*Appl. Math. and Comp. Sci., 1997, Vol.7, No.4, 881–905*

# ON 2-D POLYNOMIALS AND DISCRETE-CONTINUOUS SYSTEMS

ZDZISŁAW W. TRZASKA*

The paper deals with 2-D polynomials $Q(x,y)$ and $R(x,y)$ having real coefficients, which are generated by modified first and second numerical triangles (FMNT) and (SMNT), respectively. It has been proven that $n$-th-degree polynomials $Q_n(x,y)$ and $R_n(x,y)$ can be easily determined as solutions of third order recurrences with coefficients depending on $x$ and $y$ with appropriate initial conditions. Suitable expressions are involved and simple formulae are established to check links between $Q(x,y)$ and $R(x,y)$. Problems connected to the efficiency of the 2-D polynomials in solutions to practical problems are also considered. Examples illustrating possible applications of the presented approach are also demonstrated.

## 1. Introduction

Some recent applications of dynamic models in atmospheric sciences, biology, ecology, geology, economics and business sciences can be found in (Cronin, 1976; Jean, 1986; Liu and Sutinen, 1985; Majda, 1996; Reithofer, 1996; Tichonov and Samarskii, 1974). This general research field relates many topics in a beautiful, sometimes surprising, yet natural way. Suitable mathematical descriptions and effective solutions for many real world phenomena are related to one another. As is well-known, the instantaneous state of many physical plants usually depends on several independent variables and is described by one or several state functions with these variables as their arguments. The state functions, if sufficiently smooth, can be represented by polynomials, in some range and within some accuracy. Thus the state of any plant, no matter how it is complicated, involves the study of polynomials.

On the other hand, power polynomials play a fundamental role in pure mathematics because of their fascinating properties and numerous applications in many fields of applied mathematics (Ferri *et al.*, 1991; Klamka, 1997; Lahr, 1986; Skorobogat'ko, 1983; Trzaska, 1993b; 1996a). Moreover, they are quite often used in control theory to study such important properties of dynamical systems as stability, controllability, singularity, detectability, etc. (Barnet, 1983; Willems, 1989). The classical 1-D dynamical systems and polynomials have a rich and well-developed theory (Acosta-de-Orozc and Gomez-Calderon, 1996; Gill, 1977; Marden, 1966; Schinzel, 1982), so that many of classical, e.g. Chebyshev, Legendre and Hermite, polynomials are valuable tools in approximating complicated functions, as well as solutions to differential,

* Department of Electrical Engineering, Warsaw University of Technology, 00–661 Warsaw, Poland, e-mail: trz@iem.pw.edu.pl.

integral and difference equations. However, many other domains of the human activity explore multivariable power polynomials to reveal phenomena in our environment and to design systems with desired properties as, for instance, the synthesis of 2-D filters with appropriate frequency bands (Huang, 1972; Kaczorek, 1985; Kuchminskaya, 1995; Lewis, 1992; Morf *et al.*, 1977; Skorobogat'ko, 1983).

A new area of the theory of dynamic systems was opened in the 1970s through a series of papers (Attasi, 1973; Fornasini and Marchesini, 1976; 1978); Huang, 1972; Morf *et al.*, 1977; Roesser, 1976) and a monograph (Kaczorek, 1985) where the so-called 2-D systems have been involved, developed and well-justified as a new and suitable approach to a wide class of discrete and/or discrete-continuous systems. The study of these mathematical objects has found an established place in modern treatments of various disciplines, but the world of 2-D systems always provides many opportunities for new and exciting discoveries that are revealed by looking at them closely (Beauzamy *et al.*, 1995; Idczak, 1996; Kaczorek, 1988; Marszałek and Kekkeris, 1989; Rocha and Willems, 1991). The up-to-date research indicates that 2-D systems share many similar properties with 1-D ones, yet some differences arise due to the discrete nature of the independent variables, adding an intriguing flavor to the studies.

This paper aims at providing a comprehensive explanation of some new results concerning 2-D polynomials introduced via first and second modified triangles FMNT and SMNT, respectively, that have been involved by suitable combinatoric investigations (Edwards, 1987; Jean, 1986; Kuchminskaya, 1995; Trzaska, 1993a; 1995). As is well-known, 2-D polynomials are basic objects not only in pure mathematics (Beauzamy *et al.*, 1995; Gałkowski, 1997), but in many other neighbouring domains, especially in applied and industrial mathematics, as well as in control systems theory (Gałkowski *et al.*, 1997; Huang, 1972; Kaczorek, 1985; Lewis, 1992; Trzaska, 1996b; 1996c; 1996d; 1997). Here the 2-D polynomials are considered in the context of modified numerical triangles and their links with solutions of third and second order difference equations. One hope is that it will serve as an introduction for non-specialists in the field, and in that spirit, proofs are provided for all of the results. On the other hand, new results of varying degrees of importance do appear throughout, as well as some new extensions of known results in 1-D domain to two dimensions, so the hope is that researches in the concerned field will also find something interesting in the paper. Illustrative examples are given along the presentation and examples sometimes will serve instead of formal proofs.

The aim of this paper is to present some fundamental properties and characteristics of the $n$-th-degree polynomials $Q_n(x,y)$ and $R_n(x,y)$ introduced via FMNT and SMNT and links between them, as well as with solutions of some third and second order difference equations under appropriate initial conditions.

The paper is organized as follows. In the next section, we shall present a short review of relations and properties concerning modified numerical triangles FMNT and SMNT from the point of view of generation of 2-D polynomials $Q_n(x,y)$ and $R_n(x,y)$. In Section 3, we shall study links between these polynomials. Third-order difference equations and the relationships between their solutions and polynomials $Q_n(x,y)$ and $R_n(x,y)$ are presented in Section 4. An illustration of possible applications of the presented relationships is included in Section 5. Section 6 is devoted to a final discussion of the results and it also contains concluding remarks.

## 2. Two-Variable Polynomials and Modified Numerical Triangles

A rapidly growing interest in the studies of various forms of 2-D polynomials, also called two-variable polynomials, has been observed recently because of their numerous applications not only in the field of pure mathematics, but also in applied and/or industrial sciences, especially in the modelling and numerical analysis of physical plants like electric distributed-parameter systems, as well as in real-domain image processing and other discrete-continuous processes (Marszałek and Kekkeris, 1989; Roesser, 1976; Trzaska, 1996b). As the mathematical basis to solve various problems appearing in the studies of these systems we have the theory of recurrence equations, two-dimensional transformations and their generalizations, as well as various classes of 2-D mathematical models (Kaczorek, 1988; 1994; Pandolfi, 1984; Trzaska, 1996b; 1996c; 1996d).

In this section, we shall be concerned with power polynomials $Q_n(x,y)$ and $R_n(x,y)$ in two independent variables $x$ and $y$ and coefficients defined via modified numerical triangles FMNT and SMNT. Here we restrict ourselves to independent variables $x$ and $y$ on the real plane $\mathbb{R}^2$. Moreover, in the sequel we shall limit our attention to a non-negative subscript $n \geq 0$ only. The case of a negative subscript $n < 0$ can be treated similarly.

**Definition 1.** Monic non-zero 2-D polynomials $Q_n(x,y)$ in $\mathbb{R}^2$ which generate the first modified numerical triangle, FMNT, are defined here by the following recurrence:

$$Q_{n+2}(x,y) = (x+y)\Big[2Q_{n+1}(x,y) - (x+y)Q_n(x,y)\Big] + x^2yQ_{n-1}(x,y), \quad n = 1,2,3,\dots \tag{1}$$

with $Q_0(x,y) = 1$, $Q_1(x,y) = x+y$ and $Q_2(x,y) = x^2+3xy+y^2$ as initial elements.

By direct inspection of the above recurrence we have

$$\begin{aligned} Q_0(x,y) &= 1 \\ Q_1(x,y) &= x+y \\ Q_2(x,y) &= x^2+3xy+y^2 \\ Q_3(x,y) &= x^3+6x^2y+5xy^2+y^3 \\ Q_4(x,y) &= x^4+10x^3y+15x^2y^2+7xy^3+y^4 \\ Q_5(x,y) &= x^5+15x^4y+35x^3y^2+28x^2y^3+9xy^4+y^5 \\ &\dots\dots\dots \end{aligned} \tag{2}$$

Thus, taking into account the specific form of $Q_n(x,y)$, we can write

$$Q_n(x,y) = \sum_{k=0}^{n} a_{n,k}x^{n-k}y^k, \quad n = 0,1,2,\dots \tag{3}$$

where the coefficients $a_{n,k}$, $n = 0, 1, 2, \ldots$, $0 \le k \le n$, fulfil the relation

$$a_{n,k} = 2a_{n-1,k} + a_{n-1,k-1} - a_{n-2,k}, \quad n = 0, 1, 2, \ldots, \quad 0 \le k \le n \tag{4}$$

with $a_{0,0} = 1$ and $a_{1,0} = 1$ as initial values, and $a_{n,k} = 0$ for $k > n$ and $n < 0$.

Based on (4) we can construct the FMNT which is shown in Table 1.

Table 1. First modified numerical triangle.

| $r \setminus m$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | ... |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | |
| 1 | 1 | 1 | | | | | | |
| 2 | 1 | 3 | 1 | | | | | |
| 3 | 1 | 6 | 5 | 1 | | | | |
| 4 | 1 | 10 | 15 | 7 | 1 | | | |
| 5 | 1 | 15 | 35 | 28 | 9 | 1 | | |
| 6 | 1 | 21 | 70 | 84 | 45 | 11 | 1 | |
| ... | ... | ... | ... | ... | ... | ... | | |

Following the above line of reasoning, we can define a set of the second monic 2-D polynomials $R_n(x, y)$ with $n = 0, 1, 2, \ldots$.

**Definition 2.** Non-monic non-zero 2-D polynomials $R_n(x, y)$ in $\mathbb{R}^2$ which generate the second modified numerical triangle SMNT are defined here by the following recurrence:

$$R_{n+2}(x, y) = (x + y)\Big[2R_{n+1}(x, y) - (x + y)R_n(x, y)\Big] + x^2 y R_{n-1}(x, y), \quad n = 1, 2, 3, \ldots \tag{5}$$

with $R_0(x, y) = 0$, $R_1(x, y) = 1$ and $R_2(x, y) = 2x + y$ as the initial elements.

By direct inspection of the above recurrence we obtain

$$\begin{aligned}
R_0(x, y) &= 0 \\
R_1(x, y) &= 1 \\
R_2(x, y) &= 2x + y \\
R_3(x, y) &= 3x^2 + 4xy + y^2 \\
R_4(x, y) &= 4x^3 + 10x^2 y + 6xy^2 + y^3 \\
R_5(x, y) &= 5x^4 + 20x^3 y + 21x^2 y^2 + 8xy^3 + y^4 \\
&\ldots\ldots\ldots
\end{aligned} \tag{6}$$

Now it is evident that we can express the polynomial $R_n(x,y)$ as

$$R_n(x,y) = \sum_{r=0}^{n-1} b_{n,r} x^{n-r-1} y^r, \quad n = 0,1,2,\ldots \tag{7}$$

where the coefficients $b_{n,r}$, $n = 0,1,2,\ldots$, $0 \le r \le n$, are defined by the recurrence

$$b_{n,r} = 2b_{n-1,r} + b_{n-1,r-1} - b_{n-2,r}, \quad n = 0,1,2,\ldots, \quad 0 \le r \le n \tag{8}$$

with $b_{0,0} = 0$ and $b_{1,0} = 1$ as the initial values, $b_{n,n} = 0$, $n = 0,1,2,\ldots$, and $b_{n,r} = 0$ for $r \ge n$ and $n < 0$.

Thus, on the basis of the above expressions, we can construct the second modified numerical triangle SMNT which is presented in Table 2.

Table 2. Second modified numerical triangle.

| $l \backslash d$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | ... |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | | | | | | | |
| 1 | 1 | | | | | | | |
| 2 | 2 | 1 | | | | | | |
| 3 | 3 | 4 | 1 | | | | | |
| 4 | 4 | 10 | 6 | 1 | | | | |
| 5 | 5 | 20 | 21 | 8 | 1 | | | |
| 6 | 6 | 35 | 56 | 36 | 10 | 1 | | |
| ... | ... | ... | ... | ... | ... | ... | | |

Observe that formally both FMNT and SMNT are apparently similar to the classical Pascal triangle (Edwards, 1987; Ferri *et al.*, 1991; Lahr, 1986; Trzaska, 1993a; 1993b), but their elements cannot be evaluated directly by applying the rule corresponding to the classical Pascal triangle. They must be computed in accordance with recurrences (4) and (8), respectively. It is also interesting to emphasize that the sum of all numbers in a row of FMNT or SMNT equals $f_{2n}$ or $f_{2n-1}$, respectively, with $n = 0,1,2,\ldots$, i.e. they are equal to successive elements, with even or odd indices, respectively, of the Fibonacci sequence (Ferri *et al.*, 1991; Lahr, 1986; Trzaska, 1995; 1996a)

$$f_{n+2} = f_{n+1} + f_n, \quad n = 0,1,2,\ldots \tag{9}$$

with $f_0 = 1$ and $f_1 = 1$ as the initial values.

Thus it is easy to prove that the sum of all elements in FMNT gives the sum of Fibonacci numbers with even indices while in the case of SMNT we have the sum of all Fibonacci numbers with odd indices, so in consequence, if we add simultaneously all elements in these two triangles, we can evaluate the sum of all Fibonacci numbers, i.e. with even and odd indices from 0 to $n$, at the same time.

Let us now present some of the most useful characteristics of polynomials $Q_n(x,y)$ and $R_n(x,y)$. First, we consider specific forms of these polynomials with respect to both independent variables $x$ and $y$. Observe that all the terms of a given polynomial $Q_k(x,y)$ have, with respect to both the variables, the same degrees $k$, and in the case of a polynomial $R_k(x,y)$ all its terms also possess the same degree with respect to both the variables, but it is equal to $k-1$.

Second, it is evident from expressions (3), (4), (7) and (8) that for $x,y \in (0,\infty)$ polynomials $Q_k(x,y)$ and $R_k(x,y)$, $k=0,1,2,\ldots$, are nonnegative-definite, i.e. $Q_n(x,y) \geq 0$ and $R_n(x,y) \geq 0$ for $x,y \geq 0$. For negative $x$ and $y$ the polynomials $Q_n(x,y)$ of even degrees and the polynomials $R_k(x,y)$ of odd degrees are positive-definite, but those others (of odd and even degrees, respectively) are negative-definite. These facts are illustrated in Fig. 1 in the case of $Q_4(x,y)$ and $Q_5(x,y)$. If $x$ and $y$ have opposite signs, then the definiteness of the polynomials $Q_n(x,y)$ and $R_n(x,y)$ is much more complicated and it is left for future studies.

Further, introducing normalized values of the polynomials $Q_k(x,y)$ and $R_k(x,y)$ with respect to $x^k$, we get the so-called normalized polynomials $Q_k^{'}(q)$ and $R_k^{'}(q)$ in a normalized independent variable $q = y/x$. They take the following forms:

$$\begin{aligned} Q_0^{'}(q) &= 1 \\ Q_1^{'}(q) &= 1+q \\ Q_2^{'}(q) &= 1+3q+q^2 \\ Q_3^{'}(q) &= 1+6q+5q^2+q^3 \\ Q_4^{'}(q) &= 1+10q+15q^2+7q^3+q^4 \\ Q_5^{'}(q) &= 1+15q+35q^2+28q^3+9q^4+q^5 \\ &\cdots\cdots\cdots \end{aligned} \tag{10}$$

and

$$\begin{aligned} R_0^{'}(q) &= 0 \\ R_1^{'}(q) &= 1 \\ R_2^{'}(q) &= 2+q \\ R_3^{'}(q) &= 3+4q+q^2 \\ R_4^{'}(q) &= 4+10q+6q^2+q^3 \\ R_5^{'}(q) &= 5+20q+21q^2+8q^3+q^4 \\ &\cdots\cdots\cdots \end{aligned} \tag{11}$$

It must be underlined that polynomials (10) and (11) depend seemingly only on one independent variable $q$, but in fact $q$ depends on both $x$ and $y$. The above polynomials are characterized by a set of very interesting properties. Many details in this direction can be found in (Trzaska, 1993a; 1993b). For the sake of reasonable space limit, we present here only one of them. It employs specific zeros of these

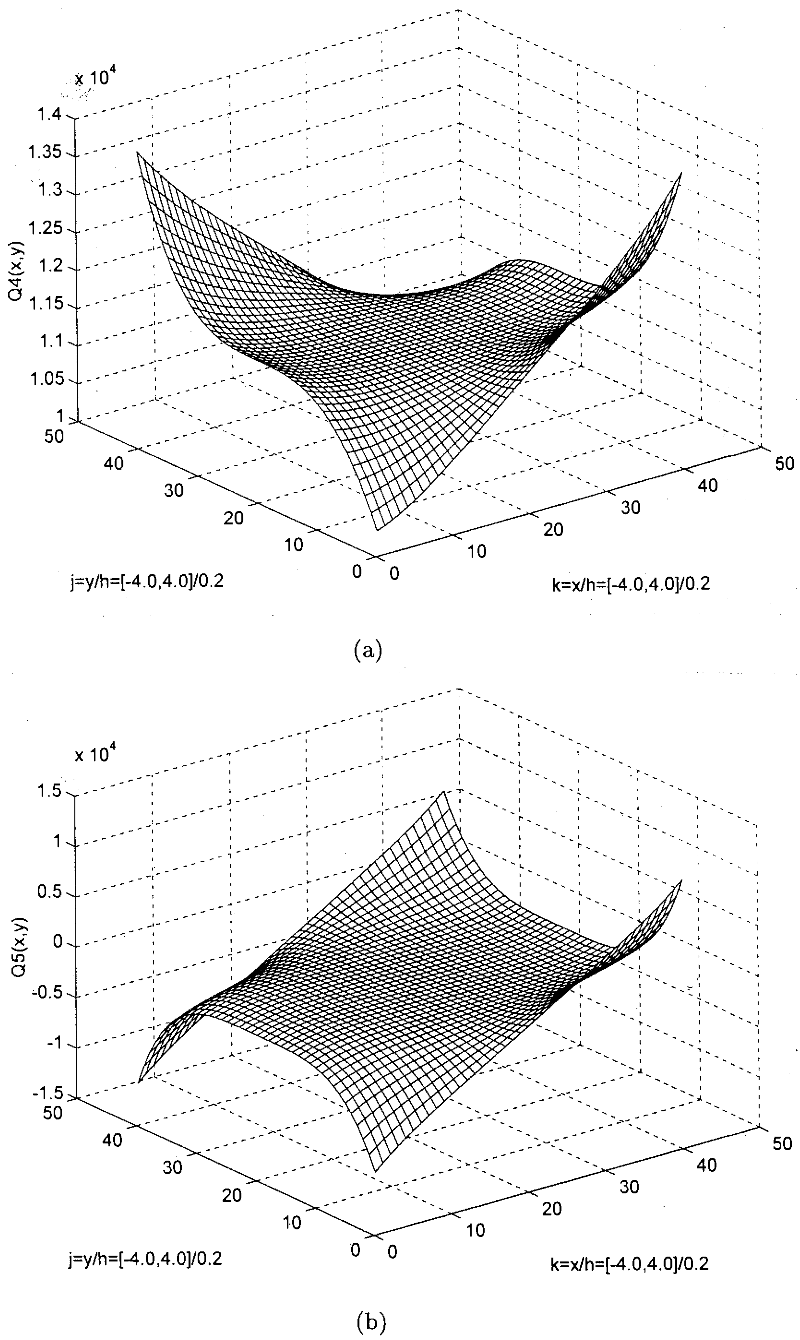


Fig. 1. Distributions of the polynomial values in the plane $(x, y)$: (a) polynomial $Q_4(x, y)$, (b) polynomial $Q_5(x, y)$.

polynomials. For $Q_n'(q) = 0$ we have

$$q_{n,k} = -4\cos^2\left(\frac{k\pi}{2n+1}\right), \quad k = 1, 2, \dots, n \tag{12}$$

as its zeros. But for $R_n'(q) = 0$ we get

$$q_{n,j} = -4\sin^2\left(\frac{j\pi}{2n}\right), \quad j = 1, 2, \dots, n-1 \tag{13}$$

as the corresponding zeros.

It is now evident that all the zeros of the above-mentioned polynomials lie in the segment $(-4, 0)$ and the zeros of $Q_n'(q)$ are different from those of $R_n'(q)$. Thus these polynomials are relatively prime (Barnet, 1983; Beauzamy *et al.*, 1995; Kaczorek, 1994). Also observe that if $n \to \infty$ and $k + j = n$, then the roots $q_{n,k}$ and $q_{n,j}$ fulfil the equation

$$q_{n,k} + q_{n,j} = -4 \tag{14}$$

Further, leaving instantaneously the particular properties of the numerical triangle elements and returning to the general problem of this section, we are able to express the sets of polynomials $Q_n(x, y)$ and $R_n(x, y)$ for $n = 0, 1, 2, \dots$ in suitable matrix forms. To perform this task, we introduce the following matrices:

$$\boldsymbol{x}_p = \operatorname{diag}\left[x^r\right]_{r=p}^{0} = \begin{bmatrix} x^p & 0 & 0 & \dots & 0 & 0 \\ 0 & x^{p-1} & 0 & \dots & 0 & 0 \\ 0 & 0 & x^{p-2} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & x & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} \tag{15}$$

$$\boldsymbol{x} = \operatorname{block}\Big[\ \left[\boldsymbol{x}_p\right]_{p=0}^{n}\ \ \boldsymbol{0}\Big] \in \mathbb{R}^{(n+1)(n+2)/2\times(n+1)}$$

If, for instance, we set $n = 0, 1$ and $2$, then we have

$$\boldsymbol{x}_0 = [1], \quad \boldsymbol{x}_1 = \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{x}_2 = \begin{bmatrix} x^2 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and from the above matrices we form the following $6 \times 3$ matrix:

$$\boldsymbol{x} = \operatorname{block}\Big[\ \left[\boldsymbol{x}_p\right]_{p=0}^{2}\ \ \boldsymbol{0}\Big] = \begin{bmatrix} 1 & 0 & 0 \\ x & 0 & 0 \\ 0 & 1 & 0 \\ x^2 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Next we construct the following vectors:

$$\begin{cases} \boldsymbol{y} = \begin{bmatrix} 1 & y & \dots & y^n \end{bmatrix}^t \\ \boldsymbol{Q}(x,y) = \begin{bmatrix} Q_0(x,y) & Q_1(x,y) & \dots & Q_n(x,y) \end{bmatrix}^t = \{Q_k(x,y)\}_{k=0}^{n} \\ \boldsymbol{R}(x,y) = \begin{bmatrix} R_1(x,y) & R_2(x,y) & \dots & R_{n+1}(x,y) \end{bmatrix}^t = \{R_k(x,y)\}_{k=1}^{n+1} \end{cases} \tag{16}$$

where the superscript $t$ denotes transposition.

Further we define the lower triangular constant matrices

$$\boldsymbol{M} = \begin{bmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 3 & 1 & & \boldsymbol{O} & & \\ 1 & 6 & 5 & 1 & & & \\ 1 & 10 & 15 & 7 & 1 & & \\ \dots & \dots & \dots & \dots & \dots & & \\ 1 & \dots & \dots & \dots & \dots & \dots & 1 \end{bmatrix}, \quad \boldsymbol{N} = \begin{bmatrix} 1 & & & & & & \\ 2 & 1 & & & & & \\ 3 & 4 & 1 & & & \boldsymbol{O} & \\ 4 & 10 & 6 & 1 & & & \\ 5 & 20 & 21 & 8 & 1 & & \\ \dots & \dots & \dots & \dots & \dots & & \\ n+1 & \dots & \dots & \dots & \dots & \dots & 1 \end{bmatrix} \tag{17}$$

together with the so-called row-shifted matrices

$$\boldsymbol{M}_{\text{rsh}} = [\text{rowshift } \boldsymbol{M}] = \begin{bmatrix} 1 & & & & & & & & & & & & & & \\ 0 & 1 & 1 & & & & & & & \boldsymbol{O} & & & & & \\ 0 & 0 & 0 & 1 & 3 & 1 & & & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 6 & 5 & 1 & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 10 & 15 & 7 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \dots & 1 \end{bmatrix} \tag{18}$$

and

$$\boldsymbol{N}_{\text{rsh}} = [\text{rowshift } \boldsymbol{N}] = \begin{bmatrix} 1 & & & & & & & & & & & & & & \\ 0 & 2 & 1 & & & & & & & \boldsymbol{O} & & & & & \\ 0 & 0 & 0 & 3 & 4 & 1 & & & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 10 & 6 & 1 & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 20 & 21 & 8 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \dots & 1 \end{bmatrix} \tag{19}$$

Then the sets of polynomials $Q_n(x,y)$ and $R_n(x,y)$ can be expressed as follows:

$$\boldsymbol{Q}(x,y) = \boldsymbol{M}_{\text{rsh}}\boldsymbol{xy}, \quad \boldsymbol{R}(x) = \boldsymbol{N}_{\text{rsh}}\boldsymbol{xy} \tag{20}$$

Next, denoting by

$$S=\begin{bmatrix}1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 1 & 1 & & O & & \\ 1 & 1 & 1 & 1 & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 \; 1\end{bmatrix}, \quad S^{-1}=\begin{bmatrix}1 & & & & & & \\ -1 & 1 & & & & & \\ 0 & -1 & 1 & & O & & \\ 0 & 0 & -1 & 1 & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & -1 \; 1\end{bmatrix} \tag{21}$$

the $n \times n$ lower triangular matrix with 1 as its elements on the diagonal and all subdiagonals, and the $n \times n$ lower triangular matrix with only the main diagonal composed of 1 and the first subdiagonal composed of $-1$, respectively, we can easily demonstrate that the matrices $\boldsymbol{M}_{\mathrm{rsh}}$ and $\boldsymbol{N}_{\mathrm{rsh}}$ fulfil the relations

$$\boldsymbol{N}_{\mathrm{rsh}} = (\boldsymbol{S}\boldsymbol{M})_{\mathrm{rsh}}, \qquad \boldsymbol{M}_{\mathrm{rsh}} = \left(\boldsymbol{S}^{-1}\boldsymbol{N}\right)_{\mathrm{rsh}} \tag{22}$$

The following example is given as an illustration.

**Example 1.** Using expresion (20) let us determine simultaneously explicit forms of all 2-D polynomials $Q_n(x,y)$ and $R_n(x,y)$ with successive degrees $n = 0, 1, 2,$ and 3. To perform this task we construct, based on (18) and (19), the corresponding row-shifted matrices $\boldsymbol{M}_{\mathrm{rsh}}$ and $\boldsymbol{N}_{\mathrm{rsh}}$. They take the forms

$$\boldsymbol{M}_{\mathrm{rsh}}=\begin{bmatrix}1 & & & & & & & & & \\ 0 & 1 & 1 & & & & & O & & \\ 0 & 0 & 0 & 1 & 3 & 1 & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 6 & 5 & 1\end{bmatrix}, \quad \boldsymbol{N}_{\mathrm{rsh}}=\begin{bmatrix}1 & & & & & & & & & \\ 0 & 2 & 1 & & & & & & O & \\ 0 & 0 & 0 & 3 & 4 & 1 & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 10 & 6 & 1\end{bmatrix}$$

Next we form the corresponding $(10 \times 4)$ matrix $\boldsymbol{x}$:

$$\boldsymbol{x} = \mathrm{block}\left[\;\left[x_p\right]_{p=0}^{3}\;\; \boldsymbol{0}\right] = \begin{bmatrix}1 & 0 & 0 & 0\\ x & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ x^2 & 0 & 0 & 0\\ 0 & x & 0 & 0\\ 0 & 0 & 1 & 0\\ x^3 & 0 & 0 & 0\\ 0 & x^2 & 0 & 0\\ 0 & 0 & x & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

and the vector

$$\boldsymbol{y} = \left[1 \;\; y \;\; y^2 \;\; y^3\right]^t$$

From (20) we obtain

$$\boldsymbol{Q}(x,y)=\begin{bmatrix}Q_0(x,y) & Q_1(x,y) & Q_2(x,y) & Q_3(x,y)\end{bmatrix}^t=\boldsymbol{M}_{\text{rsh}}\boldsymbol{x}\boldsymbol{y}$$

$$=\begin{bmatrix}1 & & & & & & & & & \\ 0 & 1 & 1 & & & & \boldsymbol{O} & & & \\ 0 & 0 & 0 & 1 & 3 & 1 & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 6 & 5 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & x & 0 & 0\\ 0 & 0 & 1 & 0\\ x^2 & 0 & 0 & 0\\ 0 & x & 0 & 0\\ 0 & 0 & 1 & 0\\ x^3 & 0 & 0 & 0\\ 0 & x^2 & 0 & 0\\ 0 & 0 & x & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\ y\\ y^2\\ y^3\end{bmatrix}$$

$$=\begin{bmatrix}1\\ x+y\\ x^2+3xy+y^2\\ x^3+10x^2y+5xy^2+y^3\end{bmatrix}$$

Similarly, we form the vector of polynomials $R_n(x,y)$, $n=1,2,3$, and 4, namely

$$\boldsymbol{R}(x,y)=\begin{bmatrix}R_1(x,y) & R_2(x,y) & R_3(x,y) & R_4(x,y)\end{bmatrix}^t=\boldsymbol{N}_{\text{rsh}}\boldsymbol{x}\boldsymbol{y}$$

$$=\begin{bmatrix}1 & & & & & & & & & \\ 0 & 2 & 1 & & & & \boldsymbol{O} & & & \\ 0 & 0 & 0 & 3 & 4 & 1 & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 10 & 6 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ x & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ x^2 & 0 & 0 & 0\\ 0 & x & 0 & 0\\ 0 & 0 & 1 & 0\\ x^3 & 0 & 0 & 0\\ 0 & x^2 & 0 & 0\\ 0 & 0 & x & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\ y\\ y^2\\ y^3\end{bmatrix}$$

$$=\begin{bmatrix}1\\ 2x+y\\ 3x^2+4xy+y^2\\ 4x^3+10x^2y+6xy^2+y^3\end{bmatrix}$$

♦

Observe that the above relations can be considered as an alternative approach to the foregoing recurrence formulae used to determine the 2-D polynomials $Q_n(x,y)$ and $R_n(x,y)$. The advantages of each of these approaches in particular applications depend on additional information supplied to a researcher.

Also note that the above relations involve some important simplifications when applied to dynamical ladder networks and other discrete-continuous 2-D systems. We shall present such problems in the next sections. In particular, we shall discuss new relations leading to other recurrence definitions. Furthermore, we shall consider the stability of the studied polynomials and we shall try to locate the polynomial zeros on the complex plane.

## 3. Fundamental Properties of 2-D Polynomials

Although 2-D polynomials have been extensively studied for a dozen years, they remain a fascinating area for exploration and still new aspects seem to exists, which can be revealed by looking at them closely. Many standard techniques from 1-D polynomials have been adopted to the study of 2-D polynomials, but due to the differences some new approaches have also been developed (Gałkowski, 1997; Trzaska, 1996a).

The number of applications of 2-D polynomials is very large, but it must be underlined that the two-variable polynomial is a notion which plays an important role not only in mathematics, but also in many neighbouring disciplines and in other research domains such as criptology, signal processing, ecology, etc. (Huang, 1972; Jean, 1986; Lions, 1996; Majda, 1996; Marszałek and Kekkeris, 1989; Trzaska, 1996d). From now on, some significant expressions start appearing. One of them concerns fundamental links between polynomials $Q_n(x,y)$ and $R_n(x,y)$. Next questions are focused on the zeros of these polynomials, as well as on alternative recurrence relations with respect to those presented in the preceding section.

### 3.1. Links between Polynomials $Q_n(x,y)$ and $R_n(x,y)$

To establish some links between polynomials $Q_n(x,y)$ and $R_n(x,y)$, we proceed as follows. First, for the clarity of presentation, we begin with an example.

**Example 2.** Applying the defined expressions for particular polynomials on the right-hand side, let us calculate the explicit form of the left-hand side for the following expression:

$$D_{3,2}(x,y) = Q_3(x,y)R_2(x,y) - R_3(x,y)Q_2(x,y)$$

To solve the problem, we use appropriate expressions which determine particular forms of the corresponding polynomials on the LHS and after some manipulations we obtain

$$\begin{aligned} RHS = &(x^3 + 6x^2y + 5xy^2 + y^3)(2x + y) \\ &- (3x^2 + 4xy + y^2)(x^2 + 3xy + y^2) = -x^4 \end{aligned}$$

Because the LHS must be the same, we have

$$D_{3,2}(x,y) = -x^4$$

♦

Generalizing this particular result, we formulate the following theorem.

**Theorem 1.** *Polynomials* $Q_n(x,y)$ *and* $R_n(x,y), n = 1,2,\dots$ *fulfil the equation*

$$Q_n(x,y)R_{n-1}(x,y) - R_n(x,y)Q_{n-1}(x,y) = -x^{2(n-1)} \tag{23}$$

*Proof.* It employs the mathematical induction and is straightforward. ■

**Theorem 2.** *Polynomials* $Q_n(x,y)$ *and* $R_n(x,y)$, $n = 1,2,\dots$ *fulfil the following system of difference equations:*

$$\begin{cases} Q_{n+1}(x,y) = (x+y)Q_n(x,y) + xyR_n(x,y) \\ R_{n+1}(x,y) = (x+y)R_n(x,y) + xQ_{n-1}(x,y) \end{cases} \tag{24}$$

*Proof.* We apply first of all expressions (3) and (7) with (4) and (8), respectively. Then we write

$$Q_{n+1}(x,y) = \sum_{l=0}^{n+1} a_{n+1,l}x^{n+1-l}y^l, \quad R_{n+1}(x,y) = \sum_{l=0}^{n} b_{n+1,l}x^{n-l}y^l \tag{25}$$

Next, making use of the results given in (Trzaska, 1993a; 1993b), we can deduce the following identities:

$$\begin{cases} a_{n+1,l} = 2a_{n,l} + a_{n,l-1} - a_{n-1,l} \\ b_{n+1,l} = 2b_{n,l} + b_{n,l-1} - b_{n-1,l}, \quad b_{n,l} = a_{n,l+1} - a_{n-1,l+1} \end{cases} \tag{26}$$

For instance, let $n = 3$ and $l = 2$. Then from FMNT and SMNT we have

$$a_{4,2} = 2a_{3,2} + a_{3,1} - a_{2,1} = 2\cdot 6 + 6 - 3 = 15$$

$$b_{4,2} = 2b_{3,2} + b_{3,1} - b_{2,1} = 2\cdot 4 + 3 - 1 = 10$$

$$b_{3,2} = a_{3,3} - a_{2,3} = 1 - 0 = 1$$

The result can be easily checked by using Tables 1 and 2, respectively.

The rest of the proof will be limited to the first of equations (25). For the other, we proceed analogously.

Substituting (26) into (25) gives

$$Q_{n+1}(x,y) = \sum_{l=0}^{n+1}(2a_{n,l} + a_{n,l} - a_{n-1,l})x^{n+1-l}y^l$$

$$= x\sum_{l=0}^{n} a_{n,l}x^{n-l}y^l + y\sum_{l=0}^{n} a_{n,l}x^{n-l}y^l + xy\sum_{l=0}^{n-1}(a_{n,l+1} - a_{n-1,l+1})x^{n-1-l}y^l$$

$$= x\sum_{l=0}^{n} a_{n,l}x^{n-l}y^l + y\sum_{l=0}^{n} a_{n,l}x^{n-l}y^l + xy\sum_{l=0}^{n-1} b_{n,l}x^{n-1-l}y^l \tag{27}$$

Now, taking into account (3) and (7) we obtain

$$
\begin{aligned}
Q_{n+1}(x,y) &= (x+y)\sum_{l=0}^{n+1} a_{n+1,l}x^{n+1-l}y^l + xy\sum_{l=0}^{n-1} b_{n,l}x^{n-1-l}y^l \\
&= (x+y)Q_n(x,y) + xyR_n(x,y)
\end{aligned}
\tag{28}
$$

The same line of reasoning can be employed for $R_{n+1}(x,y)$, and the proof is thus complete. ∎

Finally, let us write the links between $Q_n(x,y)$ and $R_n(x,y)$ in a matrix form.

**Theorem 3.** *A vector $\boldsymbol{Q}_N(x,y)$ of polynomials $Q_n(x,y)$ with $n = 0,1,2,\dots,N$ and a vector $\boldsymbol{R}(x,y)_N$ of polynomials $R_n(x,y)$ with $n = 1,2,\dots,N+1$ fulfil the following matrix identities:*

$$
\boldsymbol{Q}_N(x,y) = \boldsymbol{C}_N(x)\boldsymbol{R}_N(x,y), \quad \boldsymbol{R}_N(x,y) = \boldsymbol{D}_N(x)\boldsymbol{Q}_N(x,y) \tag{29}
$$

*where*

$$
\boldsymbol{C}_N(x) = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ -x & 1 & 0 & \dots & 0 & 0 \\ 0 & -x & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -x & 1 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}[x] \tag{30}
$$

*and*

$$
\boldsymbol{D}_n(x) = \boldsymbol{C}_n^{-1}(x) = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ x & 1 & 0 & \dots & 0 & 0 \\ x^2 & x & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x^n & x^{n-1} & x^{n-2} & \dots & x & 1 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}[x] \tag{31}
$$

*with $N = 0,1,2,\dots$.*

*Proof.* The proof follows immediately from (25) and (26). Indeed, we have

$$
R_k(x,y) = Q_{k-1}(x,y) + xR_{k-1}(x,y), \quad k = 1,2,\dots \tag{32}
$$

and

$$
Q_k(x,y) = R_{k+1} - xR_k(x,y), \quad k = 0,1,2,\dots \tag{33}
$$

Now substituting successively the above identities into (29) for $k = 0,1,\dots,n$ yields the desired equalities. For brevity, all details are omitted here. Note that (29) leads to efficient computations in practical applications of the established polynomials. ∎

Finally, it is worth underlining that the introduced set of links between $Q_n(x,y)$ and $R_n(x,y)$ is not closed because there is a number of other suitable relations between them, but for brevity they are left for a separate publication.

### 3.2. Factorizations and Zeros of Polynomials $Q_n(x,y)$ and $R_n(x,y)$

One of the most important properties of the introduced polynomials $Q_n(x,y)$ and $R_n(x,y), n = 0,1,2,\ldots$ is the factorization into appropriate monomials and the locations of the zeros in the complex plane. It is well-known that finding the locations of the zeros is one of the major problems and, in general, if even a two-variable polynomial is separable into a product of polynomials with respect to each variable, no exact algebraic solution can be given when at least one factor's degree is greater than five (Acosta-de-Orozc and Gomez-Calderon, 1996; Barnet, 1983; Schinzel, 1982). Here we shall briefly present the advantages offered by the polynomials $Q_n(x,y)$ and $R_n(x,y)$.

**Theorem 4.** *Polynomials* $Q_n(x,y)$ *with* $n = 0,1,2,\ldots$ *can be factored as follows:*

$$Q_n(x,y) = \prod_{k=1}^{n} (y + \lambda_{n,k} x) \tag{34}$$

*where* $\lambda_{n,k} = 4\cos^2 \dfrac{k\pi}{2n+1}$, $k = 1,2,\ldots,n$. *Polynomials* $R_n(x,y)$ *with* $n = 1,2,\ldots$ *can be factored as*

$$R_n(x,y) = \prod_{j=1}^{n-1} (y + \mu_{n,j} x) \tag{35}$$

*where* $\mu_{n,j} = 4\sin^2 \dfrac{j\pi}{2n}$, $j = 1,2,\ldots,n-1$.

*Proof.* We consider the real positive coefficients of the polynomials which are determined by the FMNT and SMNT. Using the results of (Trzaska, 1993a; 1993b) and the polynomials $Q'_n(x,y)$ and $R'_n(x,y)$ in normalized forms (10) and (11), we have (12) and (13). Since $q = y/x$, the zeros of $Q_n(x,y)$ are located in the complex plane on the following lines:

$$y = -4x\cos^2 \frac{k\pi}{2n+1} = -\lambda_{n,k} x, \quad k = 1,2,\ldots,n \tag{36}$$

and the zeros of $R_n(x,y)$ are located on the lines

$$y = -4x\sin^2 \frac{j\pi}{2n} = -\mu_{n,j} x, \quad j = 1,2,\ldots,n-1 \tag{37}$$

Taking into account (33) and (34), we obtain the assertion of the theorem. ∎

It is worth noticing that the zeros of $Q_n(x,y)$ and $R_n(x,y)$ have a simple geometrical interpretation as a set of $n$ straight lines passing by the second and fourth quadrants of the plane $(x,y)$ and, as $n \to \infty$, the corresponding zeros cluster on the lines $y = -4x$ and $y = 0$, respectively, in the rectangular coordinate system.

**Example 3.** Now we consider the factorization and zeros of the polynomials

$$Q_3(x,y) = x^3 + 6x^2y + 5xy^2 + y^3, \quad R_3(x,y) = 3x^2 + 4xy + y^2$$

Applying Theorem 4, we can write

$$\begin{aligned} Q_3(x,y) &= (y+\lambda_{3,1}x)(y+\lambda_{3,2}x)(y+\lambda_{3,3}x) \\ &= (y+3.2470x)(y+1.5560x)(y+0.1981x) \\ &= y^3+5xy^2+6x^2y+x^3 \end{aligned}$$

$$\begin{aligned} R_3(x,y) &= (y+\mu_{3,1}x)(y+\mu_{3,2}x) \\ &= (y+1x)(y+3x) = y^2+4xy+3x^2 \end{aligned}$$

where, from (31), we have

$$\lambda_{3,1} = 4\cos^2\frac{\pi}{7} = 3.2470, \qquad \lambda_{3,2} = 4\cos^2\frac{2\pi}{7} = 1.5560$$

$$\lambda_{3,3} = 4\cos^2\frac{3\pi}{7} = 0.1981$$

$$\mu_{3,1} = 4\sin^2\frac{\pi}{6} = 1, \qquad \mu_{3,2} = 4\sin^2\frac{2\pi}{6} = 3$$

The geometrical interpretation of the location of the zeros on the plane $(x,y)$ is shown in Fig. 2 for polynomials $Q_3(x,y)$ and $R_3(x,y)$. ♦

## 4. Applications of 2-D Polynomials

The polynomials $Q_n(x,y)$ and $R_n(x,y)$ can be effectively applied for studies of various discrete-continuous systems. Here we shall limit our attention to three examples of such possibilities.

### 4.1. Ladder Networks

Let us consider now a uniform ladder network composed of $N$ sections in which longitudinal branch elements have impedance $Y$ and the transversal branch elements have impedance $X$. We take into account a general case in which both $Y$ and $X$ vary continuously in the interval $(-\infty,+\infty)$. From the Kirchhoff laws and the voltage-current descriptions of each branch in the network we get the following system of difference equations:

$$\begin{cases} XU_{n+1}(X,Y) = (2X+Y)U_n - XU_{n-1} \\ XI_{n+1}(X,Y) = XI_n + U_n \end{cases} \tag{38}$$

with $n = 0,1,\ldots,N$ and the initial conditions

$$XU_1(X,Y) = (X+Y)U_0, \quad XI_1 = U_0, \quad I_0 = 0 \tag{39}$$

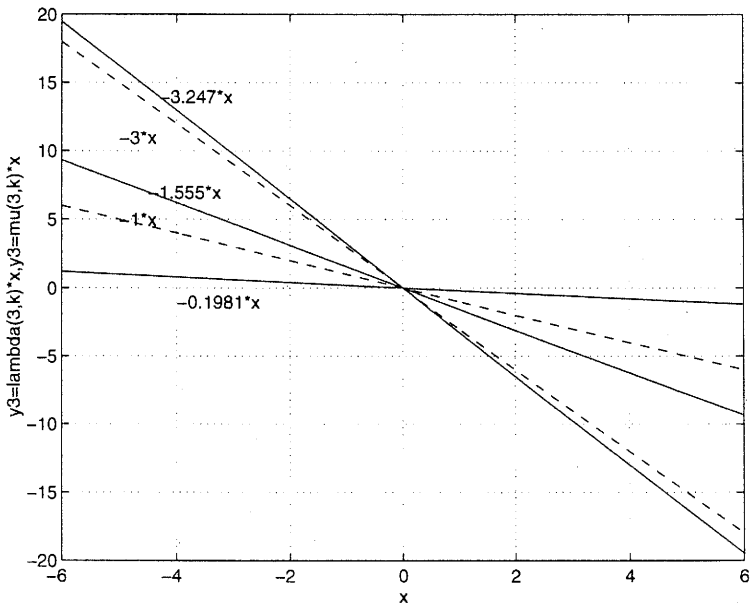


Fig. 2. Location of zeros for polynomials $Q_3(x, y)$ (continuous lines) and $R_3(x, y)$ (dashed lines).

Applying the standard procedure for solution of the difference equations, we obtain

$$\begin{cases} X^n U_n(X, Y) = Q_n(X, Y) U_0, & n = 0, 1, \ldots, N \\ X^n I_n(X, Y) = R_n(X, Y) U_0, & n = 0, 1, \ldots, N \end{cases} \tag{40}$$

where $U_0$ denotes the voltage at the output of the ladder network.

At the ladder input, we have the voltage-current relation

$$U_N(X, Y) = Z_{\text{in}}(X, Y) I_N(X, Y) \tag{41}$$

where $Z_{\text{in}}(X, Y)$ denotes the so-called input impedance of the ladder network.

Taking into account (39), it is easy to check that the input impedance is given by

$$Z_{\text{in}}(X, Y) = \frac{Q_N(X, Y)}{R_N(X, Y)} \tag{42}$$

Now it is evident that the polynomials $Q_n(X, Y)$ and $R_n(X, Y)$ describe uniquely the distribution of both the voltage and current along the ladder network and that they can be effectively used to design a network function for various changes of its parameters.

**Example 4.** It is necessary to design a one-port network described by the following driving point impedance as a function of two independent variables $X$ and $Y$:

$$Z_{\text{in}}(X,Y) = \frac{Q_3(X,Y)}{R_3(X,Y)}$$

To design the corresponding network, we develop the given function into a 2-D continuous fraction (Beauzamy *et al.*, 1995; Gill, 1977; Kuchminskaya, 1995). Using appropriate links between $Q_3(X,Y)$ and $R_3(X,Y)$, we can write

$$Z_{\text{in}}(X,Y) = \frac{YR_3(X,Y) + XQ_2(X,Y)}{R_3(X,Y)} = Y + \cfrac{1}{\cfrac{R_3(X,Y)}{XQ_2(X,Y)}}$$

$$= Y + \cfrac{1}{\cfrac{1}{X} + \cfrac{1}{\cfrac{YR_2(X,Y) + XQ_1(X,Y}{R_2(X,Y)}}}$$

$$= Y + \cfrac{1}{\cfrac{1}{X} + \cfrac{1}{Y + \cfrac{1}{\cfrac{1}{X} + \cfrac{1}{Y + \cfrac{1}{X}}}}}$$

Thus the given driving point impedance is realized by a chain connection of three identical two-ports composed of a longitudinal branch with impedance $Y$ and a transversal branch with impedance $X$. ♦

### 4.2. Solution of a Heat-Transfer Problem

As the next application of the 2-D polynomials $A_n(x,y)$ and $B_n(x,y)$, we consider a transient heat diffussion along a homogeneous thin metalic rod with non-zero initial conditions. It is assumed that one end of the rod is supplied from an ideal heat source and the other is connected with a massive thermal load characterized by a concentrated capacitance $C_0$. The system is described by the following partial differential equation:

$$\frac{\partial^2 T(z,t)}{\partial z^2} = \frac{C}{G}\frac{\partial T(z,t)}{\partial t} \tag{43}$$

where $G$ and $C$ denote respectively the per-unit length conductance and capacitance of the rod, $z \in (0,1)$ is the normalized space coordinate, and $t \in (0,\infty)$ denotes the time variable.

We shall determine the distribution of the temperature $T(G,C,z,t)$ along the rod. For that purpose, we take into account the discretization of the space coordinate with step $h = 1/N$. Applying the Laplace transform with respect to $t$ gives

$$T_{n+1}(G,C) - 2T_n(G,C) + T_{n-1}(G,C) = \frac{pC}{G}T_n(G,C) - \frac{h^2C}{G}T_n(0), \quad n = 1,2,\ldots \tag{44}$$

with $p = sh^2$, where $s$ is the Laplace operator, $T_N(G,C) = T_N$ and $T_n(0) = T_n$ are given.

Employing the procedure presented in the previous subsection, we can write

$$G^nT_n(G,C) = Q_n(G,C)T_0 - \sum_{l=1}^{n} R_{n-l}(G,C)Ch^2T_l \tag{45}$$

where at the load end we have

$$GT_1(G,C) = (G + shC_0)T_0 \tag{46}$$

and $T_0$ is to be determined.

From the boundary condition at the source end of the rod we obtain

$$G^Ns^{-1}T_N = Q_N(G,C)T_0 - \sum_{l=1}^{N-1} R_{N-l}(G,C)Ch^2T_l \tag{47}$$

Thus the Laplace transform of the temperature at the load end of the rod is expressed as follows:

$$T_0(s) = \frac{G^Ns^{-1}T_N + \sum_{l=1}^{N-1} R_{N-l}(G,C)Ch^2T_l}{Q_N(G,C)} = \frac{G^NT_N + \sum_{l=1}^{N-1} sR_{N-l}(G,C)Ch^2T_l}{(Ch^2)^N \prod_{k=1}^{N} s(s-s_k)} \tag{48}$$

where

$$s_k = -4\frac{G}{Ch^2}\cos^2\left(\frac{k\pi}{2N+1}\right), \quad k = 1,2,\ldots,N \tag{49}$$

Now it is easily seen that, since all $s_k$'s are negative, the load end temperature at the time limit $t \to \infty$ takes the value $T_N$. This analytical result agrees well with the nature of the investigated process. The Laplace transform of the rod temperature at the other points of the space coordinate discretization $n = 1,2,\ldots,N-1$ can be determined in a similar way. Then by the inverse Laplace transform we evaluate the distribution of the instantaneous temperature values along the rod.

Applying the above procedure, we can also determine the thermal flux distribution at all other points along the rod. Moreover, using this approach we are able to design a set of appropriate rod parameters for given conditions concerning the thermal sources and the heat transfer from the excitation end to the load end of the rod. The determination of the precise characteristics in this direction needs some additional studies.

### 4.3. Design of 2-D Digital Filters

To generate structures of 2-D digital filters, we consider a ratio of two-variable polynomials $Q_n(x,y)$ and $R_n(x,y)$:

$$K_n(x,y) = \frac{Q_n(x,y)}{R_n(x,y)}, \quad n = 1,2,\dots \tag{50}$$

Introducing the following transformation of independent variables:

$$x = \frac{1}{q(s_1,s_2)}, \qquad y = q(s_1,s_2) \tag{51}$$

where $(s_1,s_2) \in \mathbb{C} \times \mathbb{C}$, we obtain

$$K_n(s_1,s_2) = \frac{\prod_{k=1}^{n}\left(q^2(s_1,s_2)+\lambda_k\right)}{\prod_{k=1}^{n-1} q(s_1,s_2)\left(q^2(s_1,s_2)+\mu_k\right)} \tag{52}$$

Now, using the double bilinear transformation (Bose, 1985)

$$s_1 = \frac{1-z_1^{-1}}{1+z_1^{-1}}, \qquad s_2 = \frac{1-z_2^{-1}}{1+z_2^{-1}} \tag{53}$$

we conclude that the digital counterpart of $K_n(s_1,s_2)$ becomes

$$K_n(z_1^{-1},z_2^{-1}) = B\frac{N(z_1^{-1},z_2^{-1})}{D(z_1^{-1},z_2^{-1})} \tag{54}$$

where $N(z_1^{-1},z_2^{-1})$ and $D(z_1^{-1},z_2^{-1})$ represent polynomials in $z_1^{-1}$ and $z_2^{-1}$, and the constant scalar parameter $B$ is given by

$$B = K(s_1,s_2)\Big|_{s_1=s_2=1} \tag{55}$$

The digitalized reactance function (54) placed in the series arm of an augmented two-port structure (Bose, 1985) results in the following scattering matrix:

$$\boldsymbol{S}(z_1,z_2) = \frac{1}{1+\beta H_d(z_1^{-1},z_2^{-1})}\begin{bmatrix} 1-\beta & \beta(1+H_d(z_1^{-1},z_2^{-1})) \\ 1+H_d(z_1^{-1},z_2^{-1}) & (\beta-1)H_d(z_1^{-1},z_2^{-1}) \end{bmatrix} \tag{56}$$

where $\beta = R_1/R_2$ denotes the ratio of augmenting network resistances, and

$$H_d(z_1^{-1},z_2^{-1}) = \frac{D(z_1^{-1},z_2^{-1}) - N(z_1,z_2^{-1})}{D(z_1^{-1},z_2^{-1}) + N(z_1,z_2^{-1})} \tag{57}$$

represents a 2-D delay element.

The block-diagram representing this two-port realization is shown in Fig. 3(a). It is easy to demonstrate that by carefully choosing the parameters of the 2-D delay element it is possible to obtain the resulting realizations as generally stable.

If the function $K_n(z_1^{-1},z_2^{-1})$ is placed in the shunt arm of the two-port, a similar scattering matrix and realization result. This differs from the simulation shown in

Fig. 3(a) by reversed positions of functional blocks at the midst of the structure. The corresponding digitalized 2-D structure of a filter with a scalar input function

$$K_2(x,y) = \frac{Q_2(x,y)}{R_2(x,y)} = \frac{x^2+3xy+y^2}{2x+y} \tag{58}$$

of the ladder network is shown in Fig. 3(b). It has been obtained by applying the transformations

$$y = s, \qquad x = \frac{1}{s}, \qquad s = \frac{1+s_1 s_2}{s_1+s_2} \tag{59}$$

to the resulting transfer function

$$T(s) = \frac{1}{s^4+2s^3+4s^2+2s+2} \tag{60}$$

of the corresponding augmented two-port network.
Thus using the double bilinear transformation (53) to the reactance function of the ladder structure

$$K_2(s) = \frac{s^4+3s^2+1}{s(s^2+2)} \tag{61}$$

we get the corresponding delay unit

$$H_d(z_1^{-1}, z_2^{-1}) = \frac{-2/3z_1^{-1}z_2^{-1} - 6/5z_1^{-2}z_2^{-2} + 2/3z_1^{-3}z_2^{-3} - 2z_1^{-4}z_2^{-4}}{2 - 2/3z_1^{-1}z_2^{-1} + 6/5z_1^{-2}z_2^{-2} + 2/3z_1^{-3}z_2^{-3}} \tag{62}$$

The transfer function of the resulting digital realization is obtained by multiplying the chain matrices describing the series and shunt arms of the ladder structures. Then we obtain

$$T(z_1^{-1}, z_2^{-1}) = \frac{1 - 4z_1^{-1}z_2^{-1} + 6z_1^{-2}z_2^{-2} - 4z_1^{-3}z_2^{-3} + z_1^{-4}z_2^{-4}}{9 - 4z_1^{-1}z_2^{-1} + 10z_1^{-2}z_2^{-2} + z_1^{-4}z_2^{-4}} \tag{63}$$

The above examples show how the established polynomials $Q_n(x,y)$ and $R_n(x,y)$ can be useful in solving various problems arising in practice.

## 5. Final Remarks and Conclusions

The paper presents a new formulation in the field of 2-D polynomials $Q_n(x,y)$ and $R_n(x,y)$. In particular, we have shown some links between the considered bivariate polynomials and the so-called modified numerical triangles FMNT and SMNT. It has been proved that these triangles are advantageous in practical computations because their elements are determined by positive integers that can be easily calculated in a recurrent manner. We have demonstrated that FMNT and SMNT can be effectively used to determine a number of simple relations between the studied polynomials themselves. Both the 2-D power polynomials $Q_n(x,y)$ and $R_n(x,y)$, $n = 0,1,2,\ldots$ and modified numerical triangles FMNT and SMNT can be determined by appropriate powers of independent variables or parameters $x$ and $y$ constituting elements of real

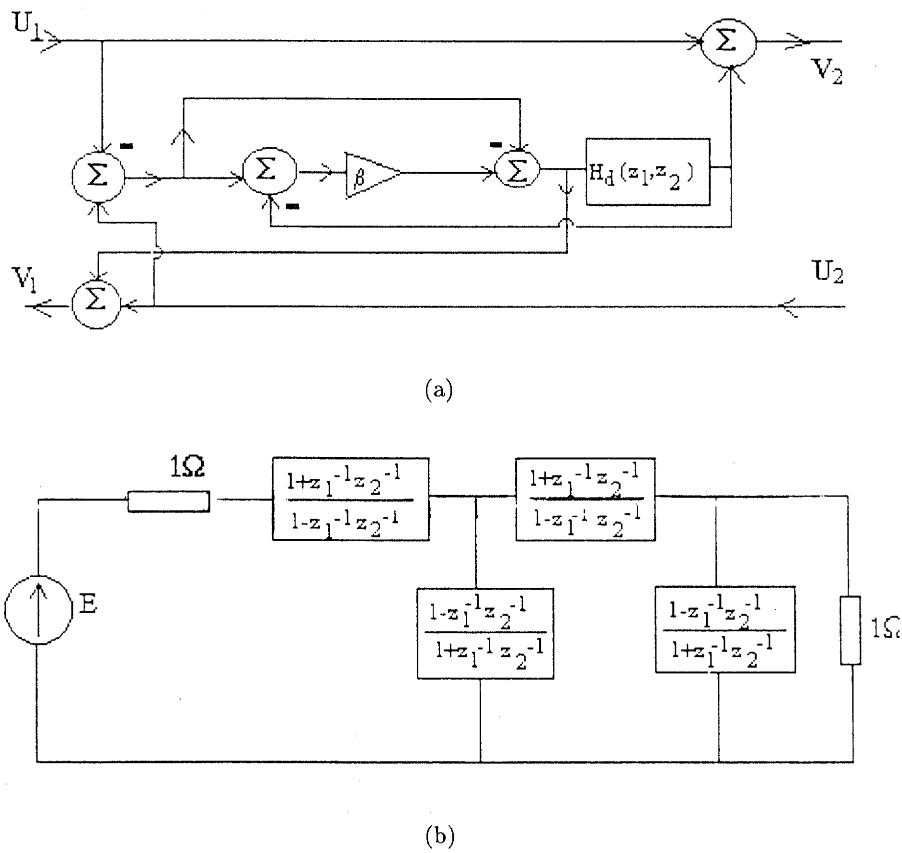


Fig. 3. Structure of a 2-D digital filter: (a) digitalized structure of a two-port, (b) structure of the 2-D digital filter realizing a second-order transfer function $K_2(x,y)$.

or complex sets. We have also demonstrated that FMNT and SMNT can be effectively used to determine with ease the $n$-th-order convergence of the corresponding 2-D continued fraction.

This is a good place to say what the links between $Q_n(x,y)$ and $R_n(x,y)$ and modified numerical triangles entail. First, they result in integer elements. Second, the recurrences lead to straightforward procedures of computing these polynomials. Further, all polynomials of successive degrees can be expressed in terms of suitable matrix expressions which are relatively simple. Finally, it must be underlined that there exists a possibility of extending the established results to the structures with parameters determined with some tolerances. However, those problems are left for future research.

It must be emphasized that the introduced polynomials are efficient in solving many discrete-continuous problems appearing in practice. Illustrations of possible applications are also presented. Three suitable examples of systems taken from different fields of electrical engineering have been solved by applying the 2-D polynomials $Q_n(x,y)$ and $R_n(x,y)$. It is worth noticing that the polynomials $Q_n(x,y)$ and $R_n(x,y)$ have no common roots (except for $Q_0(x,y)$ and $R_1(x,y)$). Thus, following the well-known factorization approach for 2-D polynomials, we have demonstrated the zero locations of the studied polynomials on the straight lines $y = -\lambda_{n,k}x$ and $y = -\mu_{n,j}x$, respectively, on the plane $(x,y)$.

Another area where interesting results may be obtained is that of the index of concentration at a low degree of both polynomials $Q_n(x,y)$ and $R_n(x,y)$ in a general case. Note that polynomials $R_n(x,y)$ have large coefficients at low degrees and are characterized by high concentrations at low degrees. But all polynomials $Q_n(x,y)$ have the leading coefficient equal to unity and it is quite likely (but not proven yet) that their indices of concentration at low degrees are not as large as in the case of $R_n(x,y)$. The determination of the precise values of the estimates does not only involve computational accuracy, but a better understanding of the problem itself. However, this question is still open and needs additional studies.

In conclusion, the world of 2-D polynomials $Q_n(x,y)$ and $R_n(x,y)$ provides many opportunities for new and exciting activities which may not be known.

## References

Acosta-de-Orozc M. and Gomez-Calderon J. (1996): *Local minimal polynomials over finite fields.* — Fibon. Quart., Vol.34, No.2, pp.139–143.

Attasi S. (1973): *Systèmes linéaires homogènes à deux indices.* — Rapport Laboria, Vol.31.

Barnet S. (1983): *Polynomials and Linear Control Systems.* — New York: Marcel Dekker.

Beauzamy B., Enflo P. and Wang P. (1995): *Quantitative estimates for polynomials in one or several variables.* — Math. Magaz., Vol.67, No.4, pp.243–257.

Bose N.K. (1985): *Digital filters. Theory and Applications.* — New York: North-Holland.

Cronin J. (1976): *Mathematics of Cell Electrophysiology.* — Lect. Notes in Pure and Appl. Math., Vol.63, New York: Marcel Dekker.

Edwards A.W.F. (1987): *Pascal's Arithmetic Triangle.* — Oxford: Oxford University Press.

Ferri G., Faccio M. and D'Amico A. (1991): *A new fast method for ladder networks characterization.* — IEEE Trans. CAS, Vol.CAS-38, No.11, pp.1377–1382.

Fornasini E. and Marchesini G. (1976): *State space realization theory of 2-D filters.* — IEEE Trans. Automat. Contr., Vol.AC-21, No.3, pp.484–492.

Fornasini E. and Marchesini G. (1978): *Doubly indexed dynamical systems: State-space models and structural properties.* — Math. Syst. Theory, Vol.12, No.1, pp.59–72.

Gałkowski K. (1997): *The resultant matrix for a family of two-variable polynomials.* — Proc. MIM'97, Vienna, pp.209–212.

Gałkowski K., Rogers E. and Owens D. (1997): *Basic systems theory for discrete linear repetitive processes using 2-DRoesser model interpretations.* — Appl. Math. Comp. Sci., Vol.7, No.1, pp.101–116.

Gill J. (1977): *A generalization of certain corresponding continued fractions.* — Bull. Cal. Math. Soc., Vol.69, No.2, pp.331–340.

Huang T.S. (1972): *Stability of two-dimensional and recursive filters.* — IEEE Trans. Audio and Electroac., Vol.Au-20, No.2, pp.158–163.

Idczak D. (1996): *The maximum principle for a continuous 2-D Roesser model with a terminal condition.* — Proc. *MMAR'96*, Międzyzdroje (Poland), pp.221–226.

Jean P.V. (1986): *A basic theorem and a fundamental approach to pattern formation on plants.* — J. Math. Biosc., Vol.16, No.4, pp.127–154.

Kaczorek T. (1985): *Two-Dimensional Linear Systems.* — Berlin: Springer.

Kaczorek T. (1988): *Singular general model of 2-D systems and its solutions.* — IEEE Trans. Automat. Contr., Vol.AC-31, No.6, pp.1060–1061.

Kaczorek T. (1994): *Linear Control Theory.* — Vols.1 and 2, New York: Wiley.

Klamka J. (1997): *Approximate controllability of delayed dynamical systems.* — Appl. Math. Comp. Sci., Vol.7, No.1, pp.5–16.

Kuchminskaya K. (1995): *The Lane-like theorem for two-dimensional continued fractions.* — Commm. in The Anal. Theory of Cont. Fract., Vol.4, No.1, pp.47–49.

Lahr J. (1986): *Fibonacci and Lucas numbers and the Morgan-Voyce polynomials in ladder networks and in electric line theory,* In: (A.N. Philippou *et al.*, Eds.) Fibonacci Numbers and Their Applications. — New York: Reidel Publishing Company, pp.141–161.

Liu P.T and Sutinen J.G. (1985): *Control Theory in Mathematical Economics.* — Lect. Not. Pure Appl. Math., Vol.47, New York: Marcel Dekker.

Lewis F.L. (1992): *A review of 2-D implicit systems.* — Automatica, Vol.28, No.3, pp.345–354.

Lions J.L. (1996): *Remarks on some mathematical problems arising in climatology.* — Proc. *ICIAM'95*, Berlin, pp.37–52.

Majda A. (1996): *Topics in Atmospheric Science.* — Lect. Not. New York University, Vol.37, New York: The New York University Press.

Marden M. (1966): *Geometry of Polynomials.* — Providence: AMS.

Marszałek W. and Kekkeris G.T. (1989): *Heat exchangers and linear image processing theory.* — Int. J. Heat and Mass Transf., Vol.32, No.12, pp.2361–2374.

Morf M., Levy B.C. and Kung S.Y. (1977): *New results in 2-D systems theory. Part I.* — Proc. IEEE, Vol.65, No.5, pp.861–872.

Pandolfi L. (1984): *Exponential stability of 2-D systems.* — Syst. Contr. Lett., Vol.4, No.6, pp.381–385.

Reithofer W. (1996): *Virtual enterprise modelling and simulation.* — Proc. *IMS Next Generation Manufacturing Systems Technical Conference*, Magdeburg, (Germany), pp.726–732.

Rocha P. and Willems J.C. (1991): *Controllability of 2-D systems.* — IEEE Trans. Automat. Contr., Vol.AC-36, No.3, pp.431–423.

Roesser R.P. (1976): *A discrete state space model for linear image processing.* — IEEE Trans. Automat. Contr., Vol.AC-21, No.3, pp.484–492.

Schinzel A. (1982): *Selected Topics on Polynomials.* — Ann Arbor: Univ. of Michigan Press.

Skorobogat'ko B.J. (1983): *The Theory of Generalized Continued Fractions and its Applications in Computational Mathematics.* — Nauka, Moscow, (in Russian).

Tichonov G.I. and Samarskii A.V. (1974): *Equations of Physical Mathematics.* — Nauka, Moscow, (in Russian).

Trzaska Z. (1993a): *Modified numerical triangles and the Fibonacci sequence.* — The Fibon. Quart., Vol.32, No.2, pp.124–129.

Trzaska Z. (1993b): *Numerical triangles, Fibonacci sequence and ladder networks: Some further results.* — Appl. Math. Lett., Vol.6, No.4, pp.55–61.

Trzaska Z. (1995): *Fundamentals and applications of the Fibonacci hyperbolic trigonometry.* — Proc. *ICIAM'95*, Hamburg, pp.586–592.

Trzaska Z. (1996a): *Fibonacci polynomials, their properties and applications.* — J. Anal. Applics., Vol.15, No.3, pp.729–746.

Trzaska Z. (1996b): *New efficient tools for modeling and simulation of 2-D heat transfer problems.* — Proc. *ESS'96*, Genoa, pp.473–477.

Trzaska Z. (1996c): *Applications of 2-D models for effective solutions of heat transfer problems.* — Proc. *CESA'96*, Lille, pp.384–389.

Trzaska Z. (1996d): *Studies of a 2-D nonlinear system.* — Proc. *MMAR'96*, Międzyzdroje, (Poland), pp.217–222.

Trzaska Z. (1997): *Studies of transients in 2-D systems.* — Proc. *MIM'97*, Vienna, pp.215–219.

Willems J.C. (1989): *Models for dynamics.* — Dynamics Reported, Vol.2, No.2, pp.171–269.