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POSITIVE SOLUTIONS TO POLYNOMIAL MATRIX EQUATIONS

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Necessary and sufficient conditions are established for the existence of positive solutions to polynomial matrix equations. Two methods are proposed for determination of such solutions. As an example of applications it is shown that the determination of the transfer matrix of a positive controller for a closed-loop system with a given transfer matrix can be reduced to finding solutions to two suitable polynomial matrix equations.

1. Introduction

Polynomial matrix equations have been considered in many papers and books (Emre and Silverman, 1981; Feinstein and Barness, 1984; Kaczorek, 1986a; 1986b; 1987; 1992; Kučera, 1972; 1979; Qianhua and Zhongjun, 1987; Solak, 1985; Šebek, 1980; 1983; 1989; Šebek and Kučera, 1981; Wolovich, 1978). Recently the positive systems theory has become a field of great interest and research (Kaczorek, 1997; Maeda and Kodama, 1981; Maeda *et al.*, 1977; Ohta *et al.*, 1984; Van den Hof, 1997). Some automatic-control problems can be reduced to finding positive polynomial matrix solutions to suitable polynomial matrix equations (Kaczorek, 1992; Kučera, 1979). The main subject of this paper is to establish conditions for the existence of positive polynomial solutions to polynomial matrix equations and to present a method for finding positive polynomial solutions to these equations. An example of application of the presented results is also presented.

2. Preliminaries and Problem Statement

Denote by $\mathbb{R}^{q\times p}$ the set of $q\times p$ real matrices and by $\mathbb{R}^{q\times p}[s]$ the set of $q\times p$ polynomial real matrices in the variable s. Consider a polynomial matrix with real coefficients in the variable s of the form

$$A(s) = A_0 + A_1 s + \cdots + A_n s^n \in \mathbb{R}^{q\times p}[s] \tag{1}$$

where $A_i \in \mathbb{R}^{q\times p}$, $i = 0, 1, \ldots, n$.

A nonnegative integer n is called the degree (denoted by deg $A(s)$) of $A(s)$ if A_n is nonzero. The matrix (1) is called regular if $q = p$ and $\det A_0 \neq 0$.

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Let $\mathbb{R}_+^{q\times p}$ be the set of $q\times p$ matrices with real nonnegative entries.

Definition 1. The polynomial matrix (1) is called *positive* if $A_i \in \mathbb{R}_+^{q\times p}$ for $i = 0, 1, \dots, n$.

Consider the polynomial matrix equation

$$A(s)X(s) = B(s) \tag{2}$$

where $A(s)$ is given by (1) and

$$\begin{aligned} B(s) &= B_0 + B_1 s + \cdots + B_m s^m, \quad B_i \in \mathbb{R}^{q\times k} \quad \text{for } i = 0, 1, \dots, m \\ X(s) &= X_0 + X_1 s + \cdots + X_r s^r, \quad X_i \in \mathbb{R}^{p\times k} \quad \text{for } i = 0, 1, \dots, r \end{aligned} \tag{3}$$

Without loss of generality it can be assumed that the matrix $A(s)$ has full row rank, since usually we have $q < p$. This assumption guarantees that the coefficient matrix $[A_0, A_1, \dots, A_n]$ has full row rank.

Note that the well-known diophantine equation (Kučera, 1979)

$$A(s)X(s) + B(s)Y(s) = C(s)$$

where $A(s) \in \mathbb{R}^{k\times t}[s]$, $B(s) \in \mathbb{R}^{k\times v}[s]$, $C(s) \in \mathbb{R}^{k\times q}[s]$ are given and $X(s) \in \mathbb{R}^{t\times q}[s]$, $Y(s) \in \mathbb{R}^{v\times q}[s]$ are unknown can be considered as a particular case of (2), since it can be written as

$$\begin{bmatrix} A(s), B(s) \end{bmatrix} \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = C(s)$$

The problem under consideration can be stated as follows. Given polynomial matrices $A(s)$ and $B(s)$, establish conditions under which there exists a positive polynomial matrix $X(s)$ satisfying (2).

A positive polynomial matrix $X(s)$ satisfying (2) is called a positive solution to (2). If the polynomial matrix $A(s)$ is regular, then without loss of generality it can be assumed that $A_0 = I_p$ (the $p\times p$ identity matrix).

Necessary and sufficient conditions will be established under which eqn. (2) has a positive polynomial solution $X(s)$ for given polynomial matrices $A(s)$ and $B(s)$.

3. Existence of a Positive Solution

From (2) it follows that the minimal degree of $X(s)$ is equal to $n-m$. Substituting (1) and (3) into (2) and comparing the coefficient at the same powers of s, we obtain for

$r + n \geq m$ and $n > r$

$$\begin{aligned} A_0 X_0 &= B_0 \\ A_1 X_0 + A_0 X_1 &= B_1 \\ A_r X_0 + A_{r-1} X_1 + \cdots + A_0 X_r &= B_r \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots& \\ A_n X_0 + A_{n-1} X_1 + \cdots + A_{n-r} X_r &= B_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots& \\ A_n X_{m-n} + \cdots + A_{m-r} X_r &= B_m \end{aligned} \tag{4}$$

Equations (4) for $r + n = m$ can be written as

$$\bar{A}\bar{X} = \bar{B} \tag{5}$$

where

$$\bar{A} = \begin{bmatrix} A_0 & 0 & 0 & \cdots & 0 & 0 \\ A_1 & A_0 & 0 & \cdots & 0 & 0 \\ A_2 & A_1 & A_0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_r & A_{r-1} & A_{r-2} & \cdots & A_1 & A_0 \\ A_{r+1} & A_r & A_{r-1} & \cdots & A_2 & A_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_n & A_{n-1} & A_{n-2} & \cdots & A_{n-r-1} & A_{n-r} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & A_n \end{bmatrix} \in \mathbb{R}^{(m+1)q \times p(r+1)}$$

$$\bar{X} = \begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ \vdots \\ X_r \end{bmatrix} \in \mathbb{R}^{p(r+1) \times k}, \quad \bar{B} = \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ \vdots \\ B_r \\ \vdots \\ B_n \\ \vdots \\ B_m \end{bmatrix} \in \mathbb{R}^{(m+1)q \times k} \tag{6}$$

Definition 2. A vector $b \in \mathbb{R}^n$ is called a *positive linear combination* of vectors $a_i \in \mathbb{R}^n$, $i = 1, \dots, k$ if there exist nonnegative scalars $c_i \in \mathbb{R}_+$, $i = 1, \dots, k$ such that

$$\sum_{i=1}^{k} c_i a_i = b$$

Definition 3. The smallest nonnegative integer t is called the nonnegative column rank of $A \in \mathbb{R}^{p \times q}$ and denoted by $\text{rank}_+ A$ if there exist t columns in A such that each column of A is a positive linear combination of the t columns (Cohen and Rothblum, 1993).

Theorem 1. *Equation (2) has a positive solution $X(s)$ if and only if one of the following equivalent conditions is satisfied:*

(i) *every column of the matrix $\bar{B}$ is a positive linear combination of columns of the matrix $\bar{A}$,*

(ii) *$b_i \in \text{cone}\,\bar{A}$ for $i = 1, \dots, k$, where $\text{cone}\,\bar{A}$ denotes the cone generated by the columns of $\bar{A}$ (Berman and Plemmons, 1994) and $\bar{B} = [b_1, \dots, b_k]$,*

(iii) $\text{rank}_+[\bar{A}, \bar{B}] = \text{rank}_+ \bar{A}$.

Proof. Let $\bar{A} = [a_1, \dots, a_z]$, $z := p(r+1)$ and $\bar{X} = [x_1, \dots, x_k]$. From (5) we have $\bar{A}x_i = b_i$ for $i = 1, \dots, k$. Equation (2) has a positive solution $X(s)$ if and only if $x_i \in \mathbb{R}_+^z$ for $i = 1, \dots, k$. By definition (Berman and Plemmons, 1994) $\text{cone}\,\bar{A}$ is the set of all positive linear combinations of the columns of $\bar{A}$. Thus eqn. (2) has a positive solution $X(s)$ if and only if $b_i \in \text{cone}\,\bar{A}$ for $i = 1, \dots, k$.

By using Definition 3 it is easy to show that eqn. (2) has a positive solution $X(s)$ if and only if condition (iii) is satisfied. ■

Let $\bar{A}_R$ be a right inverse of $\bar{A}$, i.e. $\bar{A}\bar{A}_R = I$ (the identity matrix). If $\bar{A}$ has full row rank, then $\bar{A}_R$ is given by (Kaczorek, 1992)

$$\bar{A}_R = \bar{A}^T \left[\bar{A}\bar{A}^T\right]^{-1} + \left(I - \bar{A}^T \left[\bar{A}\bar{A}^T\right]^{-1} \bar{A}\right) K \tag{7}$$

where T denotes the transposition and K is an arbitrary matrix of appropriate dimensions.

Theorem 2. *Equation (2) has a positive solution $X(s)$ of the minimal degree $r = m - n$ if there exists a right inverse $\bar{A}_R$ of $\bar{A}$ such that*

$$\bar{A}_R \bar{B} \in \mathbb{R}_+^{(r+1)p \times k} \tag{8}$$

Moreover, if eqn. (2) has a positive solution $X(s)$ and $A(s)$ is a positive matrix, then the matrix $B(s)$ is also positive.

Proof. If there exists $\bar{A}_R$, then a solution to (5) is given by

$$\bar{X} = \bar{A}_R \bar{B} \tag{9}$$

Equation (2) has a positive solution $X(s)$ of the minimal degree if (8) holds. It is easy to show that if $A(s)$ and $X(s)$ are positive polynomial matrices, then the matrix $B(s)$ is also positive. ■

Remark 1. If $(m+1)q > (r+1)p$, then the matrix $\bar{A}$ does not have full row rank and $\bar{A}_R$ does not exist, but a positive solution of the minimal degree $r = m - n$ to eqn. (2) may still exist.

In the particular case when $A(s)$ is regular, we have the following.

Theorem 3. *Let $A(s)$ be regular and $A_0 = I_p$. Equation (2) has a unique positive solution $X(s)$ of the minimal degree $r = m - n$ if and only if*

$$\sum_{i=0}^{m-1} \hat{A}^i \bar{B} \in \mathbb{R}_+^{(m+1)p \times k} \tag{10}$$

$$\hat{A} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \cdots 0 \\ A_1 & 0 & 0 & \cdots & 0 & 0 & 0 \cdots 0 \\ A_2 & A_1 & 0 & \cdots & 0 & 0 & 0 \cdots 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_r & A_{r-1} & A_{r-2} & \cdots & A_1 & 0 & 0 \cdots 0 \\ A_{r+1} & A_r & A_{r-1} & \cdots & A_2 & A_1 & 0 \cdots 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_n & A_{n-1} & A_{n-2} & \cdots & A_{n-r-1} & A_{n-r} & 0 \cdots 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & A_n & 0 \cdots 0 \end{bmatrix} \in \mathbb{R}^{(m+1)p \times (m+1)p}$$

Proof. If $A_0 = I_p$, we have $\bar{A}\bar{X} = (I - \hat{A})\hat{X}$, where $\hat{X}$ is obtained from $\bar{X}$ by addition of $m - r$ zero blocks in the positions $r+1, \ldots, m+1$. Hence we obtain

$$\hat{X} = \left[I - \hat{A}\right]^{-1} \bar{B} \tag{11}$$

It is easy to show that $\hat{A}^k = 0$ for $k \geq m$ and

$$\left[I - \hat{A}\right]^{-1} = \sum_{i=0}^{m-1} \hat{A}^i \tag{12}$$

Substituting (12) into (11), we obtain

$$\hat{X} = \sum_{i=0}^{m-1} \hat{A}^i \bar{B} \tag{13}$$

The uniqueness of the positive solution $X(s)$ follows immediately from the nonsingularity of the matrix $[I - \hat{A}]$. ■

Corollary 1. Let $A(s)$ be regular and $A_0 = I_p$. Then eqn. (2) has a unique positive solution $X(s)$ of the minimal degree $r = m - n$ if

$$A_k \in \mathbb{R}_+^{p\times p} \text{ for } k = 0, 1, \dots, n \text{ and } B_j \in \mathbb{R}_+^{q\times k} \text{ for } j = 0, 1, \dots, m \quad (14)$$

Proof. Note that (14) implies $\hat{A}^i \in \mathbb{R}_+^{(m+1)p\times(m+1)p}$, $\bar{B} \in \mathbb{R}_+^{(m+1)p\times k}$ and by (13) $\hat{X} \in \mathbb{R}_+^{(m+1)p\times k}$. Therefore eqn. (2) has a unique positive solution $X(s)$ of the minimal degree $r = m - n$. ■

4. Determination of a Positive Solution

If eqn. (2) has a positive solution, then it can be found by the use of one of the following two methods.

Method 1

The following notation will be used for elementary column operations on a polynomial matrix: $R[i \times c]$ denotes multiplication of the i-th column of the matrix by a nonzero scalar c, $R[i + j \times b(s)]$ stands for addition of the j-th column multiplied by the polynomial $b(s)$ to the i-th column of the matrix, $R[i, j]$ is the exchange of the i-th and j-th columns of the matrix.

Using elementary column operations, we perform the following reduction:

$$\begin{bmatrix} A(s) \\ \cdots \\ I_p \end{bmatrix} \to \begin{bmatrix} L(s) & 0 \\ \cdots & \cdots \\ U_1(s) & U_2(s) \end{bmatrix}$$

where $L(s)$ is the left divisor of $A(s)$ and $[U_1(s), U_2(s)]$ is a unimodular matrix of elementary column operations.

It is well-known (Kaczorek, 1992; Kučera, 1979) that eqn. (2) has a polynomial solution if $L(s)$ is a left divisor of $B(s)$, i.e. $B(s) = L(s)B_1(s)$. It is easy to check that in this case the solution is given by

$$X(s) = [U_1(s), U_2(s)] \begin{bmatrix} B_1(s) \\ K(s) \end{bmatrix} \quad (15)$$

where $K(s)$ is an arbitrary polynomial matrix of appropriate dimensions. The matrix $K(s)$ is chosen so that (15) be a minimal-degree positive polynomial solution of (2).

Example 1. Find a positive polynomial solution to eqn. (2) with

$$A(s) = \begin{bmatrix} s^2+1 & s & s \\ s & 2s+1 & 1 \end{bmatrix}, \quad B(s) = \begin{bmatrix} s^3+2s^2+2s+1 \\ 3s^2+2s+1 \end{bmatrix} \quad (16)$$

Using the elementary column operations $R[1+3\times(-s)]$, $R[2+3\times(-2s-1)]$, $R[2+1\times 2s^2]$, $R[3+1\times(-s)]$, $R[2,3]$, we reduce the matrix

$$\begin{bmatrix} A(s) \\ \cdots \\ I \end{bmatrix} = \begin{bmatrix} s^2+1 & s & s \\ s & 2s+1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

to the form

$$\begin{bmatrix} L(s) & 0 \\ U_1(s) & U_2(s) \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 1 & -s & 2s^2 \\ 0 & 0 & 1 \\ -s & s^2+1 & -2s^3-2s-1 \end{array}\right]$$

In this case

$$L(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_1(s) = B(s)$$

Using (15), we obtain

$$X(s) = \begin{bmatrix} 1 & -s & 2s^2 \\ 0 & 0 & 1 \\ -s & s^2+1 & -2s^3-2s-1 \end{bmatrix} \begin{bmatrix} s^3+2s^2+2s+1 \\ 3s^2+2s+1 \\ k(s) \end{bmatrix}$$

and for $k(s) = s$ we get the desired positive polynomial solution

$$X(s) = \begin{bmatrix} s+1 \\ s \\ 1 \end{bmatrix} \tag{17}$$

♦

Method 2

By assumption the matrix $A(s)$ has full row (normal) rank and the matrix $A(s)A^T(s)$ is invertible. Therefore there exists a right inverse matrix $A_R(s)$ of $A(s)$ of the form

$$A_R(s) = A^T(s)\left[A(s)A^T(s)\right]^{-1} + \left(I - A^T(s)\left[A(s)A^T(s)\right]^{-1}A(s)\right)\bar{K}(s) \tag{18}$$

where $\bar{K}(s)$ is an arbitrary matrix of appropriate dimensions. The matrix $\bar{K}(s)$ is chosen so that

$$X(s) = A_R(s)B(s) \tag{19}$$

is a positive polynomial solution of the minimal degree $r = m - n$ to eqn. (2).

Example 2. Using Method 2 we want to find a positive polynomial solution to eqn. (2) with (16). In this case

$$A^T(s)\big[A(s)A^T(s)\big]^{-1}B(s) = \frac{1}{\Delta(s)}\begin{bmatrix} 4s^7+4s^6+12s^5+12s^4+10s^3+10s^2+6s+2 \\ 4s^7+12s^5+4s^4+4s^3+2s^2+3s+1 \\ 4s^6+4s^5+10s^4+6s^3+4s^2+s+1 \end{bmatrix} \tag{20a}$$

and

$$\Big(I - A^T(s)\big[A(s)A^T(s)\big]^{-1}A(s)\Big)\bar{K}(s)B(s) = \frac{1}{\Delta(s)}\begin{bmatrix} 2s^2(1+2s+2s^2+s^3)(k_{21}-k_{31}-2k_{31}s+2k_{11}s^2-2k_{31}s^3) \\ \quad +2s^2(1+2s+3s^2)(k_{22}-k_{32}-2k_{32}s+2k_{12}s^2-2k_{32}s^3); \\ (1+2s+2s^2+s^3)(k_{21}-k_{31}-2k_{31}s+2k_{11}s^2-2k_{31}s^3) \\ \quad +(1+2s+3s^2)(k_{22}-k_{32}-2k_{32}s+2k_{12}s^2-2k_{32}s^3); \\ (1+2s+2s^2+s^3)(1+2s+2s^3) \\ \quad \times(-k_{21}+k_{31}+2k_{31}s-2k_{11}s^2+2k_{31}s^3)+(1+2s+3s^2) \\ \quad \times(1+2s+2s^3)(-k_{22}+k_{32}+2k_{32}s-2k_{12}s^2+2k_{32}s^3) \end{bmatrix} \tag{20b}$$

where

$$\Delta(s) = 4s^6 + 12s^4 + 4s^3 + 4s^2 + 4s + 2$$

Taking into account (20) and using (18) and (19), we obtain

$$X(s) = \frac{1}{\Delta(s)}\begin{bmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{bmatrix} \tag{21}$$

where

$$\begin{aligned} X_1(s) = {} & 1 + 3s + (7 - k_{31} - k_{32})s^2 + (9 - 4k_{31} - 4k_{32})s^3 \\ & + (13 + 2k_{11} - 6k_{31} - 7k_{32})s^4 + (11 + 4k_{11} - 7k_{31} - 8k_{32})s^5 \\ & + (8 + 4k_{11} - 6k_{31} - 4k_{32})s^6 + (2 + 2k_{11} - 4k_{31} - 6k_{32})s^7 - 2k_{31}s^8 \end{aligned}$$

$$
\begin{aligned}
X_2(s) &= 3 - k_{31} - k_{32} + (7 - 4k_{31} - 4k_{32})s + (9 + 2k_{11} - 6k_{31} - 7k_{32})s^2 \\
&+ (9 + 4k_{11} - 7k_{31} - 8k_{32})s^3 + (10 + 4k_{11} - 6k_{31} - 4k_{32})s^4 \\
&+ (12 + 2k_{11} - 4k_{31} - 6k_{32})s^5 - 2k_{31}s^6 + 4s^7
\end{aligned}
$$

$$
\begin{aligned}
X_3(s) &= -1 + k_{31} + k_{32} + (-7 + 6k_{31} + 6k_{32})s + (-11 - 2k_{11} + 14k_{31} + 15k_{32})s^2 \\
&+ (-17 - 8k_{11} + 21k_{31} + 24k_{32})s^3 + (-14 - 12k_{11} + 28k_{31} + 28k_{32})s^4 \\
&+ (-22 - 14k_{11} + 28k_{31} + 28k_{32})s^5 + (-6 - 12k_{11} + 24k_{31} + 28k_{32})s^6 \\
&+ (-12 - 8k_{11} + 16k_{31} + 8k_{32})s^7 + (-4 + 8k_{31} + 12k_{32})s^8 + 4k_{31}s^9
\end{aligned}
$$

and

$$
\bar{K} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \\ k_{31} & k_{32} \end{bmatrix} \tag{22}
$$

The matrix (22) is chosen so that (21) be a positive polynominal solution of the minimal degree $r = 1$. For $k_{12} = k_{21} = k_{22} = k_{31} = k_{32} = 0$ and $k_{11} = \frac{-(1+s-2s^2)}{2(s^2+2s^3+2s^4+s^5)}$ we obtain the same solution (17). ♦

Example 3. Find a positive polynomial solution to eqn. (2) with

$$
A(s) = \begin{bmatrix} s^2+1 & s \\ s & 2s+1 \end{bmatrix}, \quad B(s) = \begin{bmatrix} s^3+2s^2+s+1 \\ 3s^2+2s \end{bmatrix} \tag{23}
$$

In this case the matrix $A(s)$ is regular and

$$
A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}
$$

$$
B_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
$$

$$
\hat{A} = -\begin{bmatrix} 0 & 0 & 0 & 0 \\ A_1 & 0 & 0 & 0 \\ A_2 & A_1 & 0 & 0 \\ 0 & A_2 & 0 & 0 \end{bmatrix} = -\begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 1&2&0&0&0&0&0&0 \\ 1&0&0&1&0&0&0&0 \\ 0&0&1&2&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}
$$

It is easy to check that for (23) the condition (10) is satisfied. Using (13) for $r = m - n = 1$, we obtain

$$\hat{X} = (I + \hat{A} + \hat{A}^2)\bar{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 & 0 & 0 \\ 2 & 5 & -1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 2 \\ 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$X(s) = \begin{bmatrix} s+1 \\ s \end{bmatrix}$$

♦

5. Extensions and Applications

5.1. Extensions

Consider the polynomial matrix equation

$$A(s)X(s)B(s) = C(s) \tag{24}$$

where $A(s) \in \mathbb{R}^{p \times q}[s]$, $B(s) \in \mathbb{R}^{t \times k}[s]$, $C(s) \in \mathbb{R}^{p \times k}[s]$ are given and $X(s) \in \mathbb{R}^{q \times t}[s]$ is unknown. Defining

$$A(s)X(s) = D(s) \tag{25}$$

we may write (24) as

$$D(s)B(s) = C(s)$$

or

$$B^T(s)D^T(s) = C^T(s) \tag{26}$$

Solving (26) we may find $D(s)$ and then from (25) we may find the desired solution $X(s)$ of (24).

If the matrices $A(s)$ and $B(s)$ of (2) are rational, then by premultiplication of (2) by the diagonal matrix consisting of the least common denominators of $A(s)$ and $B(s)$ we obtain a suitable polynomial matrix equation. The same approach can also be applied to eqn. (24) if $A(s)$, $B(s)$ and $C(s)$ are rational matrices.

5.2. Some Applications

Consider the closed-loop system shown in Fig. 1. Let the transfer matrix $T_p = T(s)$ of the plant be given. We are looking for the transfer matrix $T_c = T_c(s)$ of a positive controller such that the transfer matrix of the closed-loop system is equal to the desired one $T_m = T_m(s)$, i.e.

$$[I + T_pT_c]^{-1}T_p = T_m \tag{27}$$

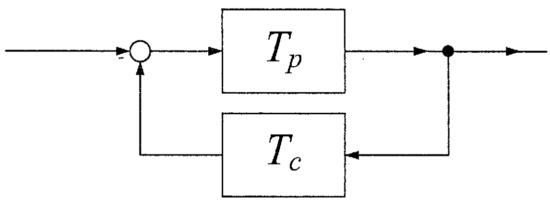

Fig. 1. The closed-loop system used to illustrate a potential application.

A controller is called positive if its transfer matrix has a positive realisation (i.e. the controller is a positive system). Let

$$T_m = N_mD_m^{-1}, \quad T_p = D_p^{-1}N_p, \quad T_c = \frac{X}{d} \tag{28}$$

where (N_m, D_m) and (N_p, D_p) are right and left coprime pairs, respectively, and d is the least common denominator of T_c. Substitution of (28) into $T_p = [I + T_pT_c]T_m$ yields the polynomial matrix equation

$$N_pXN_m = d(N_pD_m - D_pN_m) \tag{29}$$

To find a positive controller, we preassume a polynomial d (which corresponds to T_c having a positive realisation) and solve first the polynomial matrix equation

$$YN_m = d(N_pD_m - D_pN_m) \tag{30}$$

for Y. Then we may find the desired polynomial matrix X from the polynomial equation

$$N_pX = Y \tag{31}$$

Therefore the problem of finding a positive controller has been reduced to solving two polynomial matrix equations (30) and (31).

6. Conclusions

Necessary and sufficient conditions for the existence of positive solutions to polynomial matrix equations have been established. Two methods for determination of positive solutions have been proposed and illustrated by numerical examples. It has been shown that the determination of the transfer matrix of a positive controller of the closed-loop system with a given transfer matrix can be reduced to the determination of solutions to two suitable polynomial matrix equations. Some applications and numerical aspects of the proposed algorithms will be considered in a forthcoming paper.

References

Berman A. and Plemmons R.J. (1994): *Nonnegative Matrices in the Mathematical Sciences.* — Philadelphia: SIAM.

Cohen J.E. and Rothblum U.G. (1993): *Nonnegative ranks, decompositions, and factorizations of nonnegative matrices.* — Linear Algebra and its Applications, Vol.190, pp.149–168.

Emre E. and Silverman L.M. (1981): *The equation* $XR + XY = \Phi$*: A characterization of solutions.* — SIAM J. Contr. Optim., Vol.19, No.1, pp.33–38.

Feinstein J. and Bar-ness Y. (1984): *The solution of the matrix polynomial equation* $A(s)X(s) + B(s)Y(s) = C(s)$. — IEEE Trans. Autom. Contr., Vol.AC-29, No.1, pp.75–77.

Kaczorek T. (1986a): *Zero-degree solution to* $AX + BY = C$ *and invariant factors assignment problem.* — Bull. Acad. Polon. Sci., Ser. Sci. Techn., 1986, Vol.34, No.9–10, pp.553–558.

Kaczorek T. (1986b): *Zero-degree solution to the bilateral polynomial matrix equations.* — Bull. Acad. Polon. Sci., Ser. Sci. Techn., Vol.34, No.9-10, pp.547–552.

Kaczorek T. (1987): *Two-sided polynomial matrix equations.* — Bull. Pol. Acad. Techn. Sci., Vol.35, No.11–12, pp.667–671.

Kaczorek T. (1992): *Linear Control Systems.* — New York: Research Studies Press. J. Wiley.

Kaczorek T. (1997): *Realisation problem for discrete-time positive linear systems.* — Appl. Math. Comp. Sci., Vol.7, No.1, pp.117–124.

Kučera V. (1972): *A contribution to matrix equations.* — IEEE Trans. Automat. Contr., Vol.AC-17, No.6, pp.344–347.

Kučera V. (1979): *Discrete Linear Control.* — Prague: Academia Prague.

Maeda H. and Kodama S. (1981): *Positive realization of difference equations.* — IEEE Trans. Circ. Syst., Vol.CAS-28, No.1, pp.39–47.

Maeda H., Kodama S. and Kajiya F. (1977): *Compartmental system analysis: Realization of a class of linear systems with physical constraints.* — IEEE Trans. Circ. Syst., Vol.CAS-24, No.1, 1977, pp.8–14.

Ohta Y., Maeda H. and Kodama S. (1984): *Reachability, observability and realizability of continuous-time positive systems.* — SIAM J. Contr. Optim., Vol.22, No.2, pp.171–180.

Qianhua W. and Zhongjun Z. (1987): *Algorithm for obtaining the proper relatively prime matrices of polynomial matrices.* — Int. J. Contr., Vol.46, No.3, pp.769–784.

Solak M. (1985): *Differential representations of multivariable linear systems with disturbances and polynomial matrix equations* $PX + RY = W$ *and* $PX + YN = V$. — IEEE Trans. Autom. Contr., Vol.AC-30, No.7, pp.687–690.

Šebek M. (1980): *n-D polynomial matrix equations.* — IEEE Trans. Automat. Contr., Vol.AC-33, No.5, pp.499–502.

Šebek M. (1983): *2-D polynomial equations.* — Kybernetika, Vol.19, No.3, pp.212–224.

Šebek M. (1989): *Two-sided equations and skew primeness for n-D polynomial matrices.* — Syst. Contr. Lett., Vol.12, No.3, pp.331–337.

Šebek M. and Kučera V. (1981): *Matrix equations arising in regular problems.* — Kybernetika, Vol.17, No.2, pp.128–139.

Van den Hof J.M. (1997): *Realization of positive linear systems.* — Linear Algebra and its Applications, Vol.256, pp.287–308.

Wolovich W.A. (1978): *Skew prime polynomial matrices.* — IEEE Trans. Automat. Contr., Vol.AC-23, No.5, pp.880–887.

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