + \frac{\sqrt{3}}{20} \left(z^T \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} z \right)^{1/2} + \frac{1}{10} (z^T z)^{1/2} \tag{46}$$

Then

$$d(X^e_{12}) = \max_{\|z\| \le 1} \left[\frac{\sqrt{3}}{10} \left(z^T \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} z \right)^{1/2} + \frac{2}{10} (z^T z)^{1/2} \right] \tag{47}$$

Here $z^T = [z^1, z^2]$ and

$$d(X^e_{12}) = \max_{(z^1)^2 + (z^2)^2 \le 1} 0.1 \left\{ \sqrt{3} \Big[(z^1)^2 + 4(z^2)^2 \Big]^{1/2} + 2 \Big[(z^1)^2 + (z^2)^2 \Big]^{1/2} \right\} \tag{48}$$

In order to find this maximum, we use the Kuhn-Tucker conditions with the Lagrangian

$$\begin{aligned} L(z^1, z^2, \lambda) &= -\frac{\sqrt{3}}{10} \Big[(z^1)^2 + 4(z^2)^2 \Big]^{1/2} - \frac{2}{10} \Big[(z^1)^2 + (z^2)^2 \Big]^{1/2} \\ &\quad + \lambda \Big[(z^1)^2 + (z^2)^2 - 1 \Big] \end{aligned} \tag{49}$$

Consequently, we obtain the following equations:

$$\begin{cases} z^1 \left[\dfrac{\sqrt{3}}{10\sqrt{(z^1)^2 + 4(z^2)^2}} + \dfrac{2}{10\sqrt{(z^1)^2 + (z^2)^2}} - 2\lambda \right] = 0 \\ z^2 \left[\dfrac{2\sqrt{3}}{5} \dfrac{1}{\sqrt{(z^1)^2 + 4(z^2)^2}} + \dfrac{1}{5} \dfrac{1}{\sqrt{(z^1)^2 + (z^2)^2}} - 2\lambda \right] = 0 \\ \lambda \Big[(z^1)^2 + (z^2)^2 - 1 \Big] = 0 \end{cases} \tag{50}$$

From (50) it follows that a maximum in (48) is attained for $z^1 = 0$ and $z^2 = \pm 1$. Finally, $d(X^e_{12}) = 0.2(\sqrt{3} + 1) \approx 0.5464$. ♦

4. Reachability Set for 2-D Continuous-Discrete Linear Control System with Disturbances

Consider a 2-D continuous-discrete control system described by the equation

$$\dot{x}(t,k+1) = Ax(t,k) + Bu(t,k) + Cw(t,k), \quad t \in [0,T], \quad k \in [0,N] \tag{51}$$

where $u(t,k) \in \mathbb{R}^p$ is a control vector, $w(t,k) \in \mathbb{R}^q$ denotes a disturbance, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{n\times q}$ are real matrices. The boundary conditions for (51) have the form (2). Moreover, assume that $u(t,k) \in U$.

Definition 4. The *reachablity set* $X^r_{tk}(u)$ $(X^e_{tk}(u))$ is the set of all possible states of the system (51) at the moment (t,k) with boundary conditions (2) for all possible disturbances from the set W_r (W_e) when $u \in U$.

Theorem 6. *The support functions of the reachability sets of (51) with the corresponding boundary conditions are given by*

$$\begin{aligned} h\big(z \mid X^r_{tk}(u)\big) &= z^T A^k \int_0^t \frac{(t-\tau)^{k-1}}{(k-1)!} x_1(\tau)\,\mathrm{d}\tau \\ &\quad + \sum_{i=0}^{k-1} \Bigg[\frac{t^i}{i!} z^T A^i x_2(k-i) + z^T A^{k-i-1} B \int_0^t \frac{(t-\tau)^{k-i-1}}{(k-i-1)!} u(\tau,i)\,\mathrm{d}\tau \\ &\quad + \frac{t^{k-i}}{(k-i)!} \Big(z^T A^{k-i-1} C w_c + \frac{1}{2} \sum_{p=1}^{n} |z^p| \Big| \sum_{j=1}^{q} a^{ki}_{pj} \Delta w^j \Big| \Big) \Bigg] \end{aligned} \tag{52}$$

where a^{ki}_{pj} *are elements of the matrix* $A^{k-i-1}C$, *and*

$$\begin{aligned} h\big(z \mid X^e_{tk}(u)\big) &= z^T A^k \int_0^t \frac{(t-\tau)^{k-1}}{(k-1)!} x_1(\tau)\,\mathrm{d}\tau \\ &\quad + \sum_{i=0}^{k-1} \Bigg\{ \frac{t^i}{i!} z^T A^i x_2(k-i) + z^T A^{k-i-1} B \int_0^t \frac{(t-\tau)^{k-i-1}}{(k-i-1)!} u(\tau,i)\,\mathrm{d}\tau \\ &\quad + \frac{t^{k-i}}{(k-i)!} \Big[z^T A^{k-i-1} C m + \sqrt{2k-2i-1} \\ &\quad \times \Big(z^T (A^{k-i-1}C) Q (A^{k-i-1}C)^T z \Big)^{1/2} \Big] \Bigg\}, \quad z \in \mathbb{R}^n \end{aligned} \tag{53}$$

Proof. The reachability sets $X^r_{tk}(u)$ and $X^e_{tk}(u)$ may be expressed as the following sums (Krasoń, 1984; Kurzhanski, 1977):

$$X^r_{tk}(u) = p(t,k) + X^r_{tk}, \quad X^e_{tk}(u) = p(t,k) + X^e_{tk} \tag{54}$$

where $p(t,k)$ is a solution to the system

$$\dot{p}(t,k+1) = Ap(t,k) + Bu(t,k) \tag{55}$$

with the boundary conditions

$$p(t,0) = 0, \quad p(0,k) = 0 \tag{56}$$

X_{tk}^r and X_{tk}^e are the reachability sets for the system

$$\dot{x}_v(t,k+1) = Ax_v(t,k) + Cw(t,k) \tag{57}$$

when $w(t,k) \in W_r$ or $w(t,k) \in W_e$ with boundary conditions (2).

Taking into account the properties of the support function and (54), we can write

$$\begin{cases} h\big(z \mid X_{tk}^r(u)\big) = z^T p(t,k) + h\big(z \mid X_{tk}^r\big) \\ h\big(z \mid X_{tk}^e(u)\big) = z^T p(t,k) + h\big(z \mid X_{tk}^e\big) \end{cases} \tag{58}$$

for $z \in \mathbb{R}^n$. According to (Kaczorek, 1994; Kaczorek and Stajniak, 1994), the solution $p(t,k)$ takes the form

$$p(t,k) = \sum_{i=0}^{k-1} A^{k-i-1} B \int_0^t \frac{(t-\tau)^{k-i-1}}{(k-i-1)!} u(\tau,i)\,\mathrm{d}\tau \tag{59}$$

From Theorems 3 and 5, and the formulae (58), we obtain (52) and (53). ∎

Theorem 7. *The diameter of the reachability set of the system (51) with boundary conditions (2) for disturbances from W_r or W_e is equal to the diameter of the reachability set of the system (1) with the same boundary conditions and for the same disturbances.*

Proof. From (15) and (58) we have

$$d\big(X_{tk}^r(u)\big) = \max_{\|z\|\le 1} \Big\{ \Big[z^T p(t,k) + h\big(z \mid X_{tk}^r\big)\Big] + \Big[-z^T p(t,k) + h\big(-z \mid X_{tk}^r\big)\Big] \Big\} = d(X_{tk}^r) \tag{60}$$

Similarly,

$$d\big(X_{tk}^e(u)\big) = d(X_{tk}^e) \tag{61}$$

∎

5. Concluding Remarks

Definitions 1 and 3 of the reachability sets X_{tk}^r and X_{tk}^e for the 2-D continuous-discrete linear system (1) with disturbances limited to a rectangle or an ellipsoid in $\mathbb{R}^q$ for known boundary conditions (2) have been presented. The corresponding support functions of the sets X_{tk}^r and X_{tk}^e have also been established (Theorems 4 and 5). From (25) and (42) it follows that the diameters of X_{tk}^r and X_{tk}^e depend neither on the boundary conditions nor on the position of the centre of the rectangle W_r or the ellipsoid W_e. On the other hand, $d(X_{tk}^r)$ depends on the lengths of the edges Δw^j of the rectangle, and $d(X_{tk}^e)$ depends on the directions and the lengths of the axes of the ellipsoid W_e.

The analysis of both the presented examples shows that for the same system (1), but with different disturbances, the diameters of reachability sets are different.

There is no direct dependence of the type: if disturbances belong to the set of a smaller diameter, then the reachability set has a smaller diameter. From Theorem 7 it follows that the values of control u in the 2-D continuous-discrete linear control system (51) have no influence on the diameters of the reachability sets $X_{tk}^r(u)$ and $X_{tk}^e(u)$.

References

Barmish B.R., Lin Y.H. and Sankaran J. (1978): *Reachability of targets subject to disturbances bounded by convex sets.* — Int. J. Control, Vol.28, No.5, pp.771–783.

Dubnicki W. and Zorychta K. (1972): *Methods of Convex Programming.* — Warsaw: Warsaw University Publish, (in Polish).

Kaczorek T. (1994): *Singular 2-D continuous-discrete linear systems.* — Bull. Pol. Acad. Sci. Techn. Sci., Vol.42, No.3, pp.417–422.

Kaczorek T. and Stajniak A. (1994): *Local reachability and minimum energy control of 2-D continuos-discrete linear systems.* — Proc. 10-th Int. Conf. *Systems Engineering*, Conventry, U.K, pp.558–565.

Kaczorek T. (1995): *Controllability and minimum energy control of 2-D continuos-discrete linear systems.* — Appl. Math. Comp. Sci., Vol.5, No.1, pp.5–21

Krasoń E. (1984): *One differential game with delay under incomplete information.* — Ph.D. Thesis, Warsaw University of Technology, (in Polish).

Kurzhanski A.B. (1977): *Control and Observation under Uncertainty Conditions.* — Nauka, Moscow, (in Russian).

Rockafellar R.T. (1972): *Convex analysis.* — Princeton: Princeton University Press.

Schweppe F.C. (1973): *Uncertain Dynamic Systems.* — Englewood Cliffs: Prentice Hall.

Stajniak A. (1995): *Analysis of 2-D continuous-discrete systems.* — Ph.D. Thesis, Warsaw University of Technology, (in Polish).